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On some formulae in Elliptic Integrals.

Von A. Cayley in Cambridge.

I reproduce in a modified form an investigation contained in the memoir, Zolotareff, Sur la méthode d'integration de M. Tchébychef, Annalen t. V (1872) pp. 560—580.

Starting from the quartic

$$(a, b, c, d, e)(x, 1)^4, = a \cdot x - \alpha \cdot x - \beta \cdot x - \gamma \cdot x - \delta,$$

we derive from it the quartic

$$(a_1, b_1, c_1, d_1, e_1)(x_1, 1)^4 = a_1 \cdot x_1 - \alpha_1 \cdot x_1 - \beta_1 \cdot x_1 - \gamma_1 \cdot x_1 - \delta_1,$$

where, writing for shortness

$$\begin{aligned} \lambda &= -\alpha + \beta + \gamma - \delta, \\ \mu &= \quad \alpha - \beta + \gamma - \delta, \\ \nu &= \quad \alpha + \beta - \gamma - \delta, \end{aligned}$$

the roots of the new quartic are

$$\begin{aligned} \alpha_1 &= \theta + \frac{\mu\nu}{2\lambda}, \\ \beta_1 &= \theta + \frac{\nu\lambda}{2\mu}, \\ \gamma_1 &= \theta + \frac{\lambda\mu}{2\nu}, \\ \delta_1 &= \theta, \end{aligned}$$

θ being arbitrary: the differences of the roots $\alpha_1, \beta_1, \gamma_1, \delta_1$ are, it will be observed, functions of the differences of the roots $\alpha, \beta, \gamma, \delta$.

We assume $a_1 = a = 1$, nevertheless retaining in the formulae a_1 or a (each meaning 1), whenever, for the sake of homogeneity, it is convenient to do so. The relations between the remaining coefficients b_1, c_1, d_1, e_1 and b, c, d, e are of course to be calculated from the formulae $-4b = \Sigma\alpha$, $6c = \Sigma\alpha\beta$, &c. and the like formulae $-4b_1 = \Sigma\alpha_1$, $6c_1 = \Sigma\alpha_1\beta_1$, &c. We thus have

$$-4b_1 = 4\theta + \tfrac{1}{2}\ \Sigma\frac{\mu\nu}{\lambda},$$

$$6c_1 = 6\theta^2 + \tfrac{3}{2}\theta\ \Sigma\frac{\mu\nu}{\lambda} + \tfrac{1}{4}\Sigma\lambda^2,$$

$$-4d_1 = 4\theta^3 + \tfrac{3}{2}\theta^2\ \Sigma\frac{\mu\nu}{\lambda} + \tfrac{1}{2}\theta\ \Sigma\lambda^2 + \tfrac{1}{8}\lambda\mu\nu,$$

$$e_1 = \theta^4 + \tfrac{1}{2}\theta^3\ \Sigma\frac{\mu\nu}{\lambda} + \tfrac{1}{4}\theta^2\ \Sigma\lambda^2 + \tfrac{1}{8}\theta\lambda\mu\nu,$$

where $\Sigma\frac{\mu\nu}{\lambda} = \frac{1}{\lambda\mu\nu}\Sigma\lambda^2\mu^2$.

Writing for shortness

$C = ac - b^2$, $I = ae - 4bd + 3c^2$,

$D = a^2d - 3abc + 2b^3$, $J = ace - ad^2 - b^2e + 2bcd - c^3$,

$E = a^3e - 4a^2bd + 6ab^2c - 3b^4 (= a^2I - 3C^2)$, $B = \frac{-a^2I + 12C^2}{4D}$,

we have

$$\begin{aligned}
\Sigma\lambda &= -4(b+\delta),\\
\Sigma\lambda^2 &= -48C,\\
\Sigma\lambda\mu &= 24C + 8(b+\delta)^2,\\
\lambda\mu\nu &= 32D,\\
\Sigma\lambda^2\mu^2 &= 64(-a^2I + 12C^2),
\end{aligned}$$

where the last equation may be verified by means of the formula

$$(\Sigma\lambda\mu)^2 = \Sigma\lambda^2\mu^2 + 2\lambda\mu\nu\Sigma\lambda.$$

And we hence obtain

$$\begin{aligned}
a_1 &= 1,\\
b_1 &= -\theta - B,\\
c_1 &= \theta^2 + 2B\theta - 2C,\\
d_1 &= -\theta^3 - 3B\theta^2 + 6C\theta - D,\\
e_1 &= \theta^4 + 4B\theta^3 - 12C\theta^2 + 4D\theta.
\end{aligned}$$

And consequently

$$(a_1, b_1, c_1, d_1, e_1)(x_1, 1)^4 = (1, -B, -2C, -D, 0)(x_1 - \theta, 1)^4.$$

Hence also

$$I_1 = a_1e_1 - 4b_1d_1 + 3c_1^2, = -4BD + 12C^2, = a^2I;$$

$$\begin{aligned}
J_1 = a_1c_1e_1 - a_1d_1^2 - b_1^2e_1 + 2b_1c_1d_1 - c_1^3, &= -D^2 + 8C^3 - 4BCD,\\
&= -D^2 + 8C^3 + C(a^2I - 12C^2),\\
&= a^2CI - 4C^3 - D^2,\\
&= a^3J,
\end{aligned}$$

where, as regards this last equation $a^2CI - 4C^3 - D^2 = a^3J$, observe that C, D are the leading coefficients of the Hessian H and the cubicovariant Φ of the quartic function U, and hence that the identity

$-\Phi^2 = JU^3 - IU^2\mathrm{H} + 4\mathrm{H}^3$, attending only to the term in x^6, becomes $-D^2 = a^3J - a^2CI + 4C^3$, which is the equation in question.

We thus have $I_1 = I$, $J_1 = J$; viz. the functions $(a, b, c, d, e)(x, 1)^4$, $(a_1, b_1, c_1, d_1, e_1)(x_1, 1)^4$, are linearly transformable the one into the other, and that by a unimodular substitution $x_1 = \varrho x + \sigma$, $y_1 = \varrho' x + \sigma'$, where $\varrho\sigma' - \varrho'\sigma = 1$. It may be remarked that we have $(a, b, c, d, e)(x, 1)^4 = (1, 0, C, D, E)(x + b, 1)^4$; and hence the theorem may be stated in the form: the quartic functions $(1, 0, C, D, E)(x, 1)^4$, and $(1, -B, -2C, -D, 0)(x_1, 1)^4$, are transformable the one into the other by a unimodular substitution: or again substituting for E its value $a^2I - 3C^2, = -4BD + 9C^2$, the quartic functions $(1, 0, C, D, -4BD + 9C^2)(x, 1)^4$ and $(1, -B, -2C, -D, 0)(x_1, 1)^4$ are linearly transformable the one into the other by a unimodular substitution. In this last form B, C, D are arbitrary quantities; it is at once verified that the invariants I, J have the same values for the two functions respectively; and the theorem is thus self-evident.

Reverting to the expressions for $\alpha_1, \beta_1, \gamma_1, \delta_1$ we obtain

$$\alpha_1 - \delta_1 = \frac{\mu\nu}{2\lambda};\ \beta_1 - \gamma_1 = \frac{\lambda}{2\mu\nu}(\nu^2 - \mu^2), = \frac{\alpha - \delta \cdot \beta - \gamma}{\alpha_1 - \delta_1},$$

$$\beta_1 - \delta_1 = \frac{\nu\lambda}{2\mu};\ \gamma_1 - \alpha_1 = \frac{\mu}{2\nu\lambda}(\lambda^2 - \nu^2), = \frac{\beta - \delta \cdot \gamma - \alpha}{\beta_1 - \delta_1},$$

$$\gamma_1 - \delta_1 = \frac{\lambda\mu}{2\nu};\ \alpha_1 - \beta_1 = \frac{\nu}{2\lambda\mu}(\mu^2 - \lambda^2), = \frac{\gamma - \delta \cdot \alpha - \beta}{\gamma_1 - \delta_1}.$$

Hence also

$$\alpha - \delta \cdot \beta - \gamma,\quad \beta - \delta \cdot \gamma - \alpha,\quad \gamma - \delta \cdot \alpha - \beta$$
$$= \alpha_1 - \delta_1 \cdot \beta_1 - \gamma_1,\quad \beta_1 - \delta_1 \cdot \gamma_1 - \alpha_1,\quad \gamma_1 - \delta_1 \cdot \alpha - \beta,$$

which agrees with the foregoing equations $I_1 = I$ and $J_1 = J$ (since I, J are functions of the first set of quantities and I_1, J_1 the like functions of the second set; in fact $I = \frac{1}{24}(P^2 + Q^2 + R^2)$, and $J = \frac{1}{432}(Q - R)(R - P)(P - Q)$, if for a moment the quantities are called (P, Q, R).

We consider now the differential expression $\dfrac{dx}{\sqrt{x - \alpha \cdot x - \beta \cdot x - \gamma \cdot x - \delta}}$; to transform this into the elliptic form, assume

$$k^2 = -\frac{\alpha - \beta \cdot \gamma - \delta}{\gamma - \alpha \cdot \beta - \delta};\ \operatorname{sn}^2 a = \frac{\gamma - \alpha}{\gamma - \delta},$$

(where a is of course not the coefficient, $= 1$, heretofore represented by that letter: as a will only occur under the functional signs sn, cn, dn, there is no risk of ambiguity). And then further

$$x = \frac{a \operatorname{sn}^2 u - \delta \operatorname{sn}^2 a}{\operatorname{sn}^2 u - \operatorname{sn}^2 a}.$$

Forming the equations

$$k^2 \operatorname{sn}^2 a = -\frac{\alpha-\beta}{\beta-\delta}, \quad k^2 \operatorname{sn}^4 a = -\frac{\gamma-\alpha\cdot\alpha-\beta}{\gamma-\delta\cdot\beta-\delta},$$

we deduce without difficulty

$$\operatorname{sn}^2 a = \frac{\gamma-\alpha}{\gamma-\delta}, \quad \frac{\operatorname{sn}^2 u}{\operatorname{sn}^2 a} = \frac{x-\delta}{x-\alpha},$$

$$\operatorname{cn}^2 a = \frac{\alpha-\delta}{\gamma-\delta}, \quad \frac{\operatorname{cn}^2 u}{\operatorname{cn}^2 a} = \frac{x-\gamma}{x-\alpha},$$

$$\operatorname{dn}^2 a = \frac{\alpha-\delta}{\beta-\delta}, \quad \frac{\operatorname{dn}^2 u}{\operatorname{dn}^2 a} = \frac{x-\beta}{x-\alpha},$$

$$1 - k^2 \operatorname{sn}^4 a = \frac{(\alpha-\delta)(-\alpha+\beta+\gamma-\delta)}{\beta-\delta\cdot\gamma-\delta}, = \frac{\lambda(\alpha-\delta)}{\beta-\delta\cdot\gamma-\delta}$$

the use of which last equation will presently appear.

We hence obtain

$$2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \, du = -(\alpha-\delta) \operatorname{sn}^2 a \frac{dx}{(x-\alpha)^2},$$

$$\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u = \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \frac{\sqrt{x-\alpha\cdot x-\beta\cdot x-\gamma\cdot x-\delta}}{(x-\alpha)^2},$$

and consequently

$$2\,du = -\frac{(\alpha-\delta)\operatorname{sn} a}{\operatorname{cn} a \operatorname{dn} a} \frac{dx}{\sqrt{x-\alpha\cdot x-\beta\cdot x-\gamma\cdot x-\delta}},$$

or, reducing the coefficient,

$$\frac{dx}{\sqrt{x-\alpha\cdot x-\beta\cdot x-\gamma\cdot x-\delta}} = \frac{-2}{\sqrt{\gamma-\alpha\cdot\beta-\delta}}\,du,$$

which is the required formula.

We next have

$$\operatorname{sn}^2 2a = \frac{4\operatorname{sn}^2 a \operatorname{cn}^2 a \operatorname{dn}^2 a}{(1-k^2\operatorname{sn}^4 a)^2} = \frac{4\cdot\beta-\delta\cdot\gamma-\alpha}{\lambda^2}, = \frac{\gamma_1-\alpha_1}{\gamma_1-\delta_1}$$

in virtue of the foregoing values

$$\gamma_1-\alpha_1 = \frac{2\mu}{\nu\lambda}(\beta-\delta)(\gamma-\alpha) \text{ and } \gamma_1-\delta_1 = \frac{\lambda\mu}{2\nu}.$$

Moreover

$$k^2 = -\frac{\alpha-\beta.\gamma-\delta}{\gamma-\alpha\cdot\beta-\delta}, = -\frac{\alpha_1-\beta_1\cdot\gamma_1-\delta_1}{\gamma_1-\alpha_1\cdot\beta_1-\delta_1}.$$

Hence the like formulae with the same value of k^2, and with $2a$ in place of a, will be applicable to the like differential expression in x_1: viz. assuming

$$x_1 = \frac{\alpha_1 \operatorname{sn}^2 u_1 - \delta_1 \operatorname{sn}^2 2a}{\operatorname{sn}^2 u_1 - \operatorname{sn}^2 2a}$$

we have

$$\frac{dx_1}{\sqrt{x_1-\alpha_1\cdot x_1-\beta_1\cdot x_1-\gamma_1\cdot x_1-\delta_1}} = \frac{-2}{\sqrt{\gamma_1-\alpha_1\cdot\beta_1-\delta_1}}\,du_1.$$

We have thus the integral of the differential equation

$$\frac{dx_1}{\sqrt{x_1-\alpha_1\cdot x_1-\beta_1\cdot x_1-\gamma_1\cdot x_1-\delta_1}} = \frac{dx}{\sqrt{x-\alpha\cdot x-\beta\cdot x-\gamma\cdot x-\delta}}$$

(the two quartic functions being of course connected as before) viz. assuming x, x_1 functions of u, u_1 respectively as above and recollecting that $\gamma_1-\alpha_1\cdot\beta_1-\delta_1=\gamma-\alpha\cdot\beta-\delta$, we have $du_1=du$; and therefore $u_1=u+f$ (f an arbitrary constant); the required integral is thus given by the equations

$$\frac{\operatorname{sn}^2 u}{\operatorname{sn}^2 a}=\frac{x-\delta}{x-\alpha};\quad \frac{\operatorname{sn}^2(u+f)}{\operatorname{sn}^2 2a}=\frac{x_1-\delta_1}{x_1-\alpha_1};\ (f \text{ the constant of integration}).$$

Using the formula

$$\operatorname{sn}(u+f)=\frac{\operatorname{sn}^2 u-\operatorname{sn}^2 f}{\operatorname{sn} u \operatorname{cn} f \operatorname{dn} f-\operatorname{sn} f\cdot\operatorname{cn} u\operatorname{dn} u},$$

we obtain

$$\frac{x_1-\delta_1}{x_1-\alpha_1}\operatorname{sn}^2 2a=\frac{\left\{(x-\delta)\operatorname{sn}^2 a-(x-\alpha)\operatorname{sn}^2 f\right\}^2}{\left\{\sqrt{x-\alpha\cdot x-\delta}\ \operatorname{sn} a\operatorname{cn} f\operatorname{dn} f-\sqrt{x-\beta\cdot x-\gamma}\ \operatorname{sn} f\operatorname{cn} a\operatorname{dn} a\right\}^2},$$

which is the general integral.

We obtain a particular integral of a very simple form by assuming $f=a$, viz. this is

$$\frac{x_1-\delta_1}{x_1-\alpha_1}\operatorname{sn}^2 2a=\frac{\operatorname{sn}^2 a}{\operatorname{cn}^2 a\operatorname{dn}^2 a}\ \frac{(\alpha-\delta)^2}{\left\{\sqrt{x-\alpha\cdot x-\delta}-\sqrt{x-\beta\cdot x-\gamma}\right\}^2};$$

this is

$$\frac{x_1-\delta_1}{x_1-\alpha_1}\ \frac{\gamma_1-\alpha_1}{\gamma_1-\delta_1}=\frac{\gamma-\alpha\cdot\beta-\delta}{\left\{\sqrt{x-\alpha\cdot x-\delta}-\sqrt{x-\beta\cdot x-\gamma}\right\}^2},$$

or writing $\gamma-\alpha\cdot\beta-\delta=\gamma_1-\alpha_1\cdot\beta_1-\delta_1$, reducing and inverting, we have

$$\frac{x_1-\alpha_1}{x_1-\delta_1}=\frac{1}{\beta_1-\delta_1\cdot\gamma_1-\delta_1}\left\{\sqrt{x-\alpha\cdot x-\delta}-\sqrt{x-\beta\cdot x-\gamma}\right\}^2,$$

which may also be written in the equivalent forms

$$\frac{x_1-\beta_1}{x_1-\delta_1}=\frac{1}{\gamma_1-\delta_1\cdot\alpha-\delta_1}\left\{\sqrt{x-\beta\cdot x-\delta}-\sqrt{x-\gamma\cdot x-\alpha}\right\}^2,$$

$$\frac{x_1-\gamma_1}{x_1-\delta_1}=\frac{1}{\alpha_1-\delta_1\cdot\beta_1-\delta_1}\left\{\sqrt{x-\gamma\cdot x-\delta}-\sqrt{x-\alpha\cdot x-\beta}\right\}^2.$$

In fact from the first equation we have

$$\frac{\alpha_1-\delta_1\cdot\beta_1-\delta_1\cdot\gamma_1-\delta_1}{x_1-\delta_1}=(\beta_1-\delta_1)(\gamma_1-\delta_1)$$
$$-\left\{\sqrt{x-\alpha\cdot x-\delta}-\sqrt{x-\beta\cdot x-\gamma}\right\}^2.$$

where the expression on the right hand side is

$$\delta_1^2-\delta_1(\alpha_1+\beta_1+\gamma_1)+\alpha_1\delta_1+\beta_1\gamma_1-2x^2+x(\alpha+\beta+\gamma+\delta)-\alpha\delta-\beta\gamma+2\sqrt{X},$$

X having here the value

$$X=x-\alpha\cdot x-\beta\cdot x-\gamma\cdot x-\delta.$$

Writing for a moment

$$\begin{aligned} P &= \alpha\delta + \beta\gamma\,, & P_1 &= \alpha_1\delta_1 + \beta_1\gamma_1\,, \\ Q &= \beta\delta + \gamma\alpha\,, & Q_1 &= \beta_1\delta_1 + \gamma_1\alpha_1\,, \\ R &= \gamma\delta + \alpha\beta\,, & R_1 &= \gamma_1\delta_1 + \alpha_1\beta_1\,, \end{aligned}$$

then by what precedes $Q_1 - R_1$, $R_1 - P_1$, $P_1 - Q_1$ are equal to $Q-R$, $R-P$, $P-Q$ respectively; that is $P_1 - P = Q_1 - Q = R_1 - R$, $=$(suppose) Ω, a function symmetrical in regard to $\alpha_1, \beta_1, \gamma_1$; α, β, γ: the equation therefore is

$$\frac{\alpha_1-\delta_1\cdot\beta_1-\delta_1\cdot\gamma_1-\delta_1}{x_1-\delta_1} = \delta_1(\delta_1-\alpha_1-\beta_1-\gamma_1)-2x^2+x(\alpha+\beta+\gamma+\delta)+2\sqrt{X}+\Omega,$$

or the relation is symmetrical in regard to α_1, β_1, γ_1; α, β, γ: and the first form implies therefore each of the other two forms.

Cambridge, 8 May 1877.