Concerning points of continuous curves defined by certain im kleinen properties[1]).

Von

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§ 1.

Introduction.

This paper contains primarily the results of a study of the *im kleinen cut points*[2]) and *im kleinen cycle points* of a continuous curve M in euclidean space of n dimensions. If for each $\varepsilon > 0$, a domain R exists containing the point P of M and of diameter $< \varepsilon$ such that P is a cut point of the component of $M \cdot \overline{R}$ which contains P, then P is said to be *im kleinen cut point* of M; if P lies, for each ε, on some simple closed curve in M of diameter $< \varepsilon$, P is said to be an *im kleinen cycle point* of M. § 2 contains various characterizations of these types of points together with demonstrations of certain of their properties. It is there shown that the set of all im kleinen cut points of any continuous curve is a Borel set of the class F_σ (*i. e.* the sum of a countable number of closed sets); the problem of determining the Borel class, if any, of the set of all im kleinen cycle points is left open. In § 3 the possibility of the density of the non-im kleinen cut points L and the ramification points (points

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[1]) Presented to the American Mathematical Society, June 2, 1928.

[2]) The point P of a connected set M is said to be a cut point of M provided $M - P$ is not connected. The notion of an im kleinen cut point of a continuum is contained implicitly in the works of P. Urysohn and R. L. Moore, and is closely approximated in that of R. G. Lubben and C. Zarankiewicz. Cf. P. Urysohn, Über im kleinen zusammenhängende Kontinua, Math. Annalen **98** (1927), S. 296–308; [Urysohn uses the terms „unvermeidbar" (unavoidable) and „vermeidbar" (avoidable) to designate im kleinen cut points and non-im kleinen cut points, respectively]; R. L. Moore, Concerning Triods in the Plane and the Junction Points of Plane Continua, Proc. Ntl. Acad. of Sci. **14** (1928), pp. 85–88; R. G. Lubben, Concerning Connectedness near a Point Set; and C. Zarankiewicz, Sur les points de division dans les ensembles connexes, Fund. Math. **9** (1927), see proof of Theorem 14.

of Menger order >2) W of M on an arc t of M is investigated and it is found that if L is dense on t it must be uncountably dense on t and if M is cyclicly connected and W is dense on t, then L must be uncountably dense on t. In § 4 a study is made of a continuous curve M composed wholly, or almost wholly, of im kleinen cut points. Some more general theorems are established from which it follows that if M is composed wholly of im kleinen cut points, then M is a Menger regular curve which, if bounded, is for each $\varepsilon > 0$, the sum of a finite number of ε-continua no two having more than one point in common; and if M is cyclicly connected and W denotes the set of all its ramification points, then $\overline{W}$ is totally disconnected and each component of $M - \overline{W}$ is an ordinary arc-segment (*i. e.*, a simple continuous arc minus its endpoints). A special type of regular curve, called a *node curve*, is studied in § 5. These curves are defined as continua M which, for each $\varepsilon > 0$, are the sum of a finite number of ε-continua each having at most two points in common with the rest of M. The results of § 5 show that this class of curves includes all acyclic continuous curves [*i. e.*, continuous curves containing no simple closed curve, or baum curves (Menger)], and all baum im kleinen curves. In § 6 it is shown that an im kleinen cut point of a continuous curve M may be characterized as a point which is an isolated point of some irreducible cutting of M between some pair of points A and B of M, and that in a Menger regular curve the set of all non-im kleinen cut points of M is totally disconnected. In § 7 an example in 3-space is constructed of a continuous curve every subcontinuum of which is a continuous curve which has a number of interesting properties, among them being that it contains infinitely many mutually exclusive arcs all of diameter $>1/2$. In this section also is given theorems and discussion of the extension to n-space of some known theorems about Menger regular curves and continuous curves all of whose subcontinua are continuous curves in the plane.

The term continuous curve is used in this paper to designate any connected im kleinen continuum, bounded or not. The point sets considered are assumed to lie in a euclidian n-space, although it is obvious from the proofs that many of the theorems hold in more general spaces.

Definitions. A continuum will be called an ε-continuum, or in general, a set will be called an ε-set, provided that continuum or set is of diameter $< \varepsilon$, where ε denotes some positive number given in advance. A point P of continuum M is a regular point [3]) of M provided that for

[3]) *Cf.* K. Menger, Grundzüge einer Theorie der Kurven, Math. Annalen **95** (1925), S. 272–306.

each $\varepsilon > 0$, an ε-neighborhood R of P exists such that $F(R) \cdot M$ is finite, where as in this paper it will be used, $F(R)$ denotes the boundary of R. If an integer n exists such that for each ε, the neighborhood R can be chosen so that $F(R) \cdot M$ contains at most n points and if n is the smallest integer such that this property holds, then P is said to be a point of order[4]) n of M. Points of order 1 of continuum are called endpoints[5]), points of order > 2 are called ramification points[6]), and regular points which have no finite order are said to be of order w. A continuum all of whose points are regular will be called a Menger regular curve or simply a regular curve. A continuous curve M is said to cyclicly connected[7]) provided that every two points of M lie together on some simple closed curve in M. A cyclicly connected continuous curve C is called a maximal cyclic curve of a continuous curve M provided C is a subset of M but is not a proper subset of any cyclicly connected continuous curve in M. Considerable use is made in this paper of the decomposition of a continuous curve into its cyclic elements, *i. e.*, maximal cyclic curves, cut points, and endpoints, an extensive theory of which may be found in my paper *Concerning the Structure of a Continuous Curve*[8]).

The ordinary notation of Point Set Theory will be used in this paper. In general the letter M is used to denote a continuous curve or a regular curve, and, unless otherwise stated, the letters K, N, H and W denote the

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[4]) *Cf.* K. Menger, *loc. cit.*, and P. Urysohn, Comptes Rendus **175** (1922), p. 481. Urysohn uses the term 'index of a point' instead of the therm 'order of a point'.

[5]) That this definition is equivalent for the case of continuous curves to the Wilder definition [R. L. Wilder, Concerning Continuous Curves, Fund. Math. **7** (1925), pp. 340–377] was shown by H. M. Gehman. See Concerning End Point of Continuous Curves and other Continua, Trans. Amer. Math. Soc. **30** (1928). In my thesis I showed that this definition is (for plane continuous curves M) equivalent to the following simple one: P is an endpoint of M provided that P is an interior point of no arc in M. *Cf.* Concerning Continua in the Plane, Trans. Amer. Math. Soc. **29** (1927), pp. 369–400, Theorem 12. For the extension of this and other results frequently used later to n-space see W. L. Ayres, Concerning Continuous Curves in a Space of n Dimensions, Amer. Journal of Math.

[6]) *Cf.* W. Sierpinski, Comptes Rendus **160**, p. 305. Sierpinski defines a ramification point of M as a point P such that M contains 3 continua K, L and N, such that $K \cdot L = K \cdot N = L \cdot N = P$. It follows from a result of Menger's (Fund. Math. **10**) that the definition here given and Sierpinski's definition are equivalent for Menger regular curves. Rutt [Bull. Amer. Math. Soc. **33** (1927), p. 411 (abstract)] has shown them equivalent for all plane continuous curves. It appears likely that they are equivalent for continuous curves in n-space.

[7]) *Cf.* my paper Cyclicly Connected Continuous Curves, Proc. Ntl. Acad. of Sci. **13** (1927), pp. 31–38.

[8]) Amer. Journ. of Math. **50** (1928), pp. 167–194.

sets of all im kleinen cut points, im kleinen cycle points, end points, and ramification points of the curve M. The symbol $S(P, r)$ denotes the set of all points whose distance from the point P is less than the number r.

§ 2.

Im Kleinen Cut Points and Im Kleinen Cycle Points.

Theorem 1. *In order that the point P of a continuous curve M should be an im kleinen cut point of M it is necessary and sufficient that P should be a cut point of some connected open*[9]) *subset of M.*

Theorem 2. *In order that the point P of a bounded continuous curve M should be a non-im kleinen cut point (or an avoidable point in the terminology of Urysohn) it is necessary and sufficient that $M - P$ should be uniformly connected im kleinen*[10]).

Theorem 3. *Suppose R_0 and R are bounded connected open subsets of a continuous curve M and suppose $\overline{R} \subset R_0$. Then there exists a continuous curve U such that (1) $R \subset U \subset R_0$, (2) every cut point of R_0 which belongs to R is a cut point of U, and (3) every cut point of U is a cut point of R and hence is an im kleinen cut point of M.*

The proofs of Theorems 1 and 2 present no difficulties. Theorem 3 is readily established with the aid of Theorem 1 above and Theorem 1 in the paper *On continuous curves in n dimensions*[11]) by W. L. Ayres and the author.

Theorem 4. *If the im kleinen cut point P of a continuous curve M is a point of order two of M, then P is not an im kleinen cycle point of M.*

Proof. Suppose, on the contrary, that P is an im kleinen cycle point of M. There exists a neighborhood R of P such that P is a cut point of the component C of $M \cdot \overline{R}$ which contains P. Hence $C - P = C_1 + C_2$, where C_1 and C_2 are mutually separated. By supposition there exists in M a simple closed curve J which contains P and lies wholly in R. Then clearly J must belong to C, and $J - P$ must belong either to C_1 or to C_2. But this is impossible, for since P is a point of order 2 of M, P must be a point of order 1 of each of the continua $C_1 + P$ and $C_2 + P$.

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[9]) The subset R of a closed set M is said to be an open subset of M provided $M - R$ is either vacuous or closed.

[10]) A set M is uniformly connected im kleinen provided that for each $\varepsilon > 0$, a $\delta_\varepsilon > 0$ exists such that every two points x and y of M whose distance apart is $< \delta_\varepsilon$ lie in a connected subset of M of diameter $< \varepsilon$.

[11]) Bull. Amer. Math. Soc. **34** (1928), pp. 349–360.

Corollary. *Every im kleinen cycle point of M which is also an im kleinen cut point of M is a ramification point of M.*

Theorem 5. *If K, N and H respectively denote the set of all the im kleinen cut points, im kleinen cycle points, and end points of a continuous curve M, then $K+H+N=M$, $K \cdot N$ is countable, and $K \cdot H = N \cdot H = 0$.*

Proof. Let P be any point of M which belongs to neither K nor H. I shall show that P belongs to N. Let ε be any positive number. Since P does not belong to K, there exists a neighborhood R of P of diameter $< \varepsilon$ such that P is not a cut point of the component C of $M \cdot \overline{R}$ which contains P. Since P does not belong to H it follows[12]) that P is an interior point of some arc APB which belongs to C. And since P is a non-cut point of C it follows with the aid of a theorem of R. L. Moore's[13]) that $C-P$ contains an arc t from A to B. Clearly the sum of the arcs APB and t contains a simple closed curve J containing P; and since J must be of diameter $< \varepsilon$, it follows that P belongs to N. Therefore $K+H+N=M$.

By a theorem of the author's[14]) all save possibly a countable number of the points of K are points of order two of M. And by Theorem 4, no point of K which is a point of order two of M can belong to N. Hence $K \cdot N$ is countable. Obviously $K \cdot H = N \cdot H = 0$. This completes the proof of Theorem 5.

Corollary. *If M is cyclicly connected, then $H=0$ and $M=K+N$.*

Theorem 6. *Every point of a cyclicly connected continuous curve M which is not an im kleinen cycle point of M is a point of finite order of M.*

Proof. Let P be a non-im kleinen cycle point of M. There exists an $\varepsilon > 0$ such that P belongs to no simple closed curve in M of diameter $< \varepsilon$. Let R be an open set containing P and of diameter $< \varepsilon/4$, and let Q denote the component of $M \cdot \overline{R}$ which contains P. Since M is cyclicly connected, it follows that each component of $Q-P$ must contain at least one point of $F(R)$; and hence the components of $Q-P$ are finite in number. Let them be denoted by $E_1, E_2, \ldots, E_n$. I shall show that P is a point of order n of M. For each $i \leqq n$, P must be an

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[12]) See footnote [5]) and Theorem 3 above. Although C itself is not necessarily a continuous curve, it follows by Theorem 3 that C contains a continuous curve U such that $U \supset P$ and $(M-U)' \cdot P = 0$.

[13]) Concerning Continuous Curves in the Plane, Math. Zeitschr. **15** (1922), Theorem 1. Also see footnote [12]).

[14]) G. T. Whyburn, Concerning Collections of Cuttings of Connected Point Sets, Bull. Amer. Math. Soc. **35** (1929), pp. 87–104.

endpoint (and hence a point of order 1) of the continuum $E_i + P$. For by a theorem due to W. L. Ayres and the author[15]) there exists a continuous curve W_i of diameter $< \varepsilon$ containing E_i, lying in M, and such that P is not a limit point of $W_i - (E_i + P)$. Then since P is not a cut point of W_i and lies on no simple closed curve in W_i, then[16]) P is an endpoint of W_i. Hence[17]) P is a point of order 1 of W_i and since P is not a limit point of $W_i - (E_i + P)$, P is a point of order 1 of $E_i + P$. And since P is a point of order 1 of each of the continua

$$E_1 + P, E_2 + P, \ldots, E_n + P,$$

it follows that P is a point of order n of M.

Theorem 7. *In order that the point P of a continuous curve M should be a non-im kleinen cycle point of M it is necessary and sufficient that there should exist a number $\varepsilon > 0$ such that P is an endpoint of each component into which P ε-cuts M (i. e., if R is a domain of diameter $< \varepsilon$ containing P and N is the component of $\overline{R} \cdot M$ containing P, then P is an endpoint of each continuum obtained by adding it to each component of $(N - P)$). Furthermore, if P ε-cuts M into n such pieces $(n = 1, 2, 3, \ldots)$ for some one ε, then P is a point of order n of M.*

Theorem 8. *The set K of all the im kleinen cut points of any continuous curve M is an F_σ (i. e., the sum of a countable number of closed sets).*

Proof. Let K_1 denote the set of all those points of K which are condensation points of K. It follows by Theorem 3 that for each point P of K_1 and each integer $n > 0$, M contains a continuous curve U_{pn} of diameter $< 1/n$ containing P and such that P is a cut point of U_{pn} but is not a limit point of $M - U_{pn}$ and such that every cut point of U_{pn} belongs to K. By the Lindelöf Theorem, for each n, there exists a countable subset N_n of K_1 such that $K_1 \subset \sum_{P \subset N_n} I(U_{pn})$, where $I(U_{pn}) =$ set of all inner points of U_{pn}. By a theorem of Zarankiewicz[18]) for each n and each $P \subset N_n$, the set K_{pn} of cut points of the curve U_{pn} is an F_σ, and hence $K_{pn} = \sum_{j=1}^{\infty} F_{jpn}$, where F_{jpn} is closed for every j. And if Q denotes the point set $K - K_1 + \sum_{n=1}^{\infty} \sum_{P \subset N_n} \sum_{j=1}^{\infty} F_{jpn}$, then since $K - K_1$ is

[15]) *Loc. cit.* See footnote [11]).

[16]) G. T. Whyburn, Concerning Continua in the Plane, *loc. cit.*; W. L. Ayres, Concerning Continuous Curves and Correspondences, Ann. of Math. 28 (1927), p. 396. For this theorem in n dimensions see W. L. Ayres, Concerning Continuous Curves in a Space of n Dimensions, *loc. cit.* [5]).

[17]) See footnote [5]).

[18]) C. Zarankiewicz, *loc. cit.*, Theorem 17.

countable it follows that Q is an F_σ. It remains to show that Q is identical with K. Clearly $Q \subset K$, for $K_{pn} \subset K$ for each n and each $P \subset N_n$. Let P be any point of K_1. If P belongs to any set N_n then it must belong to Q, for Q contains every N_n. If P belongs to no N_n, then P must be a limit point of a sequence of points $[P_n]$, where for each n, $P_n \subset N_n$. There exists a number $d > 0$ such that P is a cut point of the component R of $\overline{M \cdot S(P, 4d)}$ which contains P. There exists an integer $n_1 > 0$ such that $1/n_1 < d$ and a point P_1 of N_{n_1} such that P belongs to $U_{p_1 n_1}$ and is an inner point of $U_{p_1 n_1}$ relative to M. Clearly $U_{p_1 n_1} \subset R$ and since P is a cut point of R and is an inner point of $U_{p_1 n_1}$ it follows[19]) readily that P is a cut point of $U_{p_1 n_1}$. Therefore P belongs to $K_{p_1 n_1}$, and hence belongs to Q. Hence Q is identical with K, and therefore K is an F_σ.

Corollary. *The set of all the non-im kleinen cut points of any continuous curve M is a G_δ (relative to M), i. e., the common part of a family of sets each open in M.*

The following additional facts concerning the Borel classification of certain types of points of a continuous curve either are already known or are easily deduced. Let K^*, H, and N^* denote the set of all cut points, endpoints and points belonging to some simple closed curve in a continuous curve M, then

(I) K^* is an F_σ (theorem of Zarankiewicz, *loc. cit.*).

(II) N^* is an F_σ (theorem of the author's, *cf.* my paper *Cyclicly Connected Continuous Curves, loc. cit.*, where it is shown that N is the sum of a countable number of continuous curves, *i. e.*, the maximal cyclic curves of M).

(III) H is a G_δ (theorem of Menger, *loc. cit.*).

(IV) $M - K^*$, and $M - N^*$ are G_δ's.

Examples are easily constructed to show that K^* is not necessarily a G_δ. It would be interesting to determine whether or not (a) H is necessarily an F_σ and (b) N^* is necessarily a G_δ.

§ 3.

Density of the Non-Im Kleinen Cut Points and Ramification Points.

Theorem 9. *Let L denote the set of all non-im kleinen cut (avoidable) points of a continuous curve M and let t be any arc in M. Then if L is dense on t it is uncountably everywhere dense on t.*

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[19]) In this connection see also, R. L. Moore, *loc. cit.* ref. [2]) Lemma 2.

Proof. By Theorem 8 and corollary, L is a G_δ relative to M. Hence if S is any arc segment in t, $S\cdot F$ is a G_δ relative to S. And since $S\cdot L$ is dense in S it follows by Young's Theorem[20]) that $S\cdot L$ has the power of the continuum. Hence L is uncountably dense on t.

Theorem 10. *Let T be any arc of a cyclicly connected continuous curve M and let L denote the set of all non-im kleinen cut points of M. Then if the set W of ramification points of M is dense on T, $L\cdot T$ has at least the power c of the continuum.*

Lemma 10a. *If T is any arc in a continuous curve M, then the set I of all points X of T such that for each $\varepsilon > 0$, M contains an ε-simple closed curve containing an arc segment in T which contains X, is a linear G_δ.*

Proof of Theorem 10. Suppose, on the contrary, that $L\cdot T$ has a cardinal number $< c$. Then by Theorem 9, L cannot be dense on T. Accordingly, there exists an interval t of T which contains no point of L. Then $t \subset K$. Let t_0 be any subarc of t of diameter $< 1/4$. Since by a theorem of the author's [see ref.[14])], $K\cdot W$ is countable, t_0 contains an interior point X_1 which is a point of order 2 of M. Since W is dense on t_0, X_1 is a limit point of $M - t_0$ but is not a limit point of any single component of $M - t_0$. Since M is a continuous curve, it is readily seen that there exists a component R_1 of $M - t_0$ of diameter $< 1/3$. As M is cyclicly connected, t_0 contains[21]) at least two limit points of R; and it is easily seen that $R_1 + t_0$ contains a simple closed curve J_1 of diameter < 1 which contains an interval t_1 of t_0 every point of which is interior to t_0 and which is of diameter $< 1/8$. Just as above it follows that M contains a simple closed curve J_2 of diameter $< 1/2$ which contains an interval t_2 of t_1 every point of which is interior to t_1 and which is of diameter $< 1/16$ and so on. Let this process be continued indefinitely. There exists a point X common to all of the intervals $t_0, t_1, t_2, \ldots$. Clearly X belongs to the set I (see Lemma 10a). And as t_0 is any interval of t, I must be dense on t. But by Lemma 10a, I is a G_δ. Hence by Young's Theorem above quoted, $I\cdot t$ is uncountable. Clearly $I\cdot t \subset N$ and since, by Theorem 5, $N\cdot K$ is countable, $I\cdot t$ must contain at least one point of L, contrary to the fact that $L\cdot t = 0$. Thus the supposition that Theorem 10 is false leads to a contradiction.

Corollary 1. *If the non-im kleinen cut points of a cyclicly connected continuous curve M are not uncountably dense on the arc t of M, then t contains an arc segment which is an open subset of M.*

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[20]) *Cf.* W. H. Young, Leipz. Ber. **55** (1903), S. 287.

[21]) G. T. Whyburn, Cyclicly Connected Continuous Curves, *loc. cit.* [7]).

Corollary 2. *If the im kleinen cycle points of a cyclicly connected continuous curve M are dense on an arc t in M, then both the avoidable points and the im kleinen cycle points of M are uncountably everywhere dense on t.*

§ 4.

Curves Composed Almost Wholly of Im Kleinen Cut Points.

Theorem 11. *If every point of a continuous curve M is either an end point or an im kleinen cut point, then M is a Menger regular curve and both the ramification points and the im kleinen cycle points of M are countable.*

Proof. That M is a regular curve was proved by the author in another paper[22]). That the ramification points of M are countable follows from the fact that no endpoint is a ramification point and the author's theorem[22]) that only a countable number of the im kleinen cut points of M are ramification points; and that the im kleinen cycle points are countable follows from Theorem 5 and the fact that no end point is an im kleinen cycle point.

Theorem 12. *In order that every subcontinuum of a continuous curve M should contain an arc segment which is an open subset of M it is necessary and sufficient that if W denotes the set of all ramification points of M then $\overline{W}$ is totally disconnected.*

Proof. The condition is obviously necessary. It is also sufficient For let Q be any subcontinuum of M and let R be a component of $Q - \overline{W} \cdot Q$. Then[23]) R contains an arc AB, and since every point of AB is a point of order two of M, it follows that no point of $AB - (A + B)$ is a limit point of $M - [AB - (A + B)]$. Hence the segment AB is an open subset of M.

Theorem 13. *If every point of a cyclicly connected continuous curve M is an im kleinen cut point of M, and W denotes the set of all ramification points of M, then* (1) *$\overline{W}$ is totally disconnected, and* (2) *every component of $M - \overline{W}$ is an arc segment.*

Proof. Suppose, contrary to (1), that $\overline{W}$ contains a continuum C. Then since by Theorem of the author's mentioned above, M is a Menger regular curve, it follows[24]) that C is a regular curve, and hence C con-

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[22]) *Cf.* G. T. Whyburn, Concerning Collections of Cuttings of Connected Point Sets, *loc. cit.*

[23]) R. L. Moore, *loc. cit.*, see footnote [13]).

[24]) K. Menger, *loc. cit.*, see footnote [3]).

tains an arc t. But then W is dense on t; and hence, by Theorem 10, t contains at least one point which is not an im kleinen cut point of M, contrary to hypothesis. Hence (1) is true.

Now let R be any component of $M-\overline{W}$. Since M is cyclicly connected, $\overline{W}$ contains[25] at least two limit points of R. Hence it is readily seen that there exists an arc AB such that A and B belong to $\overline{W}$ and which lies except for the points A and B wholly in R. And since every point of the segment $AB-(A+B)$ is a point of order two of M, clearly R must be identical with this segment.

Essentially the same argument suffices to prove the following more general theorem.

Theorem 14. *If the avoidable points of a cyclicly connected continuous curve M are not uncountably dense on any subcontinuum of M, and if W denotes the set of all ramification points of M, then* (1) *$\overline{W}$ is totally disconnected, and* (2) *every component of $M-\overline{W}$ is an arc segment.*

Theorem 15. *If every subcontinuum of a bounded cyclicly connected continuous curve M contains an arc segment which is an open subset of M, and W denotes the set of all ramification points of M, then* (1) *$\overline{W}$ is totally disconnected,* (2) *M is a Menger regular curve,* (3) *each component of $M-\overline{W}$ is an arc segment,* (4) *if L is any closed totally disconnected subset of M containing $\overline{W}$, and G denotes the collection of components of $M-L$, then for each $\varepsilon>0$ every point of L can be ε-separated*[26] *in M by a finite number of the segments of the collection G, and* (5) *for each $\varepsilon>0$, M is the sum of a finite number of mutually exlusive ε-continua plus a finite number of mutually exclusive ε-arc segments the two endpoints of each of which belong to different continua of the set just mentioned. Hence M is the sum of a finite number of ε-continua no two of which have more than one point in common*[27].

Proof. Conclusions (1), (2), and (3) are obvious from the above theorems and discussion. To prove (4), let ε be any positive number and P any point of L. Since L is closed and totally disconnected, it follows

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[25]) See footnote [21]).

[26]) For a given $\varepsilon>0$, the point P of M is said to be ε-separated in M by a set A provided $M-A=M_p+M_0$ where M_p and M_0 are mutually separated sets and M_p contains the point P and is of diameter $<\varepsilon$. *Cf.* P. Urysohn, Comptes Rendus **175** (1922), p. 481. Urysohn's definition differs from the one just given in that it requires that the set M_p+A be of diameter $<\varepsilon$.

[27]) K. Menger (Zur allgemeinen Kurventheorie, Fund. Math. **10**) has proposed the question as to whether or not every regular curve has the property mentioned in the last sentence of this theorem.

that there exists an open set R containing P and of diameter $< \varepsilon/2$ and such that $F(R) \cdot L = 0$. Then clearly $F(R) \cdot M \subset M - L = \sum_{g \varepsilon G} g$. Let G_p be the collection of all elements of G which contain at least one point of $F(R)$. Then G_p must be finite; for otherwise, since (see § 7 below) only a finite number of the segments of G are of diameter $>$ any positive number, it would follow that $F(R)$ contained a limit point of L, contrary to the fact that L is closed and $F(R) \cdot L = 0$. Hence G_p is finite and clearly G_p ε-separates P in M. And since G_p is finite, obviously it contains a subcollection which ε-separates P in M and is irreducible with respect to this property.

To prove (5), let ε be any number > 0. Since by (2) M is a regular curve, by a theorem of Menger's[28]) M is the sum of the elements of a finite collection Q of $\varepsilon/2$-continua each pair of which have at most a finite number of common points. Let E denote the set of all points which belong to at least two elements of the collection Q. The points of E are finite in number and can be ordered $P_1, P_2, P_3, \ldots, P_n$. Let L denote the set of points $\overline{W} + P_1 + P_2 + \ldots + P_n$. By (4), there exists a finite collection G_1 of the arc segments of G whose sum separates P_1 in M from $P_2 + P_3 + \ldots + P_n$ and which is irreducible with respect to this property. Each segment S of G_1 contains an arc segment U which lies, together with its end points, wholly in S; and if U_1 denotes the finite collection of segments U, it is easy to see that the sum of the segments of U_1 also separates P_1 in M from $P_2 + P_3 + \ldots + P_n$. Let V_1 denote the point set obtained by adding together all the point sets of the collection U_1. Now a similar argument shows that there exists a set V_2 which is the sum of the elements of a finite collection U_2 of arc-segments and which separates P_2 in M from $(P_1 + \overline{V}_1) + P_3 + P_4 + \ldots + P_n$; and indeed, for each i, $1 \leqq i \leqq n$, M contains a set V_i which is the sum of the elements of a finite collection U_i of arc segments selected as above and which separates P_i in M from $(P_1 + \overline{V}_1) + (P_2 + \overline{V}_2) + (P_{i-1} + \overline{V}_{i-1}) + P_{i+1} + P_{i+2} + \ldots + P_n$.

Now let U denote the collection of all the arc segments which belong to any collection U_i $(1 \leqq i \leqq n)$, and let V denote the point set $\sum_{i=1}^{n} V_i$. Let F denote the collection of all the components of $M - V$. Then each element of F is a continuum, and by a theorem proved by Kuratowski and Knaster[29]) and independently by the author[29]) it follows that F is

[28]) Grundzüge einer Theorie der Kurven, *loc. cit.*

[29]) *Cf.* Kuratowski and Knaster, Remark on a Theorem of R. L. Moore, Proc. Ntl. Acad. of Sci. **13** (1927); G. T. Whyburn, On the Separation of Connected Point Sets, Bull. Amer. Math. Soc. **33** (1927), p. 388 (abstract).

finite. And since no element of F can contain more than one point of the set E, each element of F must be of diameter $< \varepsilon$. And since each segment u in the collection U lies together with its end points in some segment of G and belongs to some collection U_i and hence is an element of an irreducible set of segments separating P_i in M from $(P_1 + \overline{V}_1) + \ldots + (P_{i-1} + \overline{V}_{i-1}) + P_{i+1} + \ldots + P_n$, it readily follows that u is of diameter $< \varepsilon$ and that not both end points of u belong to the same element of F. Hence (5) is true. And if for each u in U we let $X = \bar{u}$ and let D be the collection whose elements are the arcs X together with the elements of F, then clearly $M = \sum_D d$, and no two continua of D have more than one point in common. This completes the proof of Theorem 15.

Theorem 16. *If every point of the bounded cyclicly connected continuous curve M is an im kleinen cut point (or indeed if the non-im kleinen cut points of M are not dense on any subcontinuum of M), then M has properties* (1)—(5) *in Theorem* 15.

Theorem 17. *If every point of a bounded continuous curve M is an im kleinen cut point (or if the non-im kleinen cut points are not uncountably dense on any subcontinuum of M), then for each $\varepsilon > 0$, M is the sum of a finite number of ε-continua no two of which have more than one point in common.*

Proof. It follows by Theorem 16 that every maximal cyclic curve of M is for each ε, the sum of a finite number of ε-continua no two having more than one common point. Hence, by a theorem of the author's[30]) M itself has this same property.

Examples are easily constructed to show that (a) under the conditions of Theorem 15, neither W nor $M - K$ (the set of avoidable points) is necessarily countable, and (b) K (the im kleinen cut points) can be uncountably dense on every subcontinuum of a cyclicly connected continuous curve M and yet M not have property (1) in the statement of Theorem 15.

§ 5.

Node Curves.

Definition. A continuous curve M will be called a *node curve* provided that for each $\varepsilon > 0$, M is the sum of a finite number of ε-continua each having at most two points in common with the rest of M. It is obvious from this definition that every node curve is a bounded

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[30]) G. T. Whyburn, Concerning Menger Regular Curves, Fund. Math. 12, Theorem 2.

Menger regular curve. However, as will be apparent below, the converse is not true. Hence a node curve is a special kind of a regular curve.

Theorem 18. *In order that a continuous curve M should be a node curve it is necessary and sufficient that for each $\varepsilon > 0$, M should contain a finite set of points Q such that each component of $M - Q$ is of diameter $< \varepsilon$ and has at most two limit points in Q.*

Theorem 19. *In order that the cyclicly connected continuous curve M should be a node curve it is necessary and sufficient that for each $\varepsilon > 0$, M is the sum of a finite number of ε-continua each having exactly two points in common with the rest of M. It is likewise necessary and sufficient that M contain a finite set Q such that each component of $M - Q$ is of diameter $< \varepsilon$ and has exactly two limit points in Q.*

Theorem 20. *In order that the bounded cyclicly connected continuous curve M should be a node curve without points of order w it is necessary and sufficient that no point of M of order > 2 (i. e., no ramification point) be an im kleinen cycle point.*

Proof. The condition is sufficient. It follows by Theorem 6 that every point of M is a point of finite order of M and hence M contains no point of order w. Let ε be any positive number which for convenience later we will suppose is $< 1/3$ the diameter of M. For each point X of M which is a point of order two of M there exist two points A_x and B_x of M and a connected open subset U_x of M containing P and of diameter $< \varepsilon$ and such that $\overline{U}_x \cdot (M - U_x) = A_x + B_x$. For each point Y of M which is not a point of order two of M, since Y is not an im kleinen cycle point of M, it follows as in the proof of Theorem 6 that a domain R exists containing Y, of diameter $< \varepsilon$, and such that Y cuts the component of $M \cdot \overline{R}$ which contains Y into n components (where n is the order of P) and is an end point (point of order one) of each of them. Thus it is readily seen that a connected open subset V_y of M exists which contains Y, is of diameter $< \varepsilon$, whose M-boundary contains just n points, and which is the sum of $Y + n$ open sets $U_{yx_1}, U_{yx_2}, \ldots, U_{yx_n}$ each having just Y and some other point A_{yx} as boundary points with respect to M. Let G_0 denote the collection of sets whose elements are the sets U_x and the sets V_x. Since G_0 covers M, then by the Borel Theorem G_0 contains a finite subcollection G which also covers M. By the above properties of the sets V_x, it follows readily that there exists a finite collection $U_1, U_2, \ldots, U_m$ of open subsets of M each of diameter $< \varepsilon$ and such that (1) for each i, $1 \leq i \leq m$, the M-boundary of U_i (*i. e.*, the boundary of U_i with respect to M) consists of just two points A_i and B_i, (2) $U_i \neq U_i \cdot U_j \neq U_j$ for each i and $j \leq m$, and (3) $M \subset (\overline{U}_1 + \overline{U}_2$

$+\overline{U}_3+\ldots+\overline{U}_m)$. Let N denote the (finite) set of points $\sum_{i=1}^{m}(A_i+B_i)$. Then clearly each component of $M-N$ must be of diameter $<\varepsilon$, for it is a subset of some set $\overline{U}_i$. It remains to show that each component of $M-N$ has just two limit points in N. Suppose on the contrary that some component R of $M-N$ has as many as three limit points P_1, P_2, P_3 which belong to N.

Now clearly there exists an integer i, $1 \leqq i \leqq m$, such that $R \subset U_i$. Now since M is cyclicly connected it follows[31]) that every cut point of $\overline{U}_i$ must separate A_i and B_i in $\overline{U}_i$. And since $\overline{R}-R$ contains at least three points it readily follows that $\overline{R}-R$ contains at least one point P distinct from A_i and B_i which does not separate A_i and B_i in $\overline{U}_i$. Hence P is not a cut point of $\overline{U}_i$. Now since P belongs to N, there exists an integer j such that $A_j = P$. Now A_j belongs to U_i; and B_j cannot belong to U_i. For suppose $B_j \subset U_i$. Now $\overline{U}_i+\overline{U}_j \neq M$, for the diameter of M is $\geqq 3\varepsilon$. Let C be a component of $M-(\overline{U}_i+\overline{U}_j)$. Since M is cyclicly connected, $\overline{U}_i+\overline{U}_j$ must contain at least two limit points X_1 and X_2 of C. Obviously $X_1+X_2 \subset A_i+B_i+A_j+B_j$, and since $A_j+B_j \subset U_i$, then both A_i and B_i must be limit points of C. But since U_j contains a point of U_i but is not a subset of U_i, by (2), then U_j must contain at least one of the points A_i and B_i; and clearly this is impossible, since each of these points is a limit point of C. Therefore B_j does not belong to U_i. But now A_j is not a cut point of U_i. Hence $U_i - A_j$ is connected and contains at least one point of U_j but contains neither A_j nor B_j; and therefore U_i is a subset of U_j, contrary to (2). Thus the supposition that some component of $M-N$ has more than two limit points in N leads to a contradiction; and hence, by Theorem 18, M is a node curve.

The condition is also necessary. For let M be any node curve containing no point of order w, and let P be any point of M of order >2 of M. Let n be the order of P. Since M is a regular curve, by a theorem of Menger's[32]) there exists a set of n arcs $A_1P, A_2P, \ldots, A_nP$ belonging to M and each having P as one end point but no two having in common any point except P. Let d be a number less than each of the numbers $\delta(A_1, P), \delta(A_2, P), \ldots, \delta(A_n, P)$, where $\delta(A_i, P) =$ distance from A_i to P, and let R be a domain containing P and of diameter $< d/4$. Since M is a node curve, it is the sum of a finite collection G of continua each

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[31]) With the aid of the following easily established lemma: *If the connected open subset R of a cyclicly connected continuous curve M has just two boundary points A and B with respect to M, then every cut point of the curve $\overline{R}$ separates A and B in $\overline{R}$ and if Q denotes the set of all such points, then $Q+A+B$ is closed.*

[32]) K. Menger, *loc. cit.*, see ref. [27]).

of diameter $< d/4$ each having at most two points in common with the rest of M. Since $n > 2$ it is readily seen that P is not an interior point of any one of the continua of G relative to M, and indeed, that there exist exactly n of the continua of G each of which contains P and some segment of one of the arcs $A_i P$ having P as one of its end points. Let $E_1, E_2, \ldots, E_n$ denote these continua. Then since P is a point of order n of M, it follows that P is a point of order 1 of each of the continua $E_1, E_2, \ldots, E_n$. Therefore, since P is not a limit point of $M - (E_1 + E_2 + \ldots + E_n)$, it follows that P is not an im kleinen cycle point of M. This completes the proof.

In proving the necessity of the condition in Theorem 20 no use was made of the fact that the curve M is cyclicly connected. Hence we have the following theorem.

Theorem 21. *No node curve M contains a point of finite order > 2 which is an im kleinen cycle point.*

The above proof for the sufficiency of the condition in Theorem 20 suffices to establish the following theorem.

Theorem 22. *Every bounded node curve im kleinen is a node curve.*

Theorem 23. *Every subcontinuum of a node curve is itself a node curve.*

Theorem 24. *If P is a point of order n (n finite and > 2) of a node curve M, then there exists a positive number ε_p such that if $\varepsilon < \varepsilon_p$, and M is decomposed into a finite collection G of ε-continua each having at most two points in common with the rest of M, then P is common to exactly n of the continua of G, P is an end point of each of these n continua and is not a limit point of M minus their sum.*

Proof. By Menger's theorem there exist n subarcs $A_1P, A_2P, \ldots, A_nP$ of M from A_1 to P, A_2 to $P, \ldots, A_n$ to P, respectively, each two having just the point P in common. There exists a hypersphere S with center P which neither contains nor encloses any of the points $A_1, A_2, \ldots, A_n$. Let ε_p denote $1/2$ the radius of S. Let ε be any positive number $< \varepsilon_p$ and let M be decomposed as above into a set G of ε-continua. Let N denote the (finite) subset of M each point of which is common to at least two of the continua of G. On each of the arcs A_iP $(i = 1, 2, \ldots, n)$, in the order from P to A_i, let X_i denote the first point after P, which belongs to N. Now since $n > 2$, it is clear that P must belong to N. Each of the arc segments PX_i must lie wholly within one of the continua of G, and since each of these arc segments has at least one limit point X_i other than P which belongs to N, it is clear that no two of these segments can lie in the same continuum of G. Hence P is common to at least n continua of G. And since P is of order n, clearly P belongs to no more than n such continua, is not a limit point of M minus the sum of these n continua, and is an end point of each.

A similar proof shows the following theorem.

Theorem 25. *If P is a point of order w of the node curve M, then for each integer $n > 0$ a number $\varepsilon_{np} > 0$ exists such that if $\varepsilon < \varepsilon_{np}$ and M is decomposed into a collection G of ε-continua as described above, then P is common to at least n of the continua of G.*

The two preceding theorems give the following theorem:

Theorem 26. *Every point of a node curve M of order > 2 of M is an im kleinen cut point of M.*

Theorem 27. *The ramification points of any node curve are countable.*

Theorem 27 follows immediately from Theorem 26 and the author's theorem that the im kleinen cut points of any continuous curve M of order > 2 are countable. It also follows from Theorems 24 and 25. For if for each integer $n > 0$, we decompose M into a finite collection G_n of $1/n$-continua as above, and N_n is the set of all points common to two of the continua of G_n, then by Theorems 24 and 25, the set W of points of M of order > 2 is a subset of $\sum_{n=1}^{\infty} N_n$. And since for each n, N_n is finite, $\sum_{n=1}^{\infty} N_n$ is countable, and hence W is countable.

Theorem 28. *The im kleinen cut points of every node curve M are everywhere dense in M. Indeed, every point of $\sum_{n=1}^{\infty} N_n$ above is an im kleinen cut point.*

Every maximal cyclic curve of a bounded continuous curve M can be a node curve and yet M itself not be a node curve, as seen by the following example. Let I denote the interval $(1, 2)$ of the X-axis, let C be the circle $X^2 + Y^2 = 1$, let R be the interior of C, and for each n $(n = 1, 2, 3, \ldots)$ let C_n denote the circle $(X-1)^2 + Y^2 = 1/n^2$. Then if M denotes the continuous curve $I + C + R \cdot \sum_{n=1}^{\infty} C_n$, the set of points $C + R \cdot \sum_{n=1}^{\infty} C_n$ is the only maximal cyclic curve of M; and although, as is easily seen with the aid of Theorem 20, this set of points is a node curve, nevertheless (*cf.* Theorem 21) M itself is not a node curve. However, we may characterize a node curve in terms of its maximal cyclic curves as follows.

Theorem 29. *In order that a bounded continuous curve M should be a node curve it is necessary and sufficient that for each maximal cyclic curve C of M (1) C is a node curve, and (2) every point of C which is a limit point of some component of $M - C$ is an im kleinen cut point of C.*

Proof. The conditions are necessary. Condition (1) is necessary because by Theorem 23, every subcontinuum of M is a node curve. To show that condition (2) is necessary, let C be any maximal cyclic curve of M and P any point of C which is a limit point of some component Q of $M - C$. There exists an arc T which has P as one end point and is a subset of $Q + P$. Suppose, contrary to what we purpose to show, that P is not an im kleinen cut point of C. Then since by condition (1), C is a node curve, it follows by Theorem 26 that P is a point of order two of C. Then P is a point of order 3 of the curve $C + T$, and since, by Theorem 23, $C + T$ is a node curve, then by Theorem 21, P is not an im kleinen cycle point of $C + T$. Hence P is not an im kleinen cycle point of C; but then by Theorem 4, corollary, P must be an im kleinen cut point of C, contrary to supposition. Thus the supposition that condition (2) is not necessary leads to a contradiction.

The conditions are also sufficient. For let M be any bounded continuous curve satisfying the conditions, let ε be any positive number, and let G denote the (finite) collection of all those maximal cyclic curves of M which are of diameter $> \varepsilon/4$. In my paper *Concerning Menger Regular Curves* [33]) it was shown that M contains a continuous curve Q containing all the curves of G and such that (1) Q is the sum of the curves of G plus a finite number $t_1, t_2, t_3, \ldots, t_m$ of simple cyclic chains [34]) of cyclic elements of M such that for each i, $(1 \leqq i \leqq m)$, t_i has at most one point in common with t_{i-1}, and (2) each component of $M - Q$ is of diameter $< \varepsilon/4$ and has just one limit point in Q. It is readily seen that there exists a finite number of cyclic elements $C_1, C_2, C_3, \ldots, C_n$ of M including all the curves of the collection G, and such that if U_i is any component of $Q - \sum C_i$, then $\overline{U}_i$ is a simple cyclic chain determined by an arc $A_i B_i$ in M and having at most the points A_i and B_i in common with $\sum C_i$. Clearly the components of $Q - \sum C_i$ are finite in number. Denote them by $U_1, U_2, U_3, \ldots, U_k$. Now for each i, $(1 \leqq i \leqq n)$, let E_i denote the (finite) set of points in C_i each of which is a limit point of

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33) Fund. Math. 12, see proof of Theorem 2. For a statement of practically the same theorem see a forthcoming paper of W. L. Ayres entitled 'Concerning Arc Curves and Basic Subsets of a Continuous Curve. Second Paper'.

34) *Cf.* my paper Concerning the Structure of a Continuous Curve, *loc. cit.* 8).

some component of $Q - C_i$, and let E denote the (finite) set of points $\sum E_i$. Now since E is finite and since, by hypothesis, each point of E is an im kleinen cut point of each curve of the set $C_1, C_2, C_3, \ldots, C_n$ which contains it, it follows that there exists a number $\delta > 0$ such that if P is any point of E, C_i is any curve of the collection $C_1, C_2, \ldots, C_n$, and R is any domain containing P and of diameter $< \delta$, then P is a cut point of the component of $C_i \cdot \overline{R}$ which contains P. Let d be a positive number $< \varepsilon/4$ and $< \delta/2$. Now since for each i, $(1 \leqq i \leqq n)$, C_i is a node curve or a point, it follows by Theorem 19 that for each i, a finite subset Q_i of C_i exists such that each component of $C_i - Q_i$ is of diameter $< d$ and has just two limit points in C_i. Now for each i, $(1 \leqq i \leqq k)$, as shown in my paper *Concerning Menger Regular Curves* (*loc. cit.*) there exists on the arc $A_i B_i$ a finite set of points V_i each of which separates A_i and B_i in M and such that each component of $\overline{U}_i - V_i$ is of diameter $< d$ and has at most two limit points in V_i. Let Z denote the (finite) set of points of the curve Q such that each point P of Q belongs to at least two of the chains $\overline{U}_1, \overline{U}_2, \ldots, \overline{U}_k$. Finally, let F denote the set of points $Z + E + \sum_{i=1}^{n} Q_i + \sum_{i=1}^{k} V_i$. Then F is finite, and it is not difficult to show that no component of $M - F$ has more than two limit points in F. Therefore, by Theorem 18, M is a node curve.

Corollary 1. *Every bounded acyclic continuous curve is a node curve.*

Corollary 2. *Every bounded baum im kleinen curve is a node curve.*

Corollary 3. *If every maximal cyclic curve of a bounded continuous curve M is a baum im kleinen (or, what is equivalent, contains only a finite number of simple closed curves) then M is a node curve.*

Theorem 30. *If no point of a bounded continuous curve M is an im kleinen cycle point, then M is a node curve.*

Theorem 30 is a corollary to Theorems 29 and 20.

It is easily shown by examples: (1) that the ramification points in a node curve M may be dense on some arc in M, and (2) that the ramification points may be dense on no subcontinuum of a continuous curve M and yet M not be a node curve.

Let M be a node curve, ε any positive number, and K a finite subset of M such that (see Theorem 18) each component of $M - K$ is of diameter $< \varepsilon$ and has at most two limit points in K. Let G denote the collection of sets obtained by adding to each component of $M - K$ its limit points in K. We shall call the elements of G the *links* of the curve M. Those links of M which join on to the rest of M at two points will be called *ordinary links* of M, and those joining on at only one point will be called the *end-links* of M. By a *simple chain* X of links of M joining two links A and B of M is meant the sum of the elements of a finite collection E of links such that (1) X is a continuum (2) X contains both A and B (3) A and B are the end-links of the node curve X and are its only end-links, (4) X is irreducible with respect to the property of being the sum of the elements of a finite collection of links of M and having properties (1) — (3). The following propositions are easily deduced:

a) Every two links of a connected set H of links of M can be joined by a simple chain of links lying in H;

b) The ordinary links of M are finite in number and their sum is a continuum;

c) If X is any simple chain of links of M between two links A and B, X_A and X_B are points of $M-K$ belonging to A and B respectively, then every point of $[X-(A+B)]\cdot K$ separates X_A and X_B in X. And every link of X except A and B separates X_A and X_B in X;

d) If M is cyclicly connected, all its links are ordinary links.

§ 6.

Im Kleinen Cut Points and Irreducible Cuttings.

If A and B are points of a continuum M and K is a subset of M such that $M-K$ is the sum of two mutually separated sets M_a and M_b containing A and B respectively, then K is said to be a cutting of M between A and B; if no proper subset of K is a cutting of M between A and B, K is called an *irreducible cutting*[35]) *of M between A and B.*

Theorem 31. *In order that the point P of a continuous curve M should be an im kleinen cut point of M it is necessary and sufficient that P be an isolated point of some irreducible cutting of M between some two points A and B of M.*

Proof. The condition is sufficient. For suppose P is an isolated point of an irreducible cutting K of M between the points A and B of M. Let R be a domain containing P and such that $\overline{R}$ contains no point of $A+B+K-P$, and let N be the component of $M\cdot\overline{R}$ containing P. Then since[36]) P is a limit point of both the components R_a and R_b of

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[35]) *Cf.* G. T. Whyburn, Concerning Irreducible Cuttings of Continua, Fund. Math. **13**, pp. 42—57.

[36]) G. T. Whyburn, *loc. cit.*, Theorem 7.

$M - K$ containing A and B respectively, then N contains points of both R_a and R_b; and since $\bar{R} \cdot K = P$, it is readily seen that $N - P = N \cdot R_a + N \cdot R_b$, and $N \cdot R_a$ and $N \cdot R_b$ are mutually separated. Hence P is a cut point of N and is then an im kleinen cut point of M.

The condition is also necessary. For if P is an im kleinen cut point of M, a domain R exists such that P is a cut point of the component N of $M \cdot \bar{R}$ which contains P. Let A and B be points of N lying in R and belonging to different components of $N - P$ and such that there exists an arc AB in $M \cdot R$. Now clearly $P + F(R) \cdot M$ cuts M between A and B. Then[37]) $P + F(R) \cdot M$ contains an irreducible cutting K of M between A and B. And since there exists an arc AB in $M \cdot R$, and $K \cdot R \subset P$, it is clear that P must belong to K and must be an isolated point of K.

Theorem 32. *If K denotes the set of all the im kleinen cut points of any Menger regular curve M, then $M - K$ is totally disconnected.*

Proof. Suppose, on the contrary, that $M - K$ contains a connected set H containing two distinct points A and B. Since M is a regular curve, there exists a finite cutting Q of M, between A and B. By a theorem of the author's[38]) Q contains an irreducible cutting Q_0 of M between A and B. Since H is a connected subset of M containing both A and B, obviously it must contain at least one point P of Q_0. And since Q_0 is finite, P is an isolated point of Q_0. But then by Theorem 31, P must belong to K, contrary to the fact that $P \subset H$ and $H \cdot K = 0$.

Corollary. *Under the hypothesis of Theorem 32, K is dense on every connected subset of M.*

Theorem 32 does not remain true if the hypothesis that "M is a Menger regular curve" is replaced by the weaker one that "every subcontinuum of M is a continuous curve". This fact is demonstrated in an example due to H. M. Gehman[39]).

§ 7.

Continua All of Whose Subcontinua Are Continuous Curves and Menger Regular Curves, in n Dimensions.

In this section I shall first give a simple example in 3-space of a continuous curve every subcontinuum of which is a continuous curve and which has some rather interesting properties. Referring to a system of

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[37]) G. T. Whyburn, *loc. cit.*, Theorem 8.

[38]) *Loc. cit.*, Theorem 8.

[39]) Concerning the Subsets of a Plane Continuous Curve, Annals of Math. 27 (1925), pp. 29–46.

cylindrical coordinate axes P, Θ, Z in 3-space, let AB be the interval $(0, 1)$ of the Z-axis, let n take on in ascending order the set of values included in the set of all positive prime integers. For each n, let us subdivide AB into a set I_n of n equal subintervals by inserting a set K_n of $n-1$ points of subdivision; and in the plane $\Theta = \pi/n$, let us construct on each interval of the set I_n a semicircle having this interval as its diameter, and let C_n be the sum of all these semicircles. Let ψ denote the continuum

$$AB + \sum C_n.$$

Properties of the curve ψ:

(α) *Every subcontinuum of ψ is a continuous curve.*

This property is obvious from the construction; or it can easily be proved with the aid of a theorem of H. M. Gehman's[40]).

(β) *ψ is not a Menger regular curve.*

This property readily follows from property (γ) below.

(γ) *ψ contains infinitely many arcs AX_iB $(i = 1, 2, \ldots)$ from A to B no two having any common point except A and B.* (γ') *There exists an $\varepsilon > 0$ such that ψ contains infinitely many mutually exclusive continua of diameter $> \varepsilon$.*

To show property (γ) it is only necessary to set $AX_iB = C_i$ for each i.

(δ) *ψ contains an arcwise connected set N which is not arcwise connected im kleinen and which has a boundary point P which is not accessible from it.*

To prove (δ), we merely set $N = \sum C_n$ and let P be the point $(0, 0, 1/2\sqrt{2})$. Clearly N is not arcwise connected im kleinen at any one of its points belonging to $AB - (A + B)$, and obviously P is not accessible from N.

ε) *ψ contains a connected subset which is not arcwise connected.*

The set $N + P$ under (δ) is not arcwise connected, and hence ψ has property (ε).

(ζ) *ψ contains an arcwise connected and connected im kleinen set which is not arcwise connected im kleinen.*

The set N defined under (δ) satisfies all requirements on the set in (ζ). That this set, contrary to a statement made in the first abstract of this paper, is connected im kleinen was kindly pointed out to me by Professor R. L. Wilder. It would be interesting to determine whether a set N could

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[40]) Some Conditions Under Which a Continuum is a Continuous Curve, Ann. of Math. 27 (1926), pp. 381–384, see Theorem 2.

lie in the plane and have property (ζ), or slightly different, whether a set N could lie in the plane and be strongly connected and connected im kleinen and yet not be strongly connected im kleinen.

Gehman[41]) has given an example of a plane continuum having both properties (α) and (β) and has shown that (α) and the absence of (γ') are equivalent for bounded continua in the plane. Zarankiewicz[42]) characterizes a continuum (in n-space) having property (α) as one containing no "continuum of convergence", and in footnote states that his condition is equivalent to the absence of (γ'), (*i. e.* to Gehman's condition). The curve ψ above, of course, shows although this is true in the plane that this is not the case in n-space for $n > 2$. It has been shown by the author[43]) that no plane continuum can have both properties (α) and (δ). Knaster and Kuratowski[44]) have given an example of a plane regular curve having property (ε). Their example is somewhat more complicated.

Theorem 33. *If M is a bounded Menger regular curve and ε is any positive number, then M does not contain more than a finite number of mutually exclusive continua each of diameter $> \varepsilon$.*

Proof. Suppose, on the contrary, that M contains an infinite sequence of continua $M_1, M_2, M_3, \dots$ all of diameter $> \varepsilon$. Then since M is bounded, there exist two points A and B of M belonging to the limiting set of the sequence $M_1, M_2, \dots$. But since M is a regular curve, there exists a finite subset K of M which separates A and B in M. Clearly this is impossible, since only a finite number of the continua $M_1, M_2, \dots$ can contain points of K. Thus the supposition that Theorem 33 is false leads to a contradiction.

Theorem 34. *If M is any continuum having the property that for each $\varepsilon > 0$, M does not contain infinitely many mutually exclusive continua each of diameter $> \varepsilon$, then every subcontinuum of M is a continuous curve, and if H is any arcwise connected subset of M, then* (1) *H is arcwise connected im kleinen and* (2) *every boundary point P of H is regularly accessible*[45]) *from H.*

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[41]) See reference [39]).

[42]) Sur les Points de Division dans les Ensembles Connexes, *loc. cit.*; K is a continuum of convergence of a continuum M provided that $M - K$ contains a sequence of continua whose sequential limiting set is K.

[43]) Concerning Certain Types of Continuous Curves, Proc. Ntl. Acad. of Sci. **12** (1926), pp. 761–767.

[44]) Knaster and Kuratowski, Bull. Amer. Math. Soc. **33** (1927), p. 106.

[45]) That is, for each $\varepsilon > 0$, a $\delta_\varepsilon > 0$ exists such that every point of H whose distance from P is $< \delta_\varepsilon$ can be joined to P by an arc in $H + P$ of diameter $< \varepsilon$. See my paper Concerning the Open Subsets of a Plane Continuous Curve, Proc. Ntl. Acad. of Sci. **13** (1927), pp. 650–656.

That every subcontinuum of M is a continuous curve follows by the proof given by Gehman (*loc. cit.*) for the case of the plane, which extends to n dimensions without difficulty. That M has property (1) follows by the proof given in my paper "Concerning Certain Types of Continuous Curves"[46]) for the theorem that every arcwise connected subset of a plane continuum every subcontinuum of which is a continuous curve is arcwise connected im kleinen, in the proof of which the only plane property used is that for each $\varepsilon > 0$ the continuum contains not more than a finite number of mutually exclusive continua each of diameter $< \varepsilon$. That M has property (2) follows in a similar way by a proof given in my paper "Concerning the Complementary Domains of Continua"[47]) for the theorem that every boundary point of an arcwise connected subset of any plane continuous curve every subcontinuum of which is a continuous curve is regularly accessible from that set.

Theorem 35. *If H is any arcwise connected subset of a Menger regular curve M, then* (1) *H is arcwise connected im kleinen,* (2) *every boundary point of H is regularly accessible from H.*

Theorem 35 is an immediate consequence of Theorems 33 and 34.

Theorem 36. *If K is any closed subset of a bounded continuum M every subcontinuum of which is a continuous curve, then for each $\varepsilon > 0$, $M - K$ contains at most a finite number of components each of diameter $> \varepsilon$.*

Theorem 36 follows at once with the aid of Zarankiewicz's Theorem (*loc. cit.*) that no continuum every subcontinuum of which is a continuous curve can contain a continuum of convergence.

Theorem 37. *If R is any connected open subset of a bounded continuous curve M every subcontinuum of which is a continuous curve, then R has property S*[48]) *and every boundary point of R is regularly accessible from R.*

Proof. Suppose theorem 37 is not true. Then either R does not have property S or R has a boundary point which is not regularly accessible from R. In either case[49]) it follows that there exists an $\varepsilon > 0$

[46]) *Loc. cit.*, see reference [43]).

[47]) Ann. of Math. **29** (1928), pp. 399–411.

[48]) That is, for each $\varepsilon > 0$, R is the sum of a finite number of connected sets each of diameter $< \varepsilon$. See R. L. Moore, Fund. Math. **3** (1922), p. 232.

[49]) For the former case, see my paper Concerning the Open Subsets of a Continuous Curve, *loc. cit.*, proof of Theorem 1, p. 651; and for the latter case see my paper Concerning Menger Regular Curves, Fund. Math. **12** (1928), Fundamental Accessibility Theorem.

and R contains an infinite sequence of points $P_1, P_2, \ldots$ having a sequential limit point P belonging to $M - R$ and such that no two of these points can be joined by any arc in R of diameter $< \varepsilon$. Let C be a circle with the center P and diameter $\varepsilon/2$. For each i, R contains an arc $P_i P_{i+1}$. Since each such arc must contain points without C, it readily follows that R contains a sequence of mutually exclusive arcs $T_1, T_2, T_3, \ldots$ each of diameter $> \varepsilon/8$ such that there exists a continuum T belonging to $M - R$ and which is the sequential limiting set of the sequence of continua $T_1, T_2, \ldots$. But then T is a continuum of convergence of M, contrary to a theorem of Zarankiewicz (*loc. cit.*). This contradiction proves Theorem 37.

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