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Note on the theory of Elliptic Integrals.

Von A. CAYLEY in Cambridge.

The equation

$$\frac{M dy}{\sqrt{1-y^2 \cdot 1-k^2 y^2}} = \frac{dx}{\sqrt{1-x^2 \cdot 1-k^2 x^2}}$$

is integrable algebraically when M is rational; and so long as the modulus is arbitrary, then conversely, in order that the equation may be integrable algebraically, M must be rational: for particular values however of the modulus, the equation is integrable algebraically for values of the form M , or what is the same thing $\frac{1}{M}$, = a rational quantity \pm square root of a negative rational quantity, or say = $\frac{1}{p} (l + m \sqrt{-n})$, where l, m, n, p are integral and n is positive; we may for shortness call this a half-rational numerical value. The theory is considered by Abel in two Memoirs in the Astr. Nach. Nos. 138 & 147 (1828), being the Memoirs XIII & XIV in the Œuvres Complètes (Christiania 1839); and I here reproduce the investigation in a somewhat altered (and, as it appears to me, improved) form.

Putting the two differentials each = du , we have $x = \text{sn}(u + \alpha)$, $y = \text{sn}\left(\frac{u}{M} + \beta\right)$; and the question is whether there exists an algebraical relation between these functions; or what is the same thing, an algebraical relation between the functions $x = \text{sn } u$ and $y = \text{sn } \frac{u}{M}$.

Suppose that A and B are independent periods of $\text{sn } u$; so that $\text{sn}(u + A) = \text{sn } u$, $\text{sn}(u + B) = \text{sn } u$, and that every other period is = $m A + n B$ where m and n are integers. Then if n has successively the values $u, u + A, u + 2A$, etc. the value of x remains always the same, and if x and y are algebraically connected, y can have only a finite number of values: there are consequently integer values p', p'' for which $\text{sn } \frac{1}{M}(u + p' A) = \text{sn } \frac{1}{M}(u + p'' A)$: or writing $u - p' A$ for u and putting $p'' - p' = p$, there is an integer value p for which $\text{sn } \frac{1}{M}(u + p A) = \text{sn } \frac{1}{M} u$.

Similarly there is an integer value q for which $\operatorname{sn} \frac{1}{M}(u+qB) = \operatorname{sn} \frac{1}{M} u$; and we are at liberty to assume $q=p$; for if the original values are unequal, we have only in the place of each of them to substitute their least common multiple.

We have thus an integer p , for which

$$\operatorname{sn} \frac{1}{M}(u+pA) = \operatorname{sn} \frac{1}{M} u$$

$$\operatorname{sn} \frac{1}{M}(u+pB) = \operatorname{sn} \frac{1}{M} u.$$

There are consequently integers m, n, r, s such that

$$\frac{pA}{M} = mA + nB,$$

$$\frac{pB}{M} = rA + sB$$

equations which will constitute a single relation $\frac{p}{M} = m$, if $m = s$, $r = n = 0$; but in every other case will be two independent relations. In the case first referred to the modulus is arbitrary, and M is rational.

But excluding this case, the equations give

$$B(mA + nB) = A(rA + sB),$$

or what is the same thing

$$rA^2 - (m-s)AB - nB^2 = 0,$$

an equation which implies that the modulus has some one value out of a set of given values. The ratio $A:B$ of the two periods is of necessity imaginary, and hence the integers m, n, r, s must be such that $(m-s)^2 + nr$ is negative.

The foregoing equations may be written

$$\left(m - \frac{p}{M}\right) A + nB = 0,$$

$$rA + \left(s - \frac{p}{M}\right) B = 0,$$

whence eliminating A and B we have

$$\left(m - \frac{p}{M}\right) \left(s - \frac{p}{M}\right) - nr = 0,$$

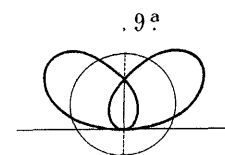
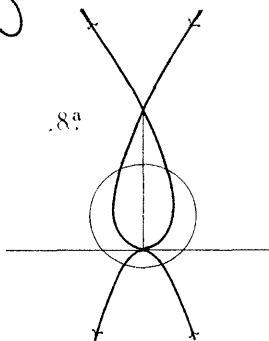
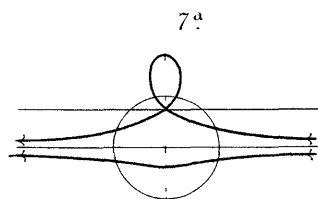
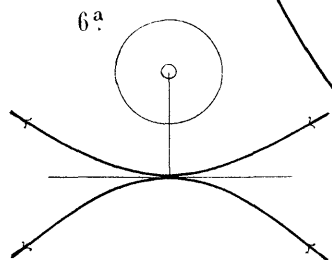
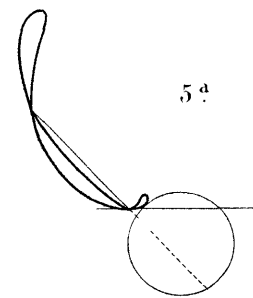
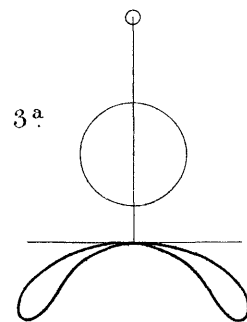
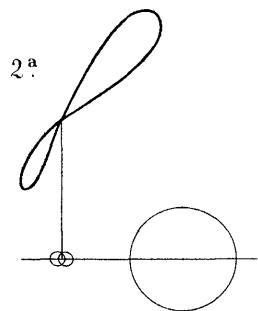
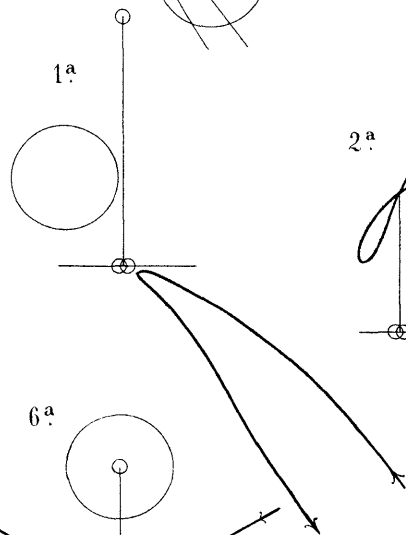
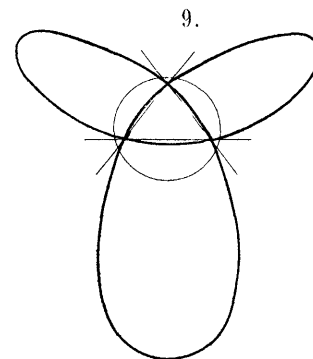
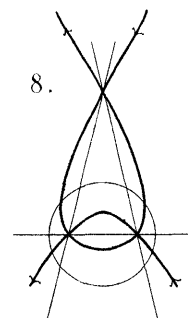
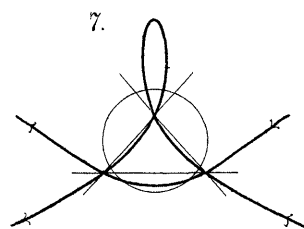
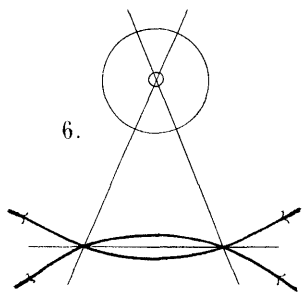
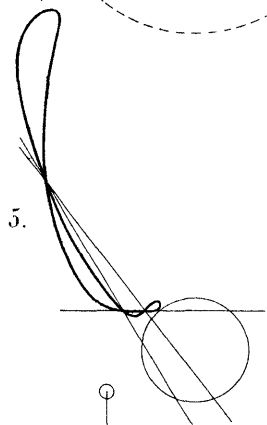
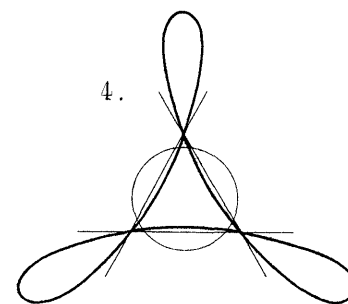
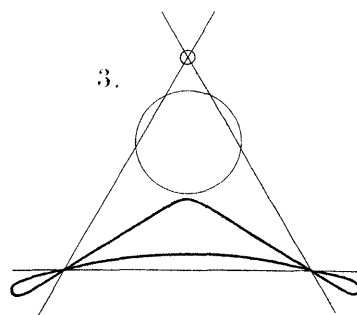
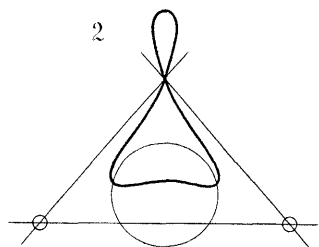
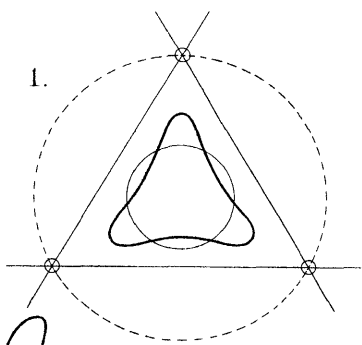
that is

$$\left(\frac{p}{M}\right)^2 - (m+s) \frac{p}{M} + ms - nr = 0.$$

and consequently

$$\frac{p}{M} = \frac{1}{2} (m+s) \pm \frac{1}{2} \sqrt{(m-s)^2 + nr}$$

where, by what precedes, the integer under the radical sign is negative: and we have thus the above mentioned theorem.



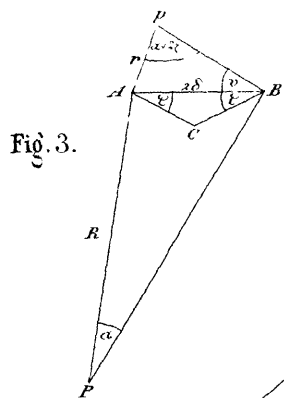
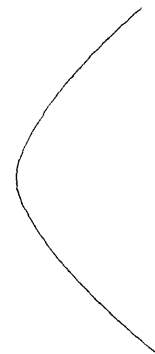
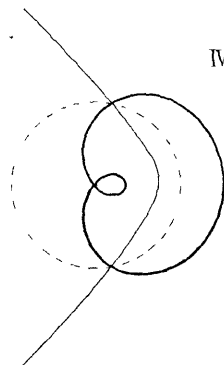
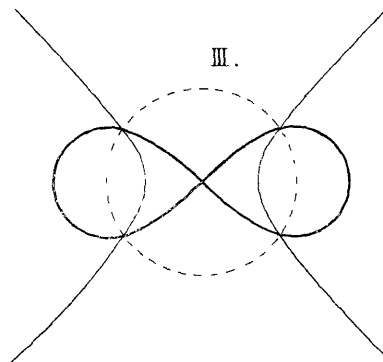
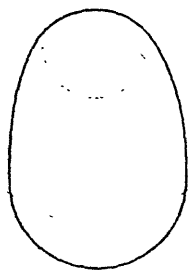
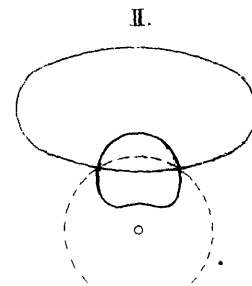
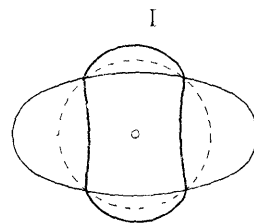
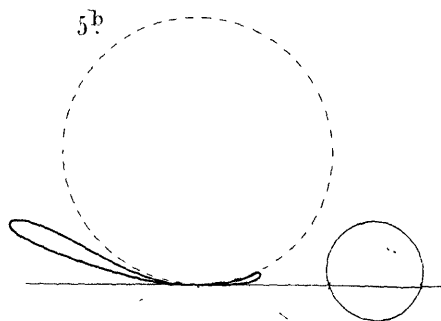
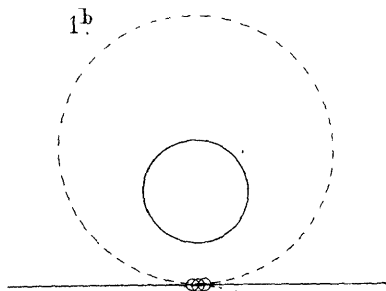


Fig. 4.

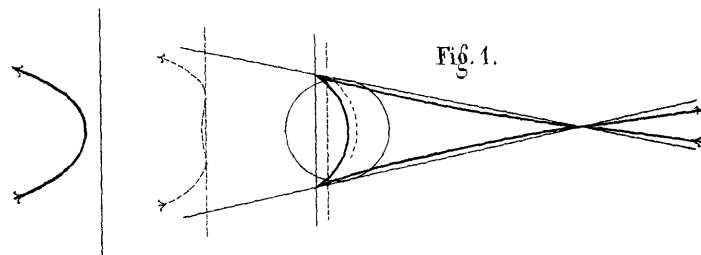
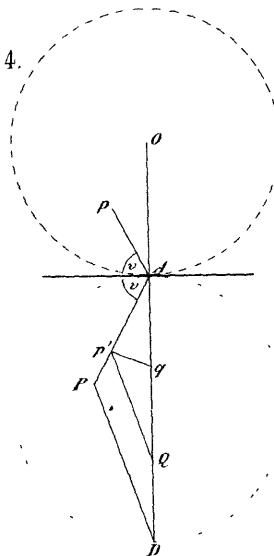


Fig. 1.

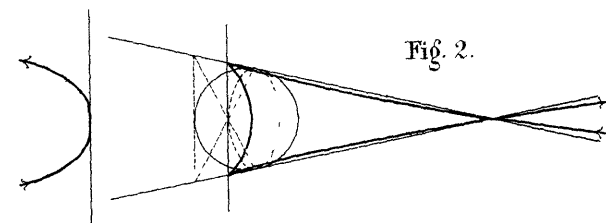


Fig. 2.

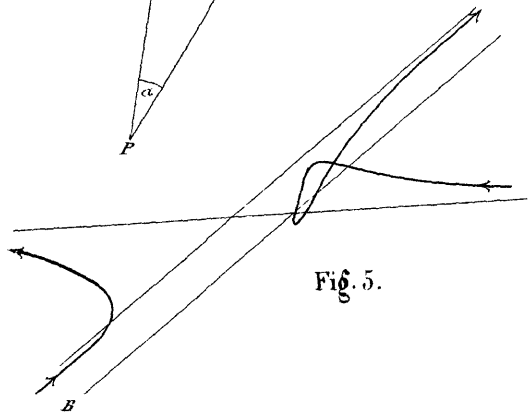


Fig. 5.

As a very general example consider the two rational transformations

$$z = (x, u, v); \text{ mod. eq. } Q(u, v) = 0; \quad \frac{Ndz}{\sqrt{1-z^2 \cdot 1-v^2z^2}} = \frac{dx}{\sqrt{1-x^2 \cdot 1-u^2x^2}}$$

$$y = (z, v, w); \text{ mod. eq. } P(v, w) = 0; \quad \frac{Mdy}{\sqrt{1-y^2 \cdot 1-w^2y^2}} = \frac{dz}{\sqrt{1-z^2 \cdot 1-v^2z^2}}$$

viz. z is taken to be a rational function of x , and of the modular fourth roots u, v ; and y to be a rational function of z , and of the modular fourth roots v, w ; the transformations being (to fix the ideas) of different orders. We have y a rational function of x , corresponding to the differential relation

$$\frac{MN \cdot y}{\sqrt{1-y^2 \cdot 1-w^2y^2}} = \frac{dx}{\sqrt{1-x^2 \cdot 1-u^2x^2}}$$

Suppose here $w^8 = u^8$, or say $w = \theta u$, θ being an eight root of unity: we then have $Q(u, v) = 0$, $P(v, \theta u) = 0$, equations which determine u , and the differential equation is then

$$\frac{MNdy}{\sqrt{1-y^2 \cdot 1-w^2y^2}} = \frac{dx}{\sqrt{1-x^2 \cdot 1-u^2x^2}}$$

an equation the algebraical integral of where is $y = \alpha$ rational function of x as above: hence by what precedes we have

$$\frac{1}{MN} = \frac{1}{2p} \{m + s \pm \sqrt{(m-s)^2 + nr}\}$$

a half-rational numerical value, as above.

To explain what the algebraical theorem implied herein is observe that the equations $Q(u, v) = 0$, $P(v, \theta u) = 0$, give for u an algebraical equation: admitting θ as an adjoint radical, suppose that an irreducible factor is $\varphi(u)$: and take u to be determined by the equation $\varphi u = 0$; then v , and consequently also any rational function $\frac{1}{MN}$ of (u, v) can be expressed as a rational integral function of u , of a degree which is at most equal to the degree of the functions φu less unity. The theorem is that in virtue of the equation $\varphi u = 0$, this rational function of u becomes equal to a half-rational numerical value as above. Thus in a simple case which actually presented itself, the equation $\varphi u = 0$ was $u^2 - 4u + 1 = 0$; and $\frac{1}{MN}$ had the value $u - 2$, which in virtue of this equation becomes $= \pm \sqrt{-3}$.

Thus if the second transformation be the identity $z = y$, $w = v$, $M = 1$; we have $v = \theta u$; and the equations are

$$y = (x, u, \theta u), \quad Q(u, \theta u) = 0, \quad \frac{Ndy}{\sqrt{1-y^2 \cdot 1-u^2y^2}} = \frac{dx}{\sqrt{1-x^2 \cdot 1-u^2x^2}}$$

and in particular if the relation between y, x be given by the cubic transformation

$$y = \frac{\frac{v+2u^3}{v}x + \frac{u^6}{v^2}x^3}{1 + vu^2(v+2u^3)x^2}$$

so that the modular equation $Q(u, v) = 0$, is $u^4 - v^4 + 2uv(1 - u^2v^2) = 0$; then writing herein $v = \theta u$, and taking θ a prime eighth root of unity, that is a root of $\theta^4 + 1 = 0$, we have

$$Q(u, \theta u) = -2\theta^3 u^2 (\theta u^2 + \theta^2 + u^4);$$

viz. disregarding the factor u^2 , the equation for u is $u^4 + \theta u^2 + \theta^2 = 0$; or if ω be an imaginary cube root of unity ($\omega^2 + \omega + 1 = 0$) this is $(u^2 - \omega\theta)(u^2 - \omega^2\theta) = 0$; so that a value of u^2 is $u^2 = -\omega\theta$.

Assuming then $\theta^4 + 1 = 0$, $v = \theta u$ and $u^2 = -\omega\theta$, we have $(v + 2u^3)v = \theta^3\omega(1 + 2\omega)$, $= \theta^3\omega(\omega - \omega^2)$; $\frac{v+2u^3}{v} = \omega - \omega^2$; $\frac{u^6}{v^2} = \omega^2$, $(v + 2u^3)v u^2 = -\omega^2(\omega - \omega^2)$, $u^8 = \omega^4\theta^4 = -\omega$ and the formula becomes

$$y = \frac{(\omega - \omega^2)x + \omega^2 x^3}{1 - \omega^2(\omega - \omega^2)x^2} \text{ giving } \frac{dy}{\sqrt{1 - y^2 \cdot 1 + \omega y^2}} = \frac{(\omega - \omega^2) dx}{\sqrt{1 - x^2 \cdot 1 + \omega x^2}}$$

(where as before $(\omega^2 + \omega + 1 = 0)$, a result which can be at once verified. We have $(\omega - \omega^2)^2 = -3$; or the coefficient $\omega - \omega^2$ in the differential equation is $=\sqrt{-3}$, which is of the form mentioned in the general theorem.

We might instead of $z = y$, have assumed between y and z the relation corresponding to any other of the six linear transformations of an elliptic integral, and thus have obtained in each case for a properly determined value of the modulus, a cubic transformation to the same modulus.

Cambridge, 10. April 1877.