On the polynomial and trigonometric approximation of measurable bounded functions on a finite interval.

Von

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Introduction.

Let $f(x)$ be a continuous function defined on a finite interval (a, b). The important role of Weierstraß' theorem dealing with polynomial (or trigonometric) approximation of such functions is well known. With this theorem is closely connected the important notion, due to Tchebycheff, of the "polynomial of the best approximation" to $f(x)$ on (a, b), of degree $\leqq n$ — it will be denoted here by $\Pi_n(x)$ —, for which the "deviation" from $f(x)$ — the so called "best approximation" $E_n(f) = \max |f(x) - \Pi_n(x)|$ on (a, b) — is the smallest possible, compared with any other polynomial of degree $\leqq n$. However, the actual construction of $\Pi_n(x)$ is attainable in a very limited number of cases only.

It is, therefore, of interest to give *for any continuous function* $f(x)$, *defined on a finite interval* (a, b), *a sequence of polynomials of degree* $n = 1, 2, \ldots$ *which, as* $n \to \infty$, *converges uniformly to* $f(x)$ *throughout the whole interval* (a, b), *and yields, for n very large, an approximation of the same order as that of the best approximation.*

This was the original object of this paper attained by considering the minimum of the integral $\int_a^b p(x) |f(x) - P_{nm}(x)|^m dx$ $(m \geqq 1)$, for $m, n \to \infty$, where $p(x)$ $(\geqq 0)$ and $f(x)$ are properly defined on (a, b) and $P_{nm}(x)$ is the required minimizing polynomial of degree $\leqq n$. The author wishes to acknowledge this part as an outgrowth of a correspondence with Professor Paul Lévy of the École Polytechnique in Paris.

In the course of the said investigation it was found possible to *extend in two ways the notion of the polynomial of the best approximation* (*in* *the above sense of Tchebycheff) to the more general class of measurable bounded functions*, making use of "measurable bounds" introduced for such functions by C. N. Haskins.

We consider in the present paper the above minimum for all possible cases: 1. m is fixed, $n \to \infty$; 2. n is fixed, $m \to \infty$; 3. $m, n \to \infty$. Thus, our results supplement and generalize those previously given by G. Pólya, D. Jackson and the writer [1]).

§ 1.

In our discussion we shall make frequent use of the following inequalities:

$$(1)\quad \left|\int_a^b f_1(x) f_2(x)\,dx\right| \leqq \left[\int_a^b |f_1(x)|^s dx\right]^{\frac{1}{s}} \left[\int_a^b |f_2(x)|^{\frac{s}{s-1}} dx\right]^{\frac{s-1}{s}} \quad (s>1)$$

$$(2)\quad \left[\int_a^b |f_1(x)+f_2(x)|^s dx\right]^{\frac{1}{s}} \leqq \left[\int_a^b |f_1(x)|^s dx\right]^{\frac{1}{s}} + \left[\int_a^b |f_2(x)|^s dx\right]^{\frac{1}{s}}$$

$$(s \geqq 1)\ ^{2})$$

$$(3)\quad \left[\int_a^b |f_1(x)-f_2(x)|^s dx\right]^{\frac{1}{s}} \geqq \left[\int_a^b |f_1(x)|^s dx\right]^{\frac{1}{s}} - \left[\int_a^b |f_2(x)|^s dx\right]^{\frac{1}{s}}$$

$$(4)\quad \int_a^b |f_1(x)|\,|f_2(x)|^{s_1} dx \leqq \left[\int_a^b |f_1(x)|\,|f_2(x)|^{s_2} dx\right]^{\frac{s_1}{s_2}} \left[\int_a^b |f_1(x)|\,dx\right]^{\frac{s_2-s_1}{s_2}} \quad (s_2 \geqq s_1 > 0)$$

$$(5)\quad |a+b|^s \leqq 2^{s-1}\left[|a|^s + |b|^s\right] \quad (s \geqq 1).$$

In $(1-4)$ the existence of the right-hand integrals (integrals are taken in the sense of Lebesgue throughout this paper) implies the existence of those on the left side.

Hereafter, the following general notations will be used: $G_n(x) = \sum_{i=0}^{n} g_i x^i$ — to denote an *arbitrary polynomial* of degree $\leqq n$, subject in some cases to certain explicitly stated conditions; A, ε — to denote respectively a suffi-

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[1]) a) G. Pólya, Sur un algorithme toujours convergent pour les polynomes de la meilleure approximation de Tchebycheff pour une fonction continue quelconque, Comptes Rendus **157** (1913), p. 840–843. b) D. Jackson, On the convergence of certain trigonometric and polynomial approximations, Transactions of the American Mathematical Society **22** (1921), p. 158–166. c) Idem, Note on the convergence of weighed trigonometric series, Bulletin of the American Mathematical Society **29** (1923), p. 259–263. d) J. Shohat, On the polynomial of the best approximation to a given continuous function, ibid. **31** (1925), p. 509–514.

[2]) F. Riesz, Über Systeme integrierbarer Funktionen, Math. Annalen **69** (1911), S. 449–497, S. 456.

ciently large or sufficiently small, but fixed positive quantity properly chosen in each case; $m_0, n_0, \ldots$ — to denote properly chosen sufficiently large numbers of given sets $(m), (n), \ldots$; τ, τ_{nm} — to denote properly chosen fixed positive quantities which remain finite as $m, n \to \infty$.

§ 2.

Definition. $p(x)$, defined on a given interval (a, b) — finite or infinite — will be called a "*characteristic function*", abbreviatedly, "*c-function*", if: α) $p(x) \geqq 0$ *in* (a, b), β) *all integrals* $\int_a^b p(x) x^i dx$ $(i = 0, 1, \ldots)$ *exist with* $\int_a^b p(x) dx > 0$. In case of finite (a, b) the integrability of $p(x)$ on (a, b) implies the existence of all the integrals above.

With any such c-function $p(x)$ we can construct a system of orthogonal and normal Tchebycheff polynomials

$$\varphi_n(p; x) \equiv \varphi_n(x) = a_n(p) x^n + \ldots \quad (n = 0, 1, 2, \ldots; \; a_n > 0) \tag{6}$$

$$\int_a^b p(x) \varphi_n(x) \varphi_m(x) dx = \begin{cases} 0, & m \neq n \\ 1, & m = n, \end{cases} \tag{7}$$

which enable us to solve the following problem.

Find the upper limit of $|\omega(G_n)| \equiv \left| \sum_{i=0}^n \alpha_i g_i \right|$ *for all polynomials* $G_n(x) = \sum_{i=0}^n g_i x^i$, *of degree* $\leqq n$, satisfying the inequality

$$\int_a^b p(x) |G_n(x)|^m dx \leqq M^m.$$

Here $p(x)$ *designates a c-function defined on* (a, b), $\alpha_i (i = 0, 1, \ldots, n)$, M (> 0) *and* m $(\geqq 2)$ *are certain given constants.*

Solution. Applying (1) we get,

$$\int_a^b p(x) G_n^2(x) dx \leqq M^2 \left[\int_a^b p(x) dx \right]^{\frac{m-2}{m}}.$$

Thus, our problem is reduced to the case $m = 2$, and we can apply the solution previously given by the writer [3]):

$$|\omega(G_n)| \leqq M \left[\int_a^b p(x) dx \right]^{\frac{m-2}{2m}} \left\{ \sum_{i=0}^n \omega^2(\varphi_i) \right\}^{\frac{1}{2}}. \tag{8}$$

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[3]) J. Shohat, On a general formula in the theory of Tchebycheff polynomials and its applications, Transactions of the American Mathematical Society 29 (1927), p. 569–583; p. 569–571.

Special case: $\omega(G_n) \equiv G_n(z)$, z real arbitrary.

$$\int_a^b p(x)\,|G_n(x)|^m\,dx \leqq M^m$$

implies:

$$|G_n(z)| \leqq M K_n^{\frac{1}{2}}(z)\Big[\int_a^b p(x)\,dx\Big]^{\frac{m-2}{2m}} \tag{9}$$

$$K_n(p;x) \equiv K_n(x) \equiv \sum_{i=0}^{n} \varphi_i^2(x). \tag{10}$$

The assumption $m \geqq 2$ is sufficient for the applications below. In case $2 > m \geqq 1$, the representation of $\omega(G_n)$ as a definite integral used in deriving (8) gives,

$$|\omega(G_n)| \leqq M\Big[\int_a^b p(x)\,dx\Big]^{\frac{m-1}{m}} \sum_{i=0}^{n} |\omega(\varphi_i)|\,\Phi_i \tag{11}$$

$$(\Phi_i = \max|\varphi_i(x)| \text{ in } (a,b)).$$

Note. *The above upper limits for* $|\omega(G_n)|$ *do not depend on* m, *if* M *and the* α_i *do not,* for in (8), (9) $\int_a^b p(x)\,dx \leqq 1$ or $\leqq \Big[\int_a^b p(x)\,dx\Big]^{\frac{1}{2}}$, according to whether $\int_a^b p(x)\,dx \leqq 1$ or >1, and similarly in (11).

§ 3.

We apply the solution of the aforesaid problem to the proof of

Theorem 1. *Given on* (a, b) *a c-function* $p(x)$ *and a measurable function* $f(x)$ *"of the clas* $[L_p^m]$*", i. e. such that* $\int_a^b p(x)\,|f(x)|^m\,dx$ *exists* $(m \geqq 1)$. *Among all polynomials of degree* $\leqq n$ *there exists at least one* $P_{nm}(x)$ *minimizing the integral* $\int_a^b p(x)\,|f(x) - P_{nm}(x)|^m\,dx$. *In case* $m > 1$ *this polynomial is unique*[4]).

[4]) This minimum problem, with more or less restricted $p(x)$ and $f(x)$, has been treated previously (loc. cit. [1])). Here the only condition imposed upon $p(x)$ is: $\int_a^b p(x)\,dx > 0$. This clearly is no restriction, since the hypothesis $\int_a^b p(x)\,dx = 0$ implies: $p(x) = 0$ almost everywhere in (a, b). Similarly, the condition of existence of $\int_a^b p(x)\,|f(x)|^m\,dx$ is imposed by the very nature of our problem, for, as it is readily seen from (1–4), the integrals $\int_a^b p(x)\,|f(x)|^m\,dx$, $\int_a^b p(x)\,|f(x) - G_n(x)|^m\,dx$ exist or do not exist simultaneously.

Proof. By (3),

$$\left[\int_a^b p(x)\,|f(x)-G_n(x)|^m\,dx\right]^{\frac{1}{m}} \geqq \left[\int_a^b p(x)\,|G_n(x)|^m\,dx\right]^{\frac{1}{m}} - \left[\int_a^b p(x)\,|f(x)|^m\,dx\right]^{\frac{1}{m}}.$$

The assumption $\int_a^b p(x)\,|G_n(x)|^m\,dx > (2A)^m$, with $A^m > \int_a^b p(x)\,|f(x)|^m\,dx$, implies: $\int_a^b p(x)\,|f(x)-G_n(x)|^m\,dx > A^m$. Therefore, we confine ourselves to polynomials $G_n(x)$ such that

$$\int_a^b p(x)\,|G_n(x)|^m\,dx \leqq (2A)^m.$$

The coefficients of such $G_n(x)$, for arbitrarily given n, are necessarily *bounded*, as we learn from (8), taking successively $\alpha_i = 1$ $(i = 0, 1, \ldots, n)$, $\alpha_j = 0$ $(j \neq i)$, and the existence of one at least minimizing polynomial $P_{nm}(x)$ follows.

As to the uniqueness of $P_{nm}(x)$ for $m > 1$, the proof is identical to that given in my aforesaid paper [5]).

The above proof gives incidentally the following

Corollary. *To an arbitrarily large $A > 0$ there correspond certain K_i $(i = 0, 1, \ldots, n)$ such that any one of the inequalities $|g_i| > K_i$ implies: $\int_a^b p(x)\,|f(x)-G_n(x)|^m\,dx > A$, where the c-function $p(x)$ and the function $f(x)$ of the class $[L_p^m]$ are arbitrary. The K_i do not depend on m, if A and M do not; $m \geqq 1$.*

We shall use the notation

$$(12)\quad I_{nm} \equiv \int_a^b p(x)\,|f(x)-P_{nm}(x)|^m\,dx = \min \int_a^b p(x)\,|f(x)-G_n(x)|^m\,dx,$$

and we have evidently,

$$(13)\qquad I_{nm} \leqq \int_a^b p(x)\,|f(x)|^m\,dx.$$

§ 4.

The following two theorems, interesting by themselves, are needed for the investigation of I_{nm} and $P_{nm}(x)$.

Theorem II. 1°. *Let $p(x)$ be non-negative and integrable on (a, b) and such that $\int_E p(x)\,dx > 0$, E denoting an arbitrary measurable set of*

[5]) Loc. cit. [1]), d), p. 511. Uniqueness, in case $m = 1$, seems to require additional conditions for $p(x)$.

points in (a, b) *with* $mE > 0$. *Let the measurable function* $f(x)$, *defined on* (a, b), *be such that* $I_m = \int_a^b p(x)|f(x)|^m dx$ *exists for every* $m > 0$[6]. *Denote by* F *the "measurable upper bound"*[7] *of* $|f(x)|$ *in* (a, b). *Then,* $\lim_{m\to\infty} I_m^{\frac{1}{m}} = F$. 2°. *If* m *monotonically increases without bounds, then* $\lim_{m\to\infty} I_m^{\frac{1}{m}}$ *exists,* $\leqq F$, *for any* $p(x)$ *non-negative and integrable in* (a, b), *and* $\left[\frac{I_m}{\int_a^b p(x)\,dx}\right]^{\frac{1}{m}}$ *is also monotonically increasing* $\left(\text{so is } I_m^{\frac{1}{m}}, \text{ if } \int_a^b p(x)\,dx \leqq I\right)$.

Proof. 1°. The proof is somewhat similar to that given by Prof. Haskins for the case $p(x) = 1$.[8] By the definition of "measurable upper bound", we write,

$$mE[|f(x)| \geqq F - \varepsilon] > 0, \quad mE[|f(x)| \geqq F + \varepsilon] = 0 \quad (a \leqq x \leqq b)$$

(with obvious modifications, in case $F = +\infty$). If F be finite, let E_ε denote the set of points x in (a, b) such that $|f(x)| \geqq F - \varepsilon$. Then,

$$I_m^{\frac{1}{m}} \geqq (F - \varepsilon)\left[\int_{E_\varepsilon} p(x)\,dx\right]^{\frac{1}{m}}.$$

On the other hand,

$$I_m^{\frac{1}{m}} \leqq (F + \varepsilon)\left[\int_a^b p(x)\,dx\right]^{\frac{1}{m}}. \tag{14}$$

Therefore, since ε does not depend on m,

$$\underline{\lim}_{m\to\infty} I_m^{\frac{1}{m}} \geqq F, \quad \overline{\lim}_{m\to\infty} I_m^{\frac{1}{m}} \leqq F. \tag{15}$$

If F be $+\infty$, denote by E_A the set of points x in (a, b) such that $|f(x)| \geqq 2A$. Then,

$$I_m^{\frac{1}{m}} > 2A\left[\int_{E_A} p(x)\,dx\right]^{\frac{1}{m}} > A \qquad (m \geqq m_0). \tag{16}$$

The inequalities (15), (16) prove our statement.

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[6]) This follows from its existence, say, for all sufficiently large integral m.

[7]) C. N. Haskins, On the measurable bounds and the distribution of functional values of summable functions. Transactions of the American Mathematical Society **17** (1916), p. 181–194, p. 184.

[8]) *Loc. cit.* [7]), p. 187–188.

2°. This follows from (14) and from the inequality (see (4))

$$I_{m_1}^{\frac{1}{m_1}} \leqq I_{m_2}^{\frac{1}{m_2}} \left[\int_a^b p(x)\,dx\right]^{\frac{m_2-m_1}{m_1 m_2}} \qquad (m_2 > m_1 > 0). \tag{17}$$

Note. The condition $\int_E p(x)\,dx > 0$ $(mE > 0)$ is indispensable, as it is seen by taking $p(x) = 0$, $f(x) = 1$ for $a < \alpha \leqq x \leqq \beta < b$ and $p(x) = 1$, $f(x) = 0$ elsewhere in (a, b).

Theorem III. *Given on a finite interval (a, b) a non-negative integrable function $p(x)$ and a family $\{f(x, m)\}$ of continuous functions, where the parameter $m (> 0)$ takes a set of values $\to \infty$. Assume: α) $\lim_{m\to\infty} f(x, m) = f(x)$ uniformly for $a \leqq x \leqq b$; β) among the points in (a, b), where $|f(x)|$ (necessarily continuous) attains its maximum F, there exists at least one, say, $x = c$, such that $\int_{(\delta)} p(x)\,dx > 0$, (δ) denoting an arbitrary sub-interval of (a, b) containing the point c. Then,* $\lim_{m\to\infty} \left[\int_a^b p(x)\,|f(x, m)|^m\,dx\right]^{\frac{1}{m}} = F$.

Proof. First, we have

$$|f(x, m) - f(x)| < \varepsilon, \qquad |f(x, m)| < F + \varepsilon \qquad (a \leqq x \leqq b;\ m \geqq m_0).$$

This assures the existence, for all m under consideration, of $\int_a^b p(x)\,|f(x, m)|^m\,dx$, with

$$i_m^{\frac{1}{m}} \equiv \left[\int_a^b p(x)\,|f(x, m)|^m\,dx\right]^{\frac{1}{m}} < (F + \varepsilon)\left[\int_a^b p(x)\,dx\right]^{\frac{1}{m}} \quad (m \geqq m_0). \tag{18}$$

On the other hand, consider the point $x = c$ and the sub-interval (δ) with the properties given above. In virtue of the continuity of $f(x)$ and the uniform convergence of $f(x, m)$, we can fixe $m = m_1$ so large and the interval (δ) so small as to have

$$|f(x, m)| \geqq F - \varepsilon \qquad (x \text{ belongs to } (\delta);\ m \geqq m_1),$$

$$i_m^{\frac{1}{m}} \geqq (F - \varepsilon)\left[\int_{(\delta)} p(x)\,dx\right]^{\frac{1}{m}} \qquad (m \geqq m_1). \tag{19}$$

(18), (19) prove our statement, for they lead to the inequalities

$$\overline{\lim_{m\to\infty}}\, i_m^{\frac{1}{m}} \leqq F, \qquad \underline{\lim_{m\to\infty}}\, i_m^{\frac{1}{m}} \geqq F.$$

We shall not dwell here upon possible generalizations of theorem III.[9])

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[9]) In theorems II and III the integrals involved can be written as Stieltjes integrals, for $\int_a^b p(x)\,f(x)\,dx = \int_a^b f(x)\,d\psi(x)$, where $\psi(x) = \int_a^x p(x)\,dx$. — As an

(Fortsetzung der Fußnote auf S. 164.)

§ 5.

Hereafter, the interval (a, b) *is assumed to be finite.* We proceed now to set forth an upper limit for $|f(x) - P_{nm}(x)|$, under the assumption: $f(x)$ *is continuous for* $a \leqq x \leqq b$. Introduce its polynomial of the best approximation on (a, b) $\Pi_n(x)$, of degree $\leqq n$, and the best approximation $E_n(f)$ (see Introduction):

$$(20) \qquad E_n(f) = \max |f(x) - \Pi_n(x)| \leqq \max |f(x) - G_n(x)| \quad \text{in } (a, b).$$

We have then, by the definition of $P_{nm}(x)$ and by (4),

$$(21) \qquad \begin{aligned} &\int_a^b p(x)\,|f(x) - P_{nm}(x)|^m\,dx \\ &= I_{nm} \leqq \int_a^b p(x)\,|f(x) - \Pi_n(x)|^m\,dx \leqq E_n^m(f) \int_a^b p(x)\,dx, \end{aligned}$$

$$(22) \qquad \int_a^b p(x)\,|P_{nm}(x) - \Pi_n(x)|^m\,dx \leqq [2E_n(f)]^m \int_a^b p(x)\,dx.$$

Assume, first, m, not being an integer, is > 2, and denote by 2μ the greatest even integer contained in m. By (4), we get from (22),

$$(23) \qquad \int_a^b p(x)\,|P_{nm}(x) - \Pi_n(x)|^{2\mu}\,dx \leqq [2E_n(f)]^{2\mu} \int_a^b p(x)\,dx,$$

and this inequality evidently holds also for

$$(24) \qquad 2\mu \leqq m < 2\mu + 2.$$

Apply (9) to the polynomial $[P_{nm}(x) - \Pi_n(x)]^\mu$ of degree $\leqq \mu n$, taking $m = 2$ and replacing n by μn. We get,

$$(25) \qquad |P_{nm}(x) - \Pi_n(x)| \leqq 2E_n(f)\left[K_{n\mu}(x) \int_a^b p(t)\,dt\right]^{\frac{1}{2\mu}}$$
$$(m \geqq 2;\ x \text{ arbitrary}),$$

$$(26) \qquad |f(x) - P_{nm}(x)| \leqq E_n(f)\left\{1 + 2\left[K_{n\mu}(x) \int_a^b p(t)\,dt\right]^{\frac{1}{2\mu}}\right\}.$$

Formulae (25), (26) *hold for any* c*-function* $p(x)$, *for any continuous function* $f(x)$. [For $1 \geqq m > 2$ we could use (11).] They yield

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illustration to theorem II may serve: 1°. The integral $c_n = \int_a^b p(x)\,|x|^n\,dx$, (a, b) finite. Here $\lim_{n\to\infty} c_n^{\frac{1}{n}} \left(= \lim \frac{c_{n+1}}{c_n}\right) = \max(|a|, |b|)$. Cf. O. Perron, Die Lehre von den Kettenbrüchen (1913), p. 3–520, p. 384–385; 2°. $\lim_{n\to\infty} \left|\int_0^\infty e^{-nx^\lambda} x^n\,dx\right|^{\frac{1}{n}}$ $(\lambda > 0)$ $= \frac{1}{\lambda}\, e^{-\frac{1}{\lambda\lambda}}$. .

the order, with respect to n, of $|f(x)-P_{nm}(x)|$ for very general $p(x)$, where the order of $K_n(x) \equiv \sum_{i=0}^{n} \varphi_i^2(x)$ is known.

Assume, to illustrate, that $p(x)$ satisfies the following

Condition P. *There exists an interval* (c, d) $(a \leqq c < d \leqq b)$ *with the following properties*: α) *it contains a finite number of points* x_i *to which correspond sufficiently small intervals* $(x_i - \delta_i,\ x_i + \delta_i)$ (the interval being one-sided if $x_i = c, d$) *in which* $\frac{p(x)}{|x-x_i|^{k_i}} > A_i$ *with certain* $k_i > -1$, $A_i > 0$; β) $p(x) \geqq p_0 > 0$ *elsewhere in* (a, b).

Then, as it has been shown by the writer[10]),

$$(27) \qquad K_n(x) < \tau n^{\sigma} \qquad (c \leqq x \leqq d),$$

σ $(>0$, determined by the $k_i)$ and τ do not depend on x, nor on n.

Therefore, *with* $p(x)$ *satisfying the condition* (P),

$$(28) \qquad |f(x)-P_{nm}(x)| < \tau_m E_n(f)\, n^{\frac{\sigma}{2\mu}} \qquad (c \leqq x \leqq d),$$

where τ_m does not depend on n, nor on x, nor on $f(x)$, and this can be rewritten, as

$$(29) \qquad |f(x)-P_{nm}(x)| < \tau_{nm} E_n(f)\, n^{\frac{\sigma}{\mu}} \qquad (c \leqq x \leqq d),$$

in case

$$(30) \qquad n, m \to \infty, \quad \frac{\log n}{m^2} \text{ remains finite.}$$

With the additional assumption

$$(31) \qquad p(x) \geqq p_0 > 0 \quad \text{for} \quad (a \leqq) c \leqq x \leqq d (\leqq b)$$

we can derive an upper limit for $|f(x)-P_{nm}(x)|$ by an entirely different method.

It makes no use of Tchebycheff polynomials, being based upon the well known Markoff-Bernstein theorem which we state as follows:

$|G_n(x)| \leqq M$ *on* (α, β) *implies*: 1°. $|G_n'(x)| \leqq Cn^2 M$ $(\alpha \leqq x \leqq \beta)$, 2°. $|G_n'(x)| \leqq CnM$ $(\alpha + \varepsilon \leqq x \leqq \beta - \varepsilon)$, *where* C (of course, not the same in 1°, 2°) *depends on* α, β *and* ε *only*.

We write, with the notations used above,

$$[P_{nm}(x) - \Pi_n(x)]^{2\mu} = \frac{d}{dx} \int_c^x [P_{nm}(t) - \Pi_n(t)]^{2\mu} dt.$$

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[10]) J. Shohat, On the development of continuous functions in series of Tchebycheff polynomials, Transactions of the American Mathematical Society 27 (1925) p. 537–550, p. 540–541.

The polynomial $Q(x) = \int\limits_{c+\varepsilon}^{x} [P_{nm}(t) - \Pi_n(t)]^{2\mu} dt$, of degree $\leqq 2n\mu + 1$, satisfies, for $c \leqq x \leqq d$, the inequality (see (23))

$$|Q(x)| \leqq \frac{1}{p_0} \int\limits_a^b p(t) [P_{nm}(t) - \Pi_n(t)]^{2\mu} dt \leqq \frac{[2E_n(f)]^{2\mu}}{p_0} \int\limits_a^b p(t)\, dt,$$

and Markoff-Bernstein theorem gives:

$$\begin{aligned} &|P_{nm}(x) - \Pi_n(x)| < \tau_{nm} n^q E_n(f), \\ (32) \qquad &|f(x) - P_{nm}(x)| \;\; < \tau_{nm} n^q E_n(f), \\ &q = 2 \text{ for } c \leqq x \leqq d; \quad q = 1 \text{ for } c + \varepsilon \leqq x \leqq d - \varepsilon. \end{aligned}$$ [11]

This is a special case of (25), (26), for, under (31),

$$(33) \quad K_n(x) < \tau n^2 \;\; (c \leqq x \leqq d), \qquad K_n(x) < \tau n \;\; (c + \varepsilon \leqq x \leqq d - \varepsilon).$$ [10]

In addition to (31) we subject now $p(x)$ to the condition

$$(34) \qquad \int\limits_x^{x+\delta} p(x)\, dx < K\delta \qquad (c \leqq x < x + \delta \leqq d),$$

where K does not depend on x, nor on δ.

Then we get a new expression for the upper limit of $|f(x) - P_{nm}(x)|$ involving the modulus of continuity $\omega(h)$ of $f(x)$ in (a, b), which proves to be especially useful in case: n is fixed, $m \to \infty$. We write:

$$f(x) - P_{nm}(x) = \frac{1}{h} \int\limits_x^{x+h} [f(x) - f(t)]\, dt + \frac{1}{h} \int\limits_x^{x+h} [f(t) - P_{nm}(t)]\, dt$$

$$+ \frac{1}{h} \int\limits_x^{x+h} [P_{nm}(t) - P_{nm}(x)]\, dt \equiv i_1 + i_2 + i_3 \quad (h > 0;\; c \leqq x < x + h \leqq d).$$

$$|i_1| \leqq \omega(h),$$

$$|i_2| \leqq \frac{1}{hp_0} \Big[\int\limits_a^b p(t) |f(t) - P_{nm}(t)|^m dt \Big]^{\frac{1}{m}} \Big[\int\limits_x^{x+h} p(t)\, dt \Big]^{\frac{m-1}{m}} \leqq \frac{K^{\frac{m-1}{m}} h^{-\frac{1}{m}} E_n(f)}{p_0},$$

$$|i_3| \leqq C n^q h M_{nm}$$

(by Markoff-Bernstein theorem; $M_{nm} = \max |P_{nm}(x)|$ in (c, d)).

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[11]) The elegant method of D. Jackson, also based essentially on Markoff-Bernstein theorem (*loc. cit.* (1-b), p. 162–164), seems to be inapplicable here directly, unless an additional assumption be made concerning the lower limit of $\Big| \int\limits_x^{x \pm \delta} p(x)\, dx \Big|$, $\delta = \dfrac{b-a}{8n^2}$, $x,\ x \pm \delta$ *varying throughout the whole interval* (a, b).

$$(35) \qquad R_{nm}(x) \equiv |f(x) - P_{nm}(x)| \leqq \omega(h) + \frac{K^{\frac{m-1}{m}} h^{-\frac{1}{m}} E_n(f)}{p_0} + Cn^q M_{nm} h$$

$$(q \text{ given in } (32);\; c \leqq x \leqq d).$$

$$|P_{nm}(x)| \leqq R_{nm}(x) + |f(x)|.$$

$$(36) \qquad \begin{aligned} M_{nm} &= |P_{nm}(\xi)| \leqq R_{nm}(\xi) + F \leqq R_{nm} + F \qquad (\xi \text{ in } (c,d)),\\ R_{nm} &= \max|R_{nm}(x)|, \quad F = \max|f(x)| \quad \text{in } (c,d). \end{aligned}$$

(35), (36) lead to

$$(37) \quad |f(x) - P_{nm}(x)| \leqq \left\{\omega(h) + \frac{K^{\frac{m-1}{m}} h^{-\frac{1}{m}} E_n(f)}{p_0} + Cn^q h F\right\} : (1 - Cn^q h)$$

$$(Cn^q h < 1),$$

$$(38) \quad |f(x) - P_{nm}(x)| \leqq E_n(f)\left\{2 + \frac{K^{\frac{m-1}{m}} h^{-\frac{1}{m}}}{p_0} + Cn^q h\right\} : (1 - Cn^q h),$$

$$(39) \quad |f(x) - P_{nm}(x)| < 2E_n(f)\left\{3 + \frac{K^{\frac{m-1}{m}} h^{-\frac{1}{m}}}{p_0}\right\}.$$

(We derive (38), (39), applying (37) to $\varphi(x) \equiv f(x) - P_{nm}(x)$ and taking $Cn^q h = \frac{1}{2}$).

In (37), (38), (39) $q = 2$ *for* $c \leqq x \leqq d$, $q = 1$ *for* $c + \varepsilon \leqq x \leqq d + \varepsilon$, C *is given by Markoff-Bernstein theorem.*

The expressions given above for the upper limit of $|f(x) - P_{nm}(x)|$ enable us to treat I_{nm} and $P_{nm}(x)$ in all the cases given in the Introduction. Formula (26) also yields some interesting applications to the theory of Tchebycheff polynomials.

§ 6.

Case I. m is fixed, $n \to \infty$.

Theorem IV. *The sequence* $\left\{I_{nm}^{\frac{1}{m}}\right\}$ *is monotonically decreasing towards zero for any* $f(x)$ *of the class* $[L_p^m]$ $(m \geqq 1)$.

Proof. We have, first, by the definition of $P_{nm}(x)$,

$$(40) \qquad I_{nm} \geqq I_{n+1,m} \geqq I_{n+2,m} \geqq \cdots.$$

On the other hand, $f(x)$ being of the class $[L_p^m]$ $(m \geqq 1)$, a *continuous function* $\varphi(x)$ can be determined such that $\int_a^b p(x)|f(x) - \varphi(x)|^m dx < \frac{\varepsilon}{2^m}$.[12])

[12]) The proof is essentially the same as that given by Hobson (The Theory of functions of a real variable, 2-d ed., 2, p. 250) for the case $p(x) = 1$.

By Weierstraß' theorem, a polynomial $Q(x)$, of sufficiently high degree n can be assigned so that $\int_a^b p(x)|\varphi(x) - Q(x)|^m dx < \frac{\varepsilon}{2^m}$. Then, using (5) and the definition of $P_{nm}(x)$, we get,

$$\int_a^b p(x)|f(x) - P_{nm}(x)|^m dx \leqq \int_a^b p(x)|f(x) - Q(x)|^m dx$$

$$\leqq 2^{m-1}\left\{\int_a^b p(x)|f(x) - \varphi(x)|^m dx + \int_a^b p(x)|\varphi(x) - Q(x)|^m dx\right\} < \varepsilon,$$

which, combined with (40), proves our statement.

Assuming $m > 1$, and writing $P_n(x), I_n$ in place of $P_{nm}(x), I_{nm}$, we associate with any $f(x)$ of the class $[L_p^m]$ the sequence of polynomials, uniquely determined,

(41) $$P_1(x),\ P_2(x),\ \ldots,\ P_n(x),\ \ldots$$

or, which is the same, the infinite series

(42) $$P_1(x) + [P_2(x) - P_1(x)] + \ldots + [P_{n+1}(x) - P_n(x)] + \ldots .$$

The sequence (41) may be spoken of as approximating $f(x)$ in the sense of the "least m-th powers" (D. Jackson), also — slightly generalizing a notion due to F. Riesz —, as "converging strongly to $f(x)$ with exponent m" [13]). We have then,

(43) $$\lim_{n\to\infty} \int_a^b p(x)|P_n(x)|^m dx = \int_a^b p(x)|f(x)|^m dx,$$

(44) $$\lim_{n\to\infty} \int_\alpha^t p(x)P_n(x)g(x)dx = \int_\alpha^t p(x)f(x)g(x)dx$$

$$\left(a \leqq \alpha,\ t \leqq b;\ g(x) \text{ of the class } \left[L_p^{\frac{m}{m-1}}\right]\right).$$

In other words, *for any $f(x)$ of the class $[L_p^m]$ $(m > 1)$ the infinite series (42), multiplied on both sides by $p(x)g(x)$, where $g(x)$ is an arbitrary function of the class $\left[L_p^{\frac{m}{m-1}}\right]$, can be integrated term by term between any two limits $(a \leqq)\alpha,\ t(\leqq b)$. The convergence is uniform, if t be variable.*

The proof of (43), (44) is essentially the same as that given by Hobson [14]) for the case $p(x) = 1$.

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[13]) *Loc. cit.* [2]), p. 464. Here we introduce $p(x)$ as a factor.

[14]) *Loc. cit.* [12]), p. 251.

With regard to the convergence of the series (42), we derive from (26)

Theorem V. *The series* (42), *or, which is the same, the sequence* $\{P_{nm}(x)\}$, *with* m (>1) *fixed and* $n \to \infty$, *converges to* $f(x)$, *assumed to be continuous in* (a, b), *uniformly over any sub-interval* (c, d) $(a \leqq c < d \leqq b)$ *where* $K_n(x) \equiv \sum_{i=0}^{n} \varphi_i^2(x) = O(n^\sigma)$ (this certainly takes place if the condition (P) is satisfied), *provided,* $\lim_{n \to \infty} E_n(f)\, n^{\frac{\sigma}{2\mu}} = 0$ (σ does not depend on n, nor on x).

From (31) we derive as sufficient condition for the uniform convergence of (41):

$$\lim_{n \to \infty} E_n(f)\, n^{\frac{q}{m}} = 0 \qquad \text{(see (32))} \tag{45}$$

quite similar to that given by D. Jackson in the case $p(x) = 1$.[15])

§ 7.

Case II. n is fixed, $m \to \infty$.

Consider the class of all $f(x)$ *measurable and bounded on* (a, b). Denote by $\overline{F}$, F respectively the upper bound and the "measurable upper bound" of $|f(x)|$ in (a, b).

The notion of the "best approximation" of functions by means of polynomials of degree $\leqq n$, established by Tchebycheff for functions *continuous* over a finite interval, can be extended to the more general class *of measurable bounded functions* $f(x)$ (under our consideration) *in two ways*, as follows. Using for the best approximation and the corresponding polynomial resp. the notations $E_n^\alpha(f)$, $E_n^\beta(f)$, $\Pi_n^\alpha(x)$, $\Pi_n^\beta(x)$, we define:

$E_n^\alpha(f) =$ *upper bound of* $|f(x) - \Pi_n^\alpha(x)|$, $E_n^\beta(f) =$ *measurable upper bound of* $|f(x) - \Pi_n^\beta(x)|$ *on* (a, b), *are each the smallest possible among all such expessions formed with an arbitrary polynomial of degree* $\leqq n$.

The existence of one at least $\Pi_n^\alpha(x)$ has been proved by Kirchberger[16]). In a similar manner we prove the existence of one at least $\Pi_n^\beta(x)$. The latter, which seems to be new, is more important, as is shown by theorems VI, VII below.

If $f(x)$ be continuous on (a, b) then

$$\overline{F} = F, \quad \Pi_n^\alpha(x) \equiv \Pi_n^\beta(x) \equiv \Pi_n(x), \quad E_n^\alpha(f) = E_n^\beta(f) = E_n(f).$$

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[15]) *Loc. cit.* [1]) b), p. 165–166.

[16]) P. Kirchberger, Über Tchebycheffsche Annäherungsmethoden, Math. Annalen **57** (1903), p. 509–540; p. 511–512.

In general,

$$\bar{F} \neq F, \quad \Pi_n^\alpha(x) \neq \Pi_n^\beta(x), \quad E_n^\alpha(f) \neq E_n^\beta(f),$$

and we have, denoting respectively by F^α, $\bar{F}^\beta$ the measurable upper bound of $|f(x) - \Pi_n^\alpha(x)|$ and the upper bound of $|f(x) - \Pi_n^\beta(x)|$, for $a \leqq x \leqq b$,

$$\begin{aligned} &E_n^\beta(f) \leqq F^\alpha \leqq E_n^\alpha(f) \leqq \bar{F}^\beta, \\ &E_n^\alpha(f) \geqq E_{n+1}^\alpha(f) \geqq \ldots; \quad E_n^\beta(f) \geqq E_{n+1}^\beta(f) \geqq \ldots . \end{aligned} \tag{46}$$

There exist, therefore,

$$\lim_{n\to\infty} E_n^\alpha(f) \geqq 0, \qquad \lim_{n\to\infty} E_n^\beta(f) = E^\beta(f) \geqq 0.$$

Theorem VI. 1°. $\lim\limits_{n\to\infty} E_n^\alpha(f) = 0$, *if* $f(x)$ *has one at least discontinuity in* (a, b). 2°. $\lim\limits_{n\to\infty} E_n^\beta(f) = 0$ *for all* $f(x)$ *measurable and bounded in* (a, b).

Proof. 1°. The assumption $\lim\limits_{n\to\infty} E_n^\alpha(f) = 0$ implies:

$$|f(x) - \Pi_n^\alpha(x)| \leqq E_n^\alpha(f) < \varepsilon \qquad (a \leqq x \leqq b;\ n \geqq n_0);$$

in other words, the infinite series of polynomials

$$\Pi_1^\alpha(x) + [\Pi_2^\alpha(x) - \Pi_1^\alpha(x)] + \ldots$$

converges to $f(x)$ uniformly for $a \leqq x \leqq b$, which necessitates the continuity of $f(x)$ throughout the whole interval (a, b), contrary to our hypothesis.

2°. Assume $E^\beta(f) > 0$. Denote by Γ_n the measurable upper bound on (a, b) of $|f(x) - G_n(x)|$. Then, for any n,

$$\Gamma_n \geqq E_n^\beta(f) \geqq E^\beta(f) > 0.$$

This contradicts a theorem of Hobson[17] which states the existence of a sequence of polynomials $Q(x)$ converging, as n, the degree of $Q(x)$, $\to \infty$, to $f(x)$ almost everywhere in (a, b).

§ 8.

Let us associate with the polynomial $G_n(x) = \sum\limits_{i=0}^{n} g_i x^i$ the point $G_n \equiv (g_0, g_1, \ldots, g_n) \equiv \left(\begin{smallmatrix} n \\ g_i \\ 0 \end{smallmatrix}\right)$ in the n-dimensional space. *The relation* $A_n \to B_n$ *means, then, that the coefficients of* $A_n(x) = \sum\limits_{i=0}^{n} a_i x^i$ *converge respectively to those of* $B_n(x) = \sum\limits_{i=0}^{n} b_i x^i$, *and thus* $A_n(x) \to B_n(x)$ *uniformly over any finite interval.*

[17]) *Loc. cit.* [12]), p. 256.

Theorem VII. *Let the c-function $p(x)$ satisfy the requirements of theorem* II, *and let $f(x)$ be measurable and bounded on (a, b). Then:* 1°. *All limit-points of the set $\{P_{nm}(x)\}$ are in a finite portion of space and coincide with the set of the polynomials of the best approximation $\Pi_n^\beta(x)$ (n is fixed, $m \to \infty$).* 2°. $\lim_{m\to\infty} I_{nm}^{\frac{1}{m}} = E_n^\beta(f)$. 3°. *If $f(x)$ be continuous on (a, b), then* $\lim_{m\to\infty} P_{nm}(x) = \Pi_n(x)$, $\lim_{m\to\infty} I_{nm}^{\frac{1}{m}} = E_n(f)$. 4°. *If m increases monotonically, so does* $\left[\frac{I_{nm}}{\int_a^b p(x)\,dx}\right]^{\frac{1}{m}}$, *and* $\lim_{m\to\infty} I_{nm}^{\frac{1}{m}}$ *exists for any c-function $p(x)$.*

Proof. 1°. 2°. F denoting, as above, the measurable upper bound of $|f(x)|$ in (a, b),

$$(47)\qquad \int_a^b p(x)\,|f(x) - P_{nm}(x)|^m\,dx < (F+\varepsilon)^m \int_a^b p(x)\,dx \qquad \text{(see (13))}.$$

It follows (making use of the Corollary to theorem I) that *the coefficients of all $P_{nm}(x)$ are bounded, as functions of m*, which proves the first part of 1°. To prove the second part, we employ an argument somewhat similar to that of G. Pólya[18]). Let $Q_n \equiv \left(\overset{n}{\underset{0}{q_i}}\right)$ denote *any one* of the limit-points in question. Denote, further, by Q the measurable upper bound of $|f(x) - Q_n(x)|$ in (a, b), and by E_ε the set of points x in (a, b) where

$$(48)\qquad |f(x) - Q_n(x)| \leqq Q - \frac{\varepsilon}{2}.$$

There exists in the set $\{m\}$ a sequence m_i ($i = 1, 2, \ldots$; $\lim_{i\to\infty} m_i = \infty$) such that $\lim_{i\to\infty} P_{nm_i}(x) = Q_n(x)$ uniformly in (a, b), and

$$|\{f(x) - P_{nm_i}(x)\} - \{f(x) - Q_n(x)\}| < \frac{\varepsilon}{2} \qquad (a \leqq x \leqq b;\ i \geqq i_0),$$

$$|f(x) - P_{nm_i}(x)| \geqq Q - \varepsilon \qquad (x \text{ in } E_\varepsilon;\ i \geqq i_0;\ \text{see (48)}),$$

$$(49)\qquad I_{nm_i}^{\frac{1}{m_i}} \geqq (Q-\varepsilon)\left[\int_{E_\varepsilon} p(x)\,dx\right]^{\frac{1}{m_i}} \qquad (i \geqq i_0).$$

By the definition of $P_{nm}(x)$, we write,

$$(50)\qquad I_{nm'}^{\frac{1}{m'}} \leqq I_{nm''}^{\frac{1}{m''}} \left[\int_a^b p(x)\,dx\right]^{\frac{m''-m'}{m'm''}} \qquad (m'' > m') \qquad \text{(see (4))},$$

$$(51)\qquad I_{nm}^{\frac{1}{m}} \leqq \left[\int_a^b p(x)\,|f(x) - \Pi_n^\beta(x)|^m\,dx\right]^{\frac{1}{m}} < (E_n^\beta(f) + \varepsilon)\left[\int_a^b p(x)\,dx\right]^{\frac{1}{m}}.$$

18) *Loc. cit.* 1) a).

We assume, further,

$$(52)\qquad \int_a^b p(x)\,dx = 1,$$

replacing, if necessary, $p(x)$ by $cp(x)$, with $c > 0$ properly chosen. Then,

$$(53)\quad I_{nm}^{\frac{1}{m}} \geqq I_{nm_i}^{\frac{1}{m_i}} \geqq (Q-\varepsilon)\left[\int_{E_\varepsilon} p(x)\,dx\right]^{\frac{1}{m_i}} \ (m > m_i;\ i \geqq i_0)\ (\text{see } (49), (50)).$$

(51), (53), where ε does not depend on m and $\lim\limits_{i\to\infty} m_i = \infty$, lead successively, making use of the definition of $\Pi_n^\beta(x)$, $E_n^\beta(f)$, to

$$(54)\qquad \overline{\lim_{m\to\infty}}\ I_{nm}^{\frac{1}{m}} \leqq E_n^\beta(f), \quad \underline{\lim_{m\to\infty}}\ I_{nm}^{\frac{1}{m}} \geqq Q,$$

$$(55)\quad Q = E_n^\beta(f) = \overline{\lim_{m\to\infty}}\ I_{nm}^{\frac{1}{m}} = \underline{\lim_{m\to\infty}}\ I_{nm}^{\frac{1}{m}};\quad \lim_{i\to\infty} P_{nm_i}(x) = Q_n(x) \equiv \Pi_n^\beta(x).$$

(55) *holds even if we reject* (52), for replacing $p(x)$ by $cp(x)$ leads to $c\,I_{nm}$ with $\lim\limits_{m\to\infty} c^{\frac{1}{m}} = 1$, while $P_{nm}(x)$ *remains unchanged.*

3° follows from 1° and 2°, $\Pi_n^\beta(x)$ here being *unique* and $\equiv \Pi_n(x)$, with $E_n^\beta(f) = E_n(f)$.

4° is proved by (50).

Theorem VII supplements and generalizes the results previously given by G. Pólya and the writer. The condition $\int\limits_E p(x)\,dx > 0$ $(mE > 0)$ is indispensable if we want theorem VII, 3° to be valid for *all* continuous functions.

With such functions we can go further, if we assume

$$(56)\qquad p(x) = \text{const. for } (a \leqq)\, c \leqq x \leqq d\, (\leqq b).$$

Applying (37) with $K = p_0$, we get, writing, in general, $E_n(f;\, a, b)$ and taking

$$(57)\qquad h = (1 + m^{-\theta})^{-m} \quad (0 < \theta < 1), \qquad h \to 0, \quad \text{as } m \to \infty:$$

$$(58)\qquad \begin{aligned} &E_n(f;\, c, d) \leqq |f(x) - P_{nm}(x)| \leqq E_n(f;\, a, b) + \eta(m) \quad (c \leqq x \leqq d) \\ &\eta(m) = O(m^{-\theta}) + \omega\left(e^{-m^{1-\theta}}\right), \quad \eta(m) \to 0, \quad \text{as } m \to \infty. \end{aligned}$$

Formula (58), in case $c = a$, $d = b$, is another proof of the result of G. Pólya: $\lim\limits_{m\to\infty} P_{nm}(x) = \Pi_n(x)$. It gives, moreover, some indication about the rapidity of this limiting process.

§ 9.

Case III. $n, m \to \infty$.

Writing m_n, I_n, $P_n(x)$, we state

Theorem VIII. 1°. $I_n^{\frac{1}{m_n}}$ *remains finite, as* $n \to \infty$, *for all* $f(x)$ *with finite measurable bounds in* (a, b). 2°. α) $\lim\limits_{n \to \infty} I_n = 0$ *for all* $f(x)$ *measurable and bounded in* (a, b). β) *For all such* $f(x)$ *the infinite series* $P_\nu(x) + [P_{\nu+1}(x) - P_\nu(x)] + \ldots$, *where* ν *is sufficiently large so that* $m_\nu > 1$, *multiplied by* $p(x)\, g(x)$ *and integrated term by term between the limits* α, t $(a \leqq \alpha,\ t \leqq b)$ *converges uniformly to* $\int\limits_\alpha^t p(x) f(x) g(x)\, dx$, *provided,* $\int\limits_a^b p(x)\,|g(x)|^{1+r} dx$ *exists with a certain* $r > 0$.

Proof. 1°. F denoting the measurable upper bound of $|f(x)|$ in (a, b), we write, by the definition of $P_n(x)$,

$$I_n < (F + \varepsilon)^{m_n} \int\limits_a^b p(x)\, dx,$$

and this proves our statement, since, as $n \to \infty$, $m_n \to \infty$, $\left[\int\limits_a^b p(x)\, dx\right]^{\frac{1}{m_n}} \to 1$.

2°. α). With $\Pi_n^\beta(x)$ and $E_n^\beta(f)$ introduced above, we write,

$$I_n^{\frac{1}{m_n}} \leqq \left[\int\limits_a^b p(x)\,|f(x) - \Pi_n^\beta(x)|^{m_n} dx\right]^{\frac{1}{m_n}} < (E_n^\beta(f) + \varepsilon)\left[\int\limits_a^b p(x)\, dx\right]^{\frac{1}{m_n}},$$

with $\lim\limits_{n \to \infty} E_n^\beta(f) = 0$.

β) $$\left|\int\limits_a^b p(x)\, g(x)\,(f(x) - P_n(x))\, dx\right|$$

$$\leqq \left[\int\limits_a^b p(x)\,|f(x) - P_n(x)|^{m_n} dx\right]^{\frac{1}{m_n}} \cdot \left[\int\limits_a^b p(x)\,|g(x)|^{\frac{m_n}{m_n - 1}} dx\right]^{\frac{m_n - 1}{m_n}},$$

and the right-hand member $\to 0$, as $n \to \infty$, for so does the first factor (by 2°, α)), while the second factor exists as a finite number for all n sufficiently large so that $\frac{m_n}{m_n - 1} < 1 + r$.

The most interesting case is that of $f(x)$ *continuous in* (a, b).

Theorem IX. *Given a sequence of exponents* m_n *such that* $\lim\limits_{n \to \infty} m_n = \infty$, $\frac{\log n}{m_n}$ *remains finite, and a c-function* $p(x)$ *defined over a finite interval* (a, b). *Then,* $f(x)$ *being an arbitrarily given continuous on* (a, b)

function, the polynomial $P_n(x)$, *of degree* $\leqq n$, *minimizing the integral* $\int_a^b p(x)\,|f(x)-P_n(x)|^{m_n}\,dx$ *converges, as* $n\to\infty$, *to* $f(x)$ *uniformly over any sub-interval* (c, d) $(a \leqq c < d \leqq b)$, *where* $\sum_{i=0}^{n} \varphi_i^2(x) = O(n^\sigma)$ $(\sigma > 0$ *independent on* $n, x)$. *Furthermore, in case* $c = a$, $d = b$, *the approximation of* $f(x)$ *by* $P_n(x)$ *on* (a, b) *is of the same order with repect to* n, *taken sufficiently large, as the best approximation* $E_n(f)$.

This follows at once from (29), since $\max|f(x)-P_n(x)|$ on (a, b) $\geqq E_n(f)$.

It suffices, therefore, to take $m_n = 2n$, $p(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}$ $(\alpha, \beta > 0)$, or, simply, $p(x) = 1$, and *we thus obtain a sequence of polynomials* $\{P_n(x)\}$, *following a simple formula, which converges to* $f(x)$, *subjected to the only condition of continuity, uniformly over the whole interval* (a, b), *the approximation being of the same order, with respect so* n, *as the best approximation.*

Corollary. *The approximation properties of* $P_n(x)$ *stated above hold for any sequence of exponents* $\{m_n\}$ *with* $\lim_{n\to\infty} m_n = \infty$, *provided,* $f(x)$ *satisfies in* (a, b) *Lipschitz's condition of an arbitrarily given order.*

In fact, such a condition of order, say, α implies:

$$(59)\qquad \begin{aligned} &E_n(f) = O(n^{-\alpha}),\\ &E_n(f)\, n^{\frac{2}{m_n}} = O\left(n^{\frac{2}{m_n}-\alpha}\right) \to 0, \quad \text{as } n\to\infty. \end{aligned}$$

The condition: $\frac{\log n}{m_n}$ remains finite for $n\to\infty$, is sufficient to assure the uniform convergence of $P_n(x)$ to $f(x)$ on (a, b). It would be interesting to find a condition relating to the mode of increase of the sequence $\{m_n\}$, which is *necessary* for such a convergence.

§ 10.

Application to Tchebycheff polynomials.

Consider our minimum problem $I_{nm} = \int_a^b p(x)|f(x)-P_{nm}(x)|^m dx$ $= \min \int_a^b p(x)|f(x)-G_n(x)|^m dx$ *for* $m = 2$. Then, as it is well known,

$$(60)\qquad P_{n,2}(x) = \sum_{i=0}^{n} A_i \varphi_i(x), \quad A_i = \int_a^b p(x) f(x) \varphi_i(x)\,dx,$$

and formula (44) gives,

$$(61)\quad \int_a^t p(x) f(x) g(x)\,dx = \sum_{n=0}^{\infty} A_n B_n, \quad B_n = \int_a^t p(x) g(x) \varphi_n(x)\,dx \ (a \leqq \alpha, t \leqq b),$$

the convergence being uniform, with respect to t, for any $g(x)$ of the class $[L_p^2]$. In particular,

$$(62)\qquad \int_a^b p(x) f^2(x)\,dx = \sum_{n=0}^{\infty} A_n^2, \qquad A_n = \int_a^b p(x) f(x) \varphi_n(x)\,dx.$$

Formulae (61), (62) represent the so-called "closure-property" for Tchebycheff polynomials. *Thus, formula* (44) *extends this property to the minimizing polynomials $P_{nm}(x)$ $(n = 1, 2, \ldots;\ m > 1)$.*

Theorem V, combined with (33), (59), (60), leads to

Theorem X. *The infinite series*

$$\sum_{n=0}^{\infty} \varphi_n(x) \int_a^b p(x) f(x) \varphi_n(x)\,dx$$

converges to $f(x)$ uniformly in any sub-interval (c, d) $(a \leqq c,\ d \leqq b)$, where $p(x) \geqq p_0 > 0$, provided, $f(x)$ has a continuous derivative in (a, b). A sufficient condition for this uniform convergence to hold inside (c, d) is: $f(x)$ satisfies Lipschitz's condition of order $> \frac{1}{2}$ in the interval (a, b).

Proof. The following remark is sufficient: for $f(x)$ having a continuous derivative in (a, b) $E_n(f) = o\left(\frac{1}{n}\right)$.

§ 11.

The results given above hold, mutatis mutandis, if we replace polynomials (of degree n) by limited trigonometric sums (of order n).

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(Eingegangen am 15. 11. 1928.)