The Inversion of a Definite Integral.

By

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The following paper is intended as a contribution towards the solution of a problem proposed by Abel*). Let $f(s)$ and $\varkappa(s, t)$ be two given functions and c a given path of integration, it is required to determine if possible, a function $\varphi(t)$ such that

$$(1) \qquad f(s) = \int_c \varkappa(s, t)\, \varphi(t)\, dt.$$

This problem is not soluble in general because a function defined by a definite integral is usually subject to certain restrictions depending on the nature of the function $\varkappa(s, t)$, accordingly there are two goals to be aimed at: we must first find necessary or sufficient conditions to be satisfied by the function $f(s)$ in order that the equation may be soluble, and then we must give a method of determining the function $\varphi(t)$ when it is known to exist.

Fredholm has remarked**) that the above equation may be considered as a particular case of the more general equation

$$\psi(s) = \lambda f(s) = \varphi(s) + \lambda \int_a^b \varkappa(s, t)\, \varphi(t)\, dt$$

in which the parameter λ is finally made infinite, but it is difficult to find conditions that the formula which he gives for the solution of this equation should have any meaning when λ tends to infinity; also this method requires that the function $f(s)$ should be given for values of s lying between a and b.

We must therefore seek another method of determining the function φ which will lead more directly to the conditions to be imposed on f.

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*) Collected Works, Vol. II, p. 67.

**) Acta Math. 1903.

This is what I have endeavoured to do in § 1, the results are not very satisfactory, but the method is one of direct calculation and so can be used to give in a descriptive way a sufficient criterion for the existence of a solution; also when the function $\varkappa(s, t)$ is subject to certain restrictions the method will certainly lead to the solution if it exists.

In § 2 a particular integral equation is considered and the solution is obtained by means of Potential Theory. The result is then used to reduce the solution of a more general integral equation to that of an integral equation of the second kind.

In § 3 the partial integral equation

$$\int_a^b f(s, x)\,\varkappa(x, t)\,dx = \int_a^b f(x, t)\,\varkappa(s, x)\,dx$$

is dealt with, it appears that if $\varkappa(s, t)$ is the Green's function for the differential equation $L_s(u) = 0$, then any solution $f(s, t)$ of this integral equation is also a solution of the partial differential equation

$$L_s(u) = L_t(u).$$

The solutions of a partial differential equation of this kind can therefore be divided into groups, each group being associated with a Green's function for a particular set of boundary conditions and every member of the group being a solution of the corresponding integral equation.

In § 4 some instances are given in which the series

$$\sum \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n}$$

represents the fundamental function $\varkappa(s, t)$ even though it is non-uniformly convergent in the neighbourhood of certain points. It follows then that when the series is multiplied by $\psi_n(s)$ and integrated term by term we shall obtain a correct result.

It is known that a series which is non-uniformly convergent for values of s in the neighbourhood of a point σ can be integrated term by term when the radius of non-uniform convergence for this point is finite*). It would be interesting if a series of the above type could be shown to satisfy this condition, or if some definite criterion for the legitimacy of the integration could be obtained.

It may be worth while to mention a very general class of series which can be integrated through a point of discontinuity.

Let $\psi_1(s) \cdots \psi_n(s) \cdots$ be a system of functions such that any

*) E. W. Hobson. The integration of series. Acta Math. 1903.

function $f(s)$ with only a finite number of discontinuities can be expanded in the form

$$\sum_0^\infty \psi_n(s) \int_a^b f(t)\, \varphi_n(t)\, dt.$$

Construct the expansion $\sum_0^\infty \psi_n(s)\, a_n(t)$ for the discontinuous function

$$\left. \begin{aligned} F(s) = +\chi(s) \quad & s > t \\ 0 \quad & s = t \\ -\chi(s) \quad & s < t \end{aligned} \right\}$$

then

$$a_n(t) = -\int_a^t \varphi_n(s)\, \chi(s)\, ds + \int_t^b \varphi_n(s)\, \chi(s)\, ds$$

and

$$\frac{d}{dt} a_n(t) = -2\chi(t)\, \varphi_n(t).$$

The expansion for $f(s)\,\chi(s)$ is

$$\begin{aligned} f(s)\,\chi(s) = & \sum_0^\infty \psi_n(s) \int_a^b f(t)\, \chi(t)\, \varphi_n(t)\, dt \\ & = -\tfrac{1}{2} \sum_0^\infty [\psi_n(s)\, f(t)\, a_n(t)]_a^b + \tfrac{1}{2} \sum \psi_n(s) \int_a^b f'(t)\, a_n(t)\, dt. \end{aligned}$$

Now

$$\begin{aligned} \sum a_n(t)\, \psi_n(s) &= +\chi(s) \quad s > t, \\ &= -\chi(s) \quad s < t \end{aligned}$$

and the series $\sum \psi_n(s)\, a_n(t)\, f'(t)$ is non-uniformly convergent in the neighbourhood of $t = s$ (since it is discontinuous), but still it can be integrated through this point, for the equation obtained by integrating term by term is

$$f(s)\,\chi(s) =$$

$$+\tfrac{1}{2}[f(b)\,\chi(s) + f(a)\,\chi(s)] + \tfrac{1}{2}\int_a^s f'(t)\,\chi(t)\, dt - \tfrac{1}{2}\int_s^b f'(t)\,\chi(t)\, dt,$$

and this is evidently satisfied.

Hence the series $\sum_0^\infty \psi_n(s)\, a_n(t)\, f'(t)$ can be integrated through its point of discontinuity $t = s$.

The remainder of this section is devoted to the consideration of the equation

$$f(s) = \varphi(s) - \int_a^b \varkappa(s, t)\, \varphi(t)\, dt$$

as b varies and it is shown that if the solution $\varphi(s)$ is known for all values of b within a range a to $a + A$, and $f(s)$ is independent of b, then the function $\varkappa(s, t)$ can be uniquely determined. The proof depends upon a theorem due to Volterra and assumes that the limitations stated in his theorem are satisfied.

In conclusion I should like to thank Prof. Hilbert for the interest he has taken in the work and for some useful suggestions.

§ 1.

The conditions to be satisfied in order that the integral equation of the first kind may be soluble.

Only a few cases are known in which a general formula has been given for obtaining the function $\varphi(t)$ from the equation

$$(1) \qquad f(s) = \int_c^d G(s, t)\, \varphi(t)\, dt$$

and in these the expression for φ takes one of two forms. In the classical cases given by Fourier, Riemann and Hankel the function φ is expressed as a definite integral similar in form to the original one, but when $G(s, t)$ is the Green's function corresponding to certain boundary conditions for a self-adjoint linear differential equation of the second order and $f(s)$ satisfies the same boundary conditions and possesses a continuous second derivate, Hilbert has shown that φ is given by operating on f with the given differential equation.

In order to solve (1) by means of a definite integral we may seek a relation of the form

$$(2) \qquad \int_c^d G(s, t)\, F(t, x)\, dt = \frac{d}{dx} H(s, x).$$

A solution will then be given by

$$\varphi(t) = \int_{x_1}^{x_2} F(t, x)\, dx$$

provided

$$H(s, x_2) - H(s, x_1) = f(s).$$

Let us now suppose that the function $G(s, t)$ is finite and integrable (in which we include the conditions for a change in the order of integration) for $c \leqq t \leqq d$ and $a \leqq s \leqq b$ and that $f(s)$ is also finite and integrable for these values of s, then we can construct two such functions F and H as follows.

Forming the symmetrical function

$$(3) \qquad \varkappa(s, t) = \int_c^d G(s, r)\, G(t, r)\, dr$$

we write

$$f_1(s) = \int_a^b \varkappa(s, t) f(t)\, dt, \cdots, f_r(s) = \int_a^b \varkappa(s, t) f_{r-1}(t)\, dt,$$

$$(4) \qquad \begin{aligned} g_r(t) &= \int_a^b G(u, t) f_r(u)\, du, \\ F(t, x) &= x g_1(t) - \frac{x^3}{1!} g_3(t) + \frac{x^5}{2!} g_5(t) - \cdots, \\ H(s, x) &= f(s) - \frac{x^2}{1!} f_2(s) + \frac{x^4}{2!} f_4(s) - \frac{x^6}{3!} f_6(s) + \cdots. \end{aligned}$$

These series are absolutely and uniformly convergent for all finite values of x, for if G and f are the maximum values of $|G(s, t)|$ and $|f(s)|$ for the given values of s and t we have

$$|f_r(s)| < |b - a|^r G^{2r} f |c - d|^r, \qquad |g_r(s)| < |b - a|^{r+1} |c - d|^r G^{2r+1} f,$$

and so the series converge like exponential series.

Observing now that

$$\int_e^d G(s, t)\, g_r(t)\, dt = \int_c^d G(s, t)\, dt \int_a^b G(u, t) f_r(u)\, du$$
$$= \int_a^b \varkappa(s, u) f_r(u)\, du = f_{r+1}(s)$$

we find that

$$(5) \qquad \int_c^d G(s, t) F(t, x)\, dt = -\frac{1}{2} \frac{d}{dx} \{H(s, x)\},$$

an equation of the required form.

We must now see whether this relation can be used to obtain a solution of equation (1). If we write

$$(6) \qquad \varphi(t) = 2 \int_0^M F(t, x)\, dx$$

and substitute in the equation we get

$$(7) \qquad \int_c^d G(s,t)\,\varphi(t)\,dt = 2\int_c^d G(s,t)\,dt\int_0^M F(t,x)\,dx$$

$$= -\int_0^M \frac{d}{dx} H(s,x)\,dx$$

$$= H(s,0) - H(s,M) = f(s) - H(s,M).$$

Two courses are now open to us, we may either write $\Psi(s)$ instead of $f(s)$ in equation (1) and endeavour to determine $f(s)$ so that

$$f(s) - H(s,M) = \Psi(s)$$

or we may try the effect of putting $M=\infty$, in which case the method will succeed if the function $f(s)$ is such that the quantity $H(s,x)$ is zero for x infinite and the processes which have been gone through are legitimate.

Every step can be examined directly if the functions $f(s)$ and $G(s,t)$ are known, hence we can determine if a particular function $f(s)$ will lead to a solution, but for most purposes it is convenient to have a definite criterion.

It has been shown by Hilbert*) that if a symmetrical function $\varkappa(s,t)$ is such that to every small positive quantily ε and every continuous function $g(s)$ there corresponds a function $h(t)$ for which

$$\int_a^b \left[g(s) - \int_a^b \varkappa(s,t)\,h(t)\,dt\right]^2 ds < \varepsilon$$

then any function $f(s)$ which is defined by an equation of the form

$$f(s) = \int_a^b \varkappa(s,t)\,\varphi(t)\,dt$$

can be expressed in an absolutely and uniformly convergent series of the function, $\psi_n(s)$ which satisfy the homogeneous equations

$$\psi_n(s) = \lambda_n \int_a^b \varkappa(s,t)\,\psi_n(t)\,dt \qquad (n = 1, 2, \cdots).$$

This theorem suggests a type of function for which the integral equation of the first kind may be soluble and we accordingly consider a function $f(s)$ which can be expanded in absolutely and uniformly convergent series

*) Göttinger Nachrichten 1904. Erste Mitteilung; following E. Schmidt (Inaugural-Dissertation, Göttingen 1905) the first supposition for $\varkappa(s,t)$ can be left.

$$f(s) = \sum_{1}^{\infty} c_n \psi_n(s) \tag{8}$$

where the quantities λ_n are supposed to be arranged according to their absolute values.*)

Multiplying by $\varkappa(s, t)$ and integrating term by term we have

$$f_1(s) = \sum_{1}^{\infty} \frac{c_n}{\lambda_n} \psi_n(s), \cdots, f_r(s) = \sum_{1}^{\infty} \frac{c_n}{\lambda_n^r} \psi_n(s).$$

Therefore

$$H(s, x) = \sum_{0}^{\infty} (-1)^r \frac{x^{2r}}{r!} \sum_{1}^{m-1} \frac{c_n}{\lambda_n^{2r}} \psi_n(s)$$
$$+ \sum_{0}^{\infty} (-1)^r \frac{x^{2r}}{r!} \sum_{m}^{\infty} \frac{c_n}{\lambda_n^{2r}} \psi_n(s).$$

Now corresponding to any small quantity ε we can choose m so that

$$\sum_{m}^{\infty} |c_n \psi_n(s)| < \varepsilon$$

also

$$|\lambda_n| > |\lambda_m| \quad \text{if} \quad n > m,$$

therefore

$$\sum_{m}^{\infty} \left| \frac{c_n}{\lambda_n^{2r}} \psi_n(s) \right| < \frac{\varepsilon}{\lambda_m^{2r}}.$$

Hence

$$\left| \sum_{0}^{\infty} (-1)^r \frac{x^{2r}}{r!} \sum_{m}^{\infty} \frac{c_n}{\lambda_n^{2r}} \psi_n(s) \right| < \sum_{0}^{\infty} \varepsilon \frac{1}{r!} \frac{x^{2r}}{\lambda_m^{2r}} < \varepsilon e^{\frac{x^2}{\lambda_m^2}}.$$

Thus

$$H(s, x) = \sum_{1}^{m-1} c_n e^{-\frac{x^2}{\lambda_n^2}} \psi_n(s) + \eta$$

where

$$\eta < \varepsilon \cdot e^{\frac{x^2}{\lambda_m^2}}.$$

Making ε tend to zero we have

$$H(s, x) = \sum_{1}^{\infty} c_n e^{-\frac{x^2}{\lambda_n^2}} \psi_n(s). \tag{9}$$

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*) It is known that the quantities λ_n are all real.

Similarly it can be shown that

$$F(t,x)=\int_a^b G(s,t)\,R(s,x)\,ds$$

where

$$R(s,x)=\sum_1^\infty c_n\,\frac{x}{\lambda_n}\,e^{-\frac{x^2}{\lambda_n^2}}\,\psi_n(s). \tag{10}$$

Now let μ be any arbitrary finite small positive quantity, then a number m can be found so that

$$\sum_m^\infty |c_n\psi_n(s)|<\frac{\mu}{2}$$

and a large quantity M can be chosen so that

$$\sum_1^m \left|c_n e^{-\frac{M^2}{\lambda_n^2}}\psi_n(s)\right|<\frac{\mu}{2}.$$

Then will

$$|H(s,M)|=\left|\sum_1^{m-1} c_n e^{-\frac{M^2}{\lambda_n^2}}\psi_n(s)+\sum_m^\infty c_n e^{-\frac{M^2}{\lambda_n^2}}\psi_n(s)\right|<\frac{\mu}{2}+\frac{\mu}{2}<\mu$$

accordingly we have the following theorem.

'*If $f(s)$ can be expanded in an absolutely and uniformly convergent series of the functions $\psi_n(s)$ then a function $\varphi(t)$ can be determined so that $\int_a^b G(s,t)\,\varphi(t)\,dt$ differs from $f(s)$ by a quantity less than μ.*'

If however we wish to make $\int_a^b G(s,t)\,\varphi(t)\,dt$ exactly equal to $f(s)$ it is necessary to make M infinite and then it is by no means certain that the integral for $\varphi(t)$ has a meaning. Two points remain to be settled.

(1) We must show that the function $F(t,x)$ given by formula (10) can be integrated with regard to x up to $x=\infty$.

(2) We must show that the order of integration in the double integral

$$\int_c^d G(s,t)\,dt\int_0^\infty F(t,x)\,dx$$

can be inverted.

The general case is rather difficult to deal with, accordingly we shall content ourselves by showing that if the function $\varphi(t)$ has a certain form our method does certainly lead to its determination.

Let us assume that the function $\varphi(t)$ can be expanded in an absolutely and uniformly convergent series of the functions $\chi_n(t)$ which satisfy the homogeneous equations

$$\chi_n(t) = \mu_n \int_c^d h(s, t)\, \chi_n(s) ds$$

where

$$h(s, t) = \int_a^b G(r, s)\, G(r, t) dr.$$

Then since

$$f(s) = \int_c^d G(s, t) \varphi(t) dt$$

and

$$\varphi(t) = \sum_1^\infty a_n \chi_n(t)$$

it follows that

(11) $$F(t, x) = \sum_1^\infty \frac{x}{\mu_n^2}\, e^{-\frac{x^2}{\mu_n^2}} a_n \chi_n(t).$$

For

$$g_1(t) = \int_a^b G(u, t) f_1(u) du$$

$$= \int_a^b G(u, t)\, du \int_c^d G(u, v)\, dv \int_a^b G(w, v) f(w) dw$$

$$= \int_a^b G(u, t) du \int_c^d G(u, v) dv \int_a^b G(w, v) dw \int_c^d G(w, r)\, \varphi(r) dr$$

$$= \int_c^d h(t, v)\, dv \int_c^d h(v, r)\, \varphi(r) dr.$$

Similarly

$$g_n(t) = \int_c^d h(t, v) g_{n-1}(v)\, dv$$

hence

$$g_1(t) = \int_c^d h(t, v)\,dv \int_c^d h(v, r)\,dr \sum_1^{\infty} a_n \chi_n(r)$$

$$= \sum_1^{\infty} \frac{a_n}{\mu_n^2} \chi_n(t)$$

and

$$g_r(t) = \sum_1^{\infty} \frac{a_n}{\mu_n^{r+1}} \chi_n(t)$$

which lead to the required result.

We must now determine $2\int_0^{\infty} F(t, x)\,dx$ and to do this we must integrate the series (11) term by term and this will be legitimate if the following sufficient conditions are satisfied.

1°. The series $u_1(x) + u_2(x) + \cdots$ should be uniformly convergent in an arbitrary interval.

2°. $\int u_n\,dx$ should exist for all values of n.

3°. $\sum_{n=1}^{\infty} \int_a^{\infty} u_n(x)\,dx$ should converge for values of a within the range of integration.

4°. A number p can be found independent of r and such that

$$\left| \sum_{n=1}^{r} \int_{x}^{\infty} u_n(x)\,dx \right| < \varepsilon$$

for all k's greater than p.

The first three conditions are evidently satisfied on account of the uniform convergence of the series $\sum_1^{\infty} a_n \chi_n(t)$. The fourth condition will be satisfied if p can be found independent of r such that

$$\left| \sum_{n=1}^{r} a_n e^{-\frac{x^2}{\mu_n^2}} \chi_n(t) \right| < \varepsilon$$

for all k's greater than p. Now since the series $\sum_1^{\infty} a_n \chi_n(t)$ is absolutely convergent we can determine a number m such that

$$\sum_{m}^{\infty} |a_n \chi_n(t)| < \frac{\varepsilon}{2}$$

and then

$$\sum_{m}^{r} \left| a_n e^{-\frac{\varkappa^2}{\mu_n^2}} \chi_n(t) \right| < \sum_{m}^{\infty} \left| a_n e^{-\frac{\varkappa^2}{\mu_n^2}} \chi_n(t) \right| < \frac{\varepsilon}{2};$$

also we can determine a number p such that

$$\left| \sum_{1}^{m} a_n e^{-\frac{\varkappa^2}{\mu_n^2}} \chi_n(t) \right| < \frac{\varepsilon}{2}$$

if $\varkappa > p$, and then we shall have

$$\left| \sum_{n=1}^{r} a_n e^{-\frac{\varkappa^2}{\mu_n^2}} \chi_n(t) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

All the conditions being satisfied the integration term by term may be effected and we have

$$2\int_0^{\infty} F(t, x)\,dx = 2\int_0^{\infty} \sum_{1}^{\infty} \frac{x}{\mu_n^2} e^{-\frac{x^2}{\mu_n^2}} a_n \chi_n(t) \cdot dx$$

$$= \sum_{0}^{\infty} a_n \chi_n(t) = \varphi(t).$$

Hence the proposed method is in this case successful. It can also be used in some cases when the limits of integration are infinite provided the functions $g_r(t)$, $f_r(s)$, are defined by the equations

$$g_1(t) = \int_a^b G(u, t)\,du \int_c^d G(u, v)\,dv \int_a^b G(w, v) f(w)\,dw$$

$$g_n(t) = \int_a^b G(u, t)\,du \int_c^d G(u, v)\, g_{n-1}(v)\,dv$$

$$f_n(s) = \int_c^d G(s, v)\,dv \int_a^b G(w, v) f_{n-1}(w)\,dw.$$

For example if

$$f(s) = \int_0^{\infty} J_0(st)\sqrt{st}\,\varphi(t)\,dt$$

where J_0 denotes the zeroeth Bessel function, it has been shown by Hankel*) that

$$\varphi(t) = \int_0^\infty J_0(st)\sqrt{st}\,f(s)\,ds$$

and our function $F(t, x)$ is found to be

$$xe^{-x^2}\int_0^\infty J_0(st)\sqrt{st}\,f(s)\,ds$$

and so the method leads to the correct result.

§ 2.

Solution of a particular equation.

We shall now consider the particular integral equation

$$f(s) = \int_{-1}^{+1} \frac{\varphi(t\,dt}{(1 - 2ts + s^2)^{\frac{n}{2}}} \quad (n = 1, 3, \cdots) \quad (-1 < s < +1)$$

and shall show that *if $f(s)$ is regular within the unit circle $|s| = 1$, the solution is given by*

$$\varphi(t) = \frac{1}{4\pi}(1 - t^2)^{\frac{n-1}{2}} \int_0^{2\pi} [nf\{t + i\sqrt{1 - t^2}\cos\alpha\}$$
$$+ 2\{t + i\sqrt{1 - t^2}\cos\alpha\} + f'\{t + i\sqrt{1 - t^2}\cos\alpha\}] \sin^{n-1}\alpha\,d\alpha.$$

Let $x_1, x_2, \cdots, x_{n+2}$ be a system of rectangular coordinates in a space of $n + 2$ dimensions and $r^2 = x_1{}^2 + x_2{}^2 + \cdots + x_{n+2}^2 = 1$ a unit hypersphere situated in this space.

The equations

$$V_1 = f(x_1 + ix_2)$$
$$V_0 = \frac{1}{r^n} f\left(\frac{x_1 + ix_2}{r^2}\right)$$

define a potential function for points inside and outside the hypersphere respectively. This potential function is continuous at the boundary and so will be due to a boundary distribution σ, the conditions of being continuous within the hypersphere and of vanishing at infinity being fulfilled since $f(s)$ is regular for $|s| < 1$.

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*) Mathematische Annalen, Bd. 8 (1875), p. 482. The function $f(s)$ must of course be such that the improper integral has a meaning.

Poisson's equation for the space we are dealing with is

$$\frac{\partial^2 V}{\partial x_1^2} + \cdots + \frac{\partial^2 V}{\partial x_{n+2}^2} + \lambda \varrho = 0$$

where λ is n times the total area of the boundary of a unit hypersphere. The corresponding equation for determining the surface density is accordingly

$$\frac{\partial V_1}{\partial n_1} + \frac{\partial V_0}{\partial n_0} + \lambda \sigma = 0$$

where the normals are drawn into the two portions of space separated by the boundary.

Substituting the above values of V for the boundary $r = 1$, we find that

$$\lambda \sigma = n f(x_1 + i x_2) + 2(x_1 + i x_2) f'(x_1 + i x_2).$$

The potential at the point $(s, 0, 0, \cdots)$ when calculated by direct integration is

$$V = \int\limits_S \frac{\sigma \, dS}{(1 - 2 x_1 s + s^2)^{\frac{n}{2}}}.$$

Put

$$\begin{aligned} x_1 &= \cos\theta \\ x_2 &= \sin\theta\cos\varphi \\ x_3 &= \sin\theta\sin\varphi\cos\chi \\ &\cdots\cdots\cdots \\ x_{n+1} &= \sin\theta\sin\varphi\sin\chi\cdots\cos\omega \\ x_{n+2} &= \sin\theta\sin\varphi\sin\chi\cdots\sin\omega \end{aligned}$$

then

$$dS = \sin^n\theta \cdot \sin^{n-1}\varphi \cdots d\theta \, d\varphi \cdots d\omega,$$

accordingly

$$V = \int_0^\pi \int_0^{2\pi} \frac{\sin^n\theta \sin^{n-1}\varphi \cdot \sigma \, d\theta \, d\varphi}{(1 - 2s\cos\theta + s^2)^{\frac{n}{2}}} \int_0^{2\pi}\int_0^{2\pi} \cdots \sin^{n-2}\chi \cdots d\chi \cdots d\omega.$$

and

$$\lambda = n \int_0^\pi \int_0^{2\pi} \sin^n\theta \sin^{n-1}\varphi \, d\theta \, d\varphi \int_0^{2\pi}\int_0^{2\pi} \cdots \sin^{n-2}\chi \cdots d\chi \cdots d\omega.$$

Now by an easy calculation we find that

$$\int_0^\pi \int_0^{2\pi} \sin^n\theta \sin^{n-1}\varphi \cdot d\theta \, d\varphi = \frac{4\pi}{n},$$

hence our equation becomes

$$f(s) = V = \frac{n}{4\pi} \int_0^{\pi} \int_0^{2\pi} \frac{\lambda \sigma \sin^n \theta \sin^{n-1} \varphi \cdot d\theta \, d\varphi}{(1 - 2s \cos \theta + s^2)^{\frac{n}{2}}}$$

i. e. if

$$f(s) = \int_0^{\pi} \frac{\varphi (\cos \theta) \sin \theta \, d\theta}{(1 - 2s \cos \theta + s^2)^{\frac{n}{2}}}$$

then

$$\varphi(\cos \theta) = \frac{n}{4\pi} \sin^{n-1} \theta \int_0^{2\pi} \lambda \sigma \sin^{n-1} \varphi \cdot d\varphi.$$

Putting $\cos \theta = t$ and substituting the value of $\lambda \sigma$ obtained above, we have

$$\varphi(t) = \frac{1}{4\pi} (1 - t^2)^{\frac{n-1}{2}} \int_0^{2\pi} \Big[n f(t + i\sqrt{1 - t^2} \cos \alpha)$$
$$+ 2(t + i\sqrt{1 - t^2} \cos \alpha) f'(t + i\sqrt{1 - t^2} \cos \alpha) \Big] \sin^{n-1} \alpha \, d\alpha$$

which is the formula stated.

If we put $n = 1$, $\varphi(t) = P_m(t)$ where P_m is the Legendre polynomial we find that $f(s) = \frac{2}{2m+1} s^m$, and our formula gives

$$\varphi(t) = P_m(t) = \frac{1}{2\pi} \int_0^{2\pi} (t + \sqrt{t^2 - 1} \cos \alpha)^m \, d\alpha$$

which is Laplace's formula for $P_m(t)$.

Our formula may also be written in another form. If we differentiate the equation

$$f(s) = \int_{-1}^{+1} \frac{\varphi(t) \, dt}{(1 - 2ts + s^2)^{\frac{n}{2}}}$$

we get

$$\chi(s) = n f(s) + 2s f'(s) = n \int_{-1}^{+1} \frac{(1 - s^2) \varphi(t) \, dt}{(1 - 2ts + s^2)^{\frac{n}{2}+1}}$$

accordingly the solution of

$$\chi(s) = \int_{-1}^{+1} \frac{n(1 - s^3)}{(1 - 2ts + s^2)^{\frac{n+2}{2}}} \varphi(t) \, dt$$

is given by

$$\varphi(t) = \frac{(1-t^2)^{\frac{n-1}{2}}}{4\pi} \int_0^{2\pi} \chi(t+\sqrt{t^2-1}\cos\alpha)\sin^{n-1}\alpha\cdot d\alpha.$$

The result may be obtained by another method which also applies when n is an even integer.

The function $\frac{(1-s^2)(1-t^2)^{\frac{n-1}{2}}}{(1-2ts+s^2)^{\frac{n+2}{2}}}$ only differs by a constant factor from $\frac{\partial G}{\partial \nu}$ where G is the Green's function for the hypersphere, and the formula

$$V = \int_0^{\pi} \chi(x_1 + \sqrt{r^2 - x_1^2}\cos\alpha)\sin^{n-1}\alpha\cdot d\alpha$$

represents a potential function which is a function of r and x_1 only and which takes the value

$$\int_0^{\pi} \chi(s)\sin^{n-1}\alpha\cdot d\alpha$$

at the point $(s, 0, 0, \cdots)$.

The values which this potential function takes at points on the hypersphere will be given by

$$V(t) = \int_0^{\pi} \chi(t+\sqrt{t^2-1}\cos\alpha)\sin^{n-1}\alpha\cdot d\alpha$$

and Green's formula $V = \int_s \frac{\partial G}{\partial \nu} V(t)\,ds$ gives us the required relation, hence we have the following theorem:

If $\chi(s)$ is regular within the unit circle $|s| = 1$, the integral equation

$$\chi(s) = \int_{-1}^{+1} \frac{n(1-s^2)}{(1-2ts+s^2)^{\frac{n+2}{2}}}\varphi(t)\,dt \qquad (n = 1, 2, 3, \cdots, \infty)$$

is satisfied by

$$\varphi(t) = \frac{(1-t^2)^{\frac{n-1}{2}}}{2\pi} \int_0^{\pi} \chi(t+\sqrt{t^2-1}\cos\alpha)\sin^{n-1}\alpha\cdot d\alpha.$$

The previous relation may be deduced from this; hence if

$$f(s)=\int_{-1}^{+1}\frac{\varphi(t)\,dt}{(1-2ts+s^2)^{\frac{n}{2}}}\qquad (n=1,2,3,\cdots)\qquad (-1<s<+1)$$

then

$$\varphi(t)=\frac{1}{2\pi}(1-t^2)^{\frac{n-1}{2}}\int_0^\pi\Big[nf\big(t+\sqrt{t^2-1}\cos\alpha\big)$$
$$+2\big(t+\sqrt{t^2-1}\cos\alpha\big)f'\big(t+\sqrt{t^2-1}\cos\alpha\big)\Big]\sin^{n-1}\alpha\cdot d\alpha.$$

An interesting case occurs when $n=2$, for if $s+\frac{1}{s}=2z$ the first equation may be written

$$F(z)=sf(s)=\frac{1}{2}\int_{-1}^{+1}\frac{\varphi(t)}{z-t}\,dt$$

and the second equation gives

$$\varphi(t)=\frac{1}{i\pi}\Big[\big(t+\sqrt{t^2-1}\cos\alpha\big)f\big(t+\sqrt{t^2-1}\cos\alpha\big)\Big]_0^\pi.$$

The function $F(z)$ is in general a many-valued function of the form $\frac{1}{2}\varphi(z)\operatorname{Log}\frac{z+1}{z-1}+\psi(z)$ and since the two values of the logarithm differ by $2i\pi$ we see at once how the above formula will give the correct result.

The formula which we have obtained may be used to make the solution of an integral equation of the first kind depend upon that of an integral equation of the second kind.

Let

$$\chi(s)=\int_{-1}^{+1}\left[G(s,t)+\frac{n(1-s^2)}{(1-2ts+s^2)^{\frac{n+2}{2}}}\right]\varphi(t)\,dt$$

then if

$$\psi(t)=\frac{(1-t^2)^{\frac{n-1}{2}}}{2\pi}\int_0^\pi\chi\big(t+\sqrt{t^2-1}\cos\alpha\big)\sin^{n-1}\alpha\cdot d\alpha$$

we shall have

$$\psi(t)=\varphi(t)+\int_{-1}^{+1}\varkappa(s,t)\,\varphi(t)\,dt$$

where

$$\varkappa(s,t)=\frac{(1-s^2)^{\frac{n-1}{2}}}{2\pi}\int_0^\pi G\big(s+\sqrt{s^2-1}\cos\alpha,\,t\big)\sin^{n-1}\alpha\cdot d\alpha.$$

Now this is an integral equation of the second kind and so can be solved by Fredholm's method provided the determinantal function δ is not equal to zero.

§ 3.

A partial integral equation.

The theory of the integral equation of the second kind

$$f(s) = \varphi(s) - \lambda \int_a^b \varkappa(s, t)\, \varphi(t)\, dt \tag{1}$$

is closely connected with that of a certain partial integral equation

$$\int_a^b \varkappa(s, x)\, f(x, t)\, dx = \int_a^b \varkappa(x, t)\, f(s, x)\, dx. \tag{2}$$

This equation has many remarkable properties, for instance if $f(s, t)$ and $g(s, t)$ are two solutions then the function

$$h(s, t) = \int_a^b f(s, x)\, g(x, t)\, dx$$

is also a solution.

For if we multiply both sides by $g(r, s)$ and integrate with regard to s between a and b we have, assuming that the order of integration can be reversed,

$$\int_a^b \int_a^b g(r, s)\, \varkappa(s, x)\, f(x, t)\, ds\, dx = \int_a^b \varkappa(x, t)\, h(r, x)\, dx$$

or

$$\int_a^b \int_a^b \varkappa(r, s)\, g(s, x)\, f(x, t)\, ds\, dx = \int_a^b \varkappa(x, t)\, h(r, x)\, dx$$

since $g(r, s)$ is a solution of (1).

Hence

$$\int_a^b \varkappa(r, s)\, h(s, t)\, ds = \int_a^b \varkappa(x, t)\, h(r, x)\, dx$$

which proves the proposition.

The function $\varkappa(s, t)$ itself is evidently a solution of (2) hence we can form at once a number of other solutions, namely

$$\varkappa\varkappa(s, t) = \int_a^b \varkappa(s, x)\, \varkappa(x, t)\, dx,$$

$$\varkappa\varkappa\varkappa(s, t) = \int_a^b \varkappa(s, x)\, \varkappa\varkappa(x, t)\, dx,$$

. . . .

The solving function $K(s, t)$ of the equation (1) is also a solution for we have the equations

$$\varkappa(s, t) = K(s, t) - \lambda \int_a^b \varkappa(s, r)\, K(r, t)\, dr,$$

$$K(s, t) = \varkappa(s, t) + \lambda \int_a^b K(s, r)\, \varkappa(r, t)\, dr.$$

Again, if we seek a solution of the form $\varphi(s)\,\psi(t) = f(s, t)$ we find at once that we must have

$$\varphi(s) - \lambda \int_a^b \varkappa(s, t)\, \varphi(t)\, dt = 0,$$

$$\psi(t) - \lambda \int_a^b \varkappa(s, t)\, \psi(s)\, ds = 0$$

and it is known*) that these equations are satisfied by functions $\varphi(s)$ and $\psi(s)$ when λ is one of the roots of the determinantal equation

$$\delta(\lambda) = 0.$$

Now let us consider the particular case in which the function $\varkappa$ is the Green's function corresponding to boundary conditions I . . . IV **) for the self-adjoint linear differential equation of the second order

$$L_s(u) = 0.$$

It is then evident that a function $f(s, t)$ which satisfies the integral equation

$$\int_a^b G(s, x)\, f(x, t)\, dx = \int_a^b G(x, t)\, f(s, x)\, dx \tag{3}$$

must satisfy the same boundary conditions as $G(s, x)$.

For instance if we take boundary conditions I, viz.

$$G(a, x) = G(b, x) = 0$$

we have, putting $s = a$ in the above equation,

$$0 = \int_a^b G(x, t)\, f(0, x)\, dx.$$

But if $\varphi(t)$ is a function which has a continuous second derivate and which satisfies the given boundary conditions, the solution of the equation

$$\varphi(t) = \int_a^b G(x, t)\, f(x)\, dx$$

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*) Plemelj, Zur Theorie der Fredholmschen Funktionalgleichung. Monatshefte für Mathematik und Physik. XV. Jahrg., p. 93 u. p. 337).

**) Cf. Hilbert, Gött. Nachr. 1904. Zweite Mitteilung.

is given by $f(x) = -L_x\{\varphi(x)\}$ and this solution is unique, accordingly we must have $f(0, x) = 0$.

If we refer to the definitions of the functions $G(s, t)$ in Hilbert's paper it is easy to see that they are all solutions of the partial differential equation

(4) $$L_s(u) = L_t(u).$$

We can now show that a continuous solution of equation (3) is also a solution of this partial differential equation. If we write

$$\varphi(s, t) = \int_a^b G(s, x) f(x, t)\, dx = \int_a^b G(x, t) f(s, x)\, dx$$

we have

$$f(s, t) = -L_s\varphi(s, t)$$
$$= -\int_a^b G(x, t)\, L_s f(s, x)\, dx.$$

But since $f(s, t)$ satisfies the given boundary conditions we have

$$f(s, t) = -\int_a^b G(x, t)\, L_x \cdot f(s, x)\, dx$$

therefore

$$0 = \int_a^b G(x, t)\, [L_s f(s, x) - L_x f(s, x)]\, dx.$$

Now the equation $0 = \int_a^b G(x, t)\, \psi(t)\, dt$ only possesses one solution viz. $\psi(t) = 0$, hence we have

$$L_s f(s, x) - L_x f(s, x) = 0.$$

Thus corresponding to each Green's function for $L_x(u) = 0$ there is a group of solutions of the partial differential equation, and if $f(s, t)$, $g(s, t)$ are any two members of this group the function

$$h(s, t) = \int_a^b f(s, x)\, g(x, t)\, dx$$

will also belong to the group.

If $\psi_n(s)$ is an 'Eigenfunktion' belonging to the function $G(s, t)$ the quantity $\psi_n(s)\,\psi_n(t)$ is a solution of (3) and so

$$\int_a^b f(s, x)\, \psi_n(x)\, \psi_n(t)\, dx$$

is a solution and is of the form $\varphi(s)\,\psi_n(t)$, hence we must have $\varphi(s) = \mu_n \psi_n(s)$, that is

$$\mu_n \psi_n(s) = \int_a^b f(s, x)\, \psi_n(x)\, dx.$$

This equation shows that the functions $\psi_n(s)$ are also Eigenfunctions for the 'Kern' $f(s, t)$.

There are other types of integral equations which are satisfied by certain groups of solutions of a partial differential equation. For example we can show that if $F(x, y, z)$ is a solution of Laplace's equation

$$\frac{1}{\pi}\int_0^\pi F(x, y, z + i\varrho \cos\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} F(x + \varrho\cos\varphi,\ y + \varrho \sin\varphi,\ z)\, d\varphi$$

provided the integrals have a meaning.

In other words the integral of a solution of Laplace's equation round a circle of radius ϱ is equal to the integral of the same solution multiplied by a certain definite function, along an axis drawn through the centre of the circle and perpendicular to it, the integration extending from the centre of one point sphere passing through the given circle to the centre of the other.

A simple proof of this relation may be obtained by remarking that each integral represents the solution of

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial V}{\partial \varrho} = 0$$

which reduces to $F(x, y, z)$ when $\varrho = 0$, and this solution is known to be unique.

The two paths of integration appear to be related in some way to the characteristics of the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0;$$

if the relation could be expressed in a more definite form it might suggest how similar integral equations could be obtained for more general partial differential equations.

§ 4.

Construction of an integral equation possessing assigned solutions.

The problem of determining a function $\varkappa(s, t)$ so that the equation

$$\varphi(s) = \lambda \int_a^b \varkappa(s, t)\, \varphi(t)\, dt$$

may be satisfied for a given set of functions $\psi_n(s)$ and for a given set of values of λ, has been considered by Hilbert. If the functions $\psi_n(s)$ possess the orthogonal properties,

$$\int_a^b \psi_m(s)\, \psi_n(s)\, ds = 0 \qquad m \neq n$$
$$= 1 \qquad m = n$$

and the series $\sum \frac{1}{\lambda_n} \psi_n(s)\, \psi_n(t)$ is uniformly convergent it will furnish a solution of the problem. It is however not necessary for this series to be uniformly convergent as the following example will show.

Choosing the Legendre polynomials for our given set of functions we remark that they satisfy the equations

$$\frac{d}{ds}\left\{(1-s^2)\frac{du}{ds}\right\} + n(n+1)u = 0$$

and so we can obtain a simple function developable in the form

$$\sum A_n P_n(s)\, P_n(t)$$

by finding a suitable solution of the partial differential equation

$$\frac{d}{ds}\left\{(1-s^2)\frac{\partial \varkappa}{\partial s}\right\} = \frac{d}{dt}\left\{(1-t^2)\frac{\partial \varkappa}{\partial t}\right\}.$$

Assuming a solution of the form $\varkappa = F(s^2+t^2)$ we obtain

$$\varkappa(s, t) = \frac{1}{\sqrt{1-s^2-t^2}}.$$

We shall now show that the function

$$\varkappa(s, t) = \frac{1}{\sqrt{1-s^2-t^2}} \qquad s^2+t^2 < 1$$
$$= 0 \qquad s^2+t^2 > 1$$

gives us a solution of our problem.

If we integrate the expansion

$$P_n\left(st+\sqrt{(1-s^2)(1-t^2)}\cos\alpha\right) = P_n(s)\,P_n(t) + 2\sum_1^n \frac{(n-m)!}{(n+m)!} P_n^m(s) P_n^m(t) \cos m\alpha$$

between 0 and π we obtain

$$\int_0^\pi P_n\left(st+\sqrt{(1-s^2)(1-t^2)}\cos\alpha\right) d\alpha = \pi P_n(s)\,P_n(t).$$

Putting $x = st + \sqrt{(1-s^2)(1-t^2)}\cos\alpha$ this relation gives

$$\int \frac{P_n(x)\,dx}{\sqrt{1-x^2-s^2-t^2+2stx}} = \pi P_n(s)\,P_n(t),$$

the integral being taken over the values of x between -1 and $+1$ for which the quantity under the square root is positive.

When $t = 0$, this gives

$$\int_{-\sqrt{1-s^2}}^{+\sqrt{1-s^2}} \frac{P_n(x)\,dx}{\sqrt{1-x^2-s^2}} = \pi P_n(s) P_n(0)$$

which shows that with the above function $\varkappa(s, t)$ the functions $P_n(s)$ are solutions of the homogeneous integral equation.

The corresponding expansion is obtained by making use of a theorem of Darboux's.*)

Sufficient conditions that a function $f(s)$ may be expanded in a convergent series of Legendre polynomials for values of s lying between -1 and $+1$ are that

(1) The integrals $\frac{2n+1}{2}\int_{-1}^{+1} ds\, f(s)\, P_n(s)$ should have a meaning, this requires that if $f(s)$ becomes infinite within the range it should become infinite to an order less than unity.

(2) If $P_n(s)$ becomes infinite at one of the points ± 1 it should become infinite to an order less than $\frac{3}{4}$.

(3) $f(s)$ should satisfy the conditions laid down by Dirichlet for a function developable in a Fourier series, i. e. it should only have a limited number of maxima and minima and of discontinuities within the range.

When these conditions are satisfied the series

$$\sum_0^\infty \frac{2n+1}{2} P_n(s) \int_{-1}^{+1} f(s)\, P_n(s)\, ds$$

will converge to the value $f(s)$ at all points where $f(s)$ is continuous, and to the value

$$\frac{1}{2} \sum [f(s+0) + f(s-0)]$$

at any point where $f(s)$ is discontinuous.

Applying this theorem to the function

$$\begin{aligned} f(s) &= (1-s^2-x^2-t^2+2stx)^{-\frac{1}{2}} \qquad & 1-s^2-x^2-t^2+2stx > 0 \\ &= 0 & 1-s^2-x^2-t^2+2stx < 0 \end{aligned}$$

*) Approximation des fonctions de très grands nombres. Journ. de Math. (3e série) tome IV, (1878), p. 393.

we obtain the series

$$\frac{\pi}{2}\sum_0^\infty (2n+1)\,P_n(s)\,P_n(t)\,P_n(x).$$

Putting $x=0$ the expansion becomes

$$\frac{\pi}{2}\sum_0^\infty (-1)^n \frac{1\cdot 3\cdots 2n-1}{2\cdot 4\cdots 2n}(4n+1)P_{2n}(s)\,P_{2n}(t) = \begin{matrix}(1-s^2-t^2)^{-\frac{1}{2}} & 1>s^2+t^2\\ 0 & 1<s^2+t^2;\end{matrix}$$

the particular case when $t=0$ has already been given by Heine.*)

Other expansions may be obtained in a similar way, for instance if $t^2 = s^2$

$$\int\limits_{-\sqrt{1-t^2}}^{+\sqrt{1-t^2}} \frac{ds}{\sqrt{1-s^2-t^2}} \int\limits_{-\sqrt{1-s^2}}^{+\sqrt{1-s^2}} \frac{P_n(x)\,dx}{\sqrt{1-x^2-s^2}} = \pi\,P_n(0)\int\limits_{-\sqrt{1-t^2}}^{+\sqrt{1-t^2}} \frac{P_n(s)}{\sqrt{1-s^2-t^2}} = \pi^2 P_n^2(0)\,P_n(t).$$

The left hand side becomes on changing the order of integration

$$\int\limits_{-1}^{-t} P_n(x)\,dx \int\limits_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \frac{ds}{\sqrt{(1-s^2-t^2)(1-x^2-s^2)}} + \int\limits_{-t}^{+t} P_n(x)\,dx \int\limits_{-\sqrt{1-t^2}}^{+\sqrt{1-t^2}} \frac{ds}{\sqrt{(1-s^2-t^2)(1-x^2-s^2)}}$$

$$+\int\limits_{t}^{1} P_n(x)\,dx \int\limits_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \frac{ds}{\sqrt{(1-s^2-t^2)(1-x^2-s^2)}};$$

hence if $4K$ is the period of the Jacobian elliptic functions

$$\frac{\pi^2}{4}\sum_0^\infty (4n+1)\,\frac{1^2\cdot 3^2\cdots(2n-1)^2}{2^2\cdot 4^2\cdots(2n)^2}\,P_{2n}(s)\,P_{2n}(t) = \frac{1}{\sqrt{1-t^2}}\,K\left(\sqrt{\frac{1-s^2}{1-t^2}}\right) \quad s^2>t^2$$

$$= \frac{1}{\sqrt{1-s^2}}\,K\left(\sqrt{\frac{1-t^2}{1-s^2}}\right) \quad s^2<t^2.$$

Putting $t=0$ we have the expansion

$$K'(s) = \frac{\pi^2}{4}\sum_0^\infty (-1)^n (4n+1)\,\frac{1^3\cdot 3^3\cdots(2n-1)^3}{2^3\cdot 4^3\cdots(2n)^3}\,P_{2n}(s).$$

If the solution of the integral equation of the second kind

$$f(s) = \varphi(s) - \int_a^b \varkappa(s,t)\,\varphi(t)\,dt$$

is known for values of b contained within the interval a to $a+A$, the function $\varkappa(s,t)$ can be uniquely determined.

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*) Handbuch der Kugelfunctionen Bd. 1, p. 85.

For when $b = a$ we have $\varphi(s) = f(s)$, and so the equation may be written

$$\varphi(s, b) - \varphi(s, a) = \int_a^b \varphi(t, b)\, \varkappa(s, t)\, dt$$

and is therefore of the form

$$F(b) - F(a) = \int_a^b \varphi(t, b)\, \psi(t)\, dt.$$

Now Volterra*) has shown that if in an integral equation of this kind $F(b)$ and $F'(b)$ remain finite and continuous for values of b between a and $a + A$, and the functions

$$\varphi(t, b) \quad \text{and} \quad \frac{\partial \varphi}{\partial b} = H(t, b)$$

are always finite for $b > t > a$, $a + A > b > a$, and are integrable, and if the lower limit of the absolute value of $\varphi(b, b)$ is different from zero, there will exist one and only one finite and continuous function $\psi(t)$ which satisfies the functional equation for values of b between a and $a + A$ and this function will be given by

$$\varphi(b) = \frac{F'(b)}{\varphi(h, b)} - \frac{1}{\varphi(b, b)} \int_a^b F'(x) \sum_0^\infty S_i(x, b)\, dx$$

where

$$S_0(x, b) = \frac{H(x, b)}{\varphi(b, b)}$$

$$S_i(x, b) = \int_b^x S_{i-j}(x, \xi)\, S_{j-1}(\xi, b)\, d\xi.$$

Applying this theorem to our equation we see that the function $\varkappa(s, t)$ can be uniquely determined provided the above conditions are satisfied.

Göttingen, March 6th 1906.

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*) Sopra alcune quistioni di inversione di integrali definiti. Annali di Matematica 1897.