On the Structure of Sets of Points of Classes One, Two, and Three.

Von

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I. Introduction.

The Classification of Sets of Points.

The classification, defined by Lebesgue[1]), of sets of points is based upon the Baire classification of functions. Any set is closed or F of class α if it can be considered as the set $E\ (a \leqq f \leqq b)$ relative to a function f of class α, at most. A set is open or O of class α if it can be considered as the set $E\ (a < f < b)$ relative to a function f of class α, at most.

The following are some of the properties of the Lebesgue classification:

1. If a set is F of class α its complement is O of class α, and conversely;

2. A finite sum or product of sets F of class α, at most, gives a set F of class α, at most, and a finite sum or product of sets O of class α, at most, gives a set O of class α, at most;

3. The product of an enumerable infinity of sets F of class α, at most, is F of class α, at most, and the sum of an enumerable infinity of sets O of class α, at most, is O of class α, at most;

4. A set F of class α is O of class $\alpha + 1$, at most, and a set O of class α is F of class $\alpha + 1$, at most;

5. The sum of an enumerable infinity of sets F of class α, at most, is F of class $\alpha + 2$, at most, and the product of an enumerable infinity of sets O of class α, at most, is O of class $\alpha + 2$, at most.

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1) Journal de Mathématiques 1905, pp. 156, 157.

For the sets which are both F and O of class α de la Vallée Poussin[2]) has introduced the term 'ambiguous' or in notation, A of class α. By the introduction of new and powerful methods de la Vallée Poussin was able to simplify and extend the theories of Baire and Lebesgue.

Two things are evident relative to the Lebesgue classification, a set belongs to two classes, open and closed, and its class depends directly on the class of a function. A classification which is not double and which depends only on the set itself is desirable. The sigma-delta-systems of Lebesgue[3]) and Hausdorff[4]) attain this latter aim. These classifications are based on the representation of sets in terms of open sets and closed sets.

In the Lebesgue classification the sets of class zero are open or O_0 and closed or F_0. The sets open of class one or O_1 are sums of sets F_0 and the sets closed of class one or F_1 are products of sets O_0. The sets open of class two or O_2 are sums of sets F_1 and the sets closed of class two or F_2 are products of sets O_1. This can be continued for all finite classes. For the first transfinite class ω the sets A'_ω and A''_ω are defined as the sum and product, respectively, of sets A of all finite classes. Then the sets open of class ω or O_ω are sums of sets A''_ω and the sets closed of class ω or F_ω are products of sets A'_ω. The process can then be continued for all transfinite classes.

The Hausdorff system is equivalent. Let G denote an open set and F a closed set. If E_δ denotes a product and E_σ a sum of sets E there are two systems of sets:

$$1.\quad G, G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, G_{\delta\sigma\delta\sigma}, \ldots,$$

$$2.\quad F, F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, F_{\sigma\delta\sigma\delta}, \ldots.$$

For the first system the classes are designated by the symbol (α) and for the second by $[\alpha]$. The open sets G are of class (1), the sets G_δ are of class (2), and so on for all finite classes (n). To the class (ω) belong the products of sets of classes $(1), (2), (3), (4), \ldots, (n), \ldots$. A sum of sets of class (ω) is of class $(\omega+1)$ and the process can be continued for all transfinite classes. In like manner the sets F are of class $[1]$, the sets F_σ are of class $[2]$, and the sets $F_{\sigma\delta}$ are of class $[3]$. The sets of class $[\omega]$ are sums of sets of classes $[1], [2], [3], \ldots, [n], \ldots$. These systems have been called delta- and sigma-systems, respectively. Certain relations between the two systems follow from the fact that an open set O is F_σ and a closed set F is G_δ.

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[2]) Intégrales de Lebesgue, Paris 1916, pp. 150, 151.

[3]) de la Vallée Poussin, loc. cit. pp. 138, 139.

[4]) Math. Annalen 77, pp. 430—432.

These classifications of Lebesgue and Hausdorff are closely related. The sets of class $(\alpha+1)$ in the Hausdorff classification are open of class α in the Lebesgue system if α is even and are closed of class α if α is odd. The sets of class $[\alpha+1]$ in the Hausdorff classification are closed of class α if α is even and are open of class α if α is odd.

The classifications of Lebesgue and Hausdorff are successful in defining classes without relying on the Baire classification of functions. The double character of these classifications has already been pointed out, and in either of them the classes overlap. From the properties cited for the Lebesgue classification it follows that a set open of class α is closed of class $\alpha-1$, α, or $\alpha+1$ and a set closed of class α is open of class $\alpha-1$, α, or $\alpha+1$. These sets, though they have the same open class, or the same closed class, are distinctly different. To make a distinction between these sets is a part of the purpose of this paper.

It is only natural to suppose that there are certain structural characteristics peculiar to the sets of a particular class. What are those structural characteristics possessed in common by all the sets of a given class? In this investigation an answer to this question has been found for sets of classes one and two, and in part for sets of class three.

II. The Structure of Sets of Points.

The Type of a Set.

In the Lebesgue classification the class of a set is defined in terms of its representation with closed sets and open sets. If the class be defined in terms of the simplest representation with closed sets and open sets, that is, the representation requiring not more alternate sums and products than any other such representation, then the classes are distinct and do not overlap.

The distinction to be made between those sets which are of the same open class but are of different closed classes is based on the notion of the "type" of a set. If a set is open of class α and is closed of class β it is of "type" (α, β), and conversely.

Of two sets A and B of types (α', β') and (α, β), respectively, A is of lower type if $\alpha' \leqq \alpha$ and $\beta' \leqq \beta$ and the equalities do not hold simultaneously. Of two sets A and B of types (α, β) and (β, α), respectively, it cannot be said that either is of lower type than the other. However, the two types are distinct.

Corresponding to the properties cited for the Lebesgue classification are the following properties of types:

1. Every finite sum or product of sets of type (α', β'), where $\alpha' \leqq \alpha$ and $\beta' \leqq \beta$, is of type (α, β), at most;

2. Every enumerably infinite sum or product of sets of type (α', β'), where $\alpha' \leqq \alpha$ and $\beta' \leqq \beta$, is of type $(\alpha+1, \beta+1)$, at most;

3. The removal of a subset of type (α, β) from a set of the same type leaves a subset of type (γ, γ), at most, where γ is the larger of α and β;

4. The removal of a subset of type (α, β) from a set of lower type leaves a subset of type (β, α), at most.

A Property of Sets of Form G_δ.

For two sets which are of form G_δ the following theorem holds:

Theorem 1. *If two sets of form G_δ are everywhere dense*[5]) *in a perfect set they have a common point.*

Assume the contrary, that two sets A and B of form $\prod G_m$ and $\prod \bar{G}_m$, respectively, have no common points and are everywhere dense in a perfect set H. Since A is a subset of G_m and B is a subset of $\bar{G}_m$, both G_m and $\bar{G}_m$ must be everywhere dense in H while their complementary closed sets are nowhere dense in H. If $U_m = G_m - \bar{G}_m$ and $V_m = \bar{G}_m - G_m$, these sets are nowhere dense in H. By definition, A is the product of sets $(U_m + G_m \cdot \bar{G}_m)$. By hypothesis, $A \cdot B = \prod G_m \cdot \bar{G}_m$ is a null set. Hence, any point of A must be in every U_m for m sufficiently large. Therefore, A may be expressed as the sum of sets $A_n = \prod_{m=n}^{\infty} U_m$ each nowhere dense in H. Therefore, A is of the first category relative to H. The complement of A is evidently of the first category relative to H. Since a set and its complement cannot both be of the first category, the desired contradiction is obtained.

From this theorem it is easy to demonstrate the following propositions:

1. If a subset of any set A of form $F_{\sigma\delta}$ is of the second category relative to a perfect set H in which a subset of any set B of form G_δ is everywhere dense, then A and B have a common point;

2. There exists a perfect set H relative to which the subset $A \cdot H$ of any set A of type $(2,1)$ or $(2,2)$ is of the second category.

By definition, a set of type $(2,1)$ is of form $F_{\sigma\delta}$ and G_δ and a set of type $(2,2)$ is of form $F_{\sigma\delta}$ and $G_{\delta\sigma}$. Hence, the complement, B, is of form $\bar{G}_{\delta\sigma}$.

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[5]) Acta Mathematica 30 (1906), pp. 10—12.

Assume the contrary. Then $B \cdot H$ is everywhere of the second category relative to H, and by Theorem 1 any subset G_δ of A is nowhere dense in any perfect set H. But such a set is enumerable. Therefore, A is enumerable and of form F_σ, contrary to the hypothesis.

Structure of the Sets of Type (1, 1).

Theorem 2. *The necessary and sufficient condition that any set A which is neither open nor closed be of type* $(1, 1)$ *is that if a subset of A or its complement B be everywhere dense in a perfect set H then $B \cdot H$ or $A \cdot H$, respectively, is nowhere dense in H.*

By definition, both the set A and its complement B of type $(1, 1)$ are of form G_δ. Any perfect set H is of form G_δ and the subsets $A \cdot H$ and $B \cdot H$ are then of the same form. By Theorem 1, the condition is necessary, since A and B have no common point.

Consider the closed derived set of order zero, A^0, which is the sum of a perfect set H and a reducible set[6] N. The set $A \cdot H$ is everywhere dense in H and by hypothesis $B \cdot H$ is nowhere dense in H. If G be the open set determined by all those portions of H containing no point of B, the set $G \cdot H$ is a subset of A. If $A_1 = A - G \cdot H$, the set $A_1 \cdot H_1$ is everywhere dense in the perfect subset H_1 of A_1^0. If G_1 be the open set determined by all those portions of H_1 containing no point of $B \cdot H_1$, $G_1 \cdot H_1$ is a subset of A_1 and of A. Let $A_2 = A_1 - G_1 \cdot H_1$.

The continuation of this process gives rise to a sequence of closed sets A_α^0 such that each is a subset of its predecessors. Let A_μ^0 be the closed set of points common to all the sets A_α^0, where $\alpha < \mu$ of the second kind. Defining the corresponding sets H_u and G_μ the process may be continued for transfinite ordinals.

By the fundamental principle of Cantor[7] there exists a least number β such that the sets $A_\beta^0, A_{\beta+1}^0, A_{\beta+2}^0, \ldots$, are identical. It follows that the corresponding perfect sets $H_\beta, H_{\beta+1}, H_{\beta+2}, \ldots$, are null sets, since $G_\beta \cdot H_\beta$ must be a null set when $A_\beta = A_{\beta+1}$ and G_β is not a null set if H_β exists. Also A_β is a reducible set N_β when H_β is a null set. By definition, the sets $G \cdot H, G_1 \cdot H_1, G_2 \cdot H_2, \ldots$, are non-overlapping. Every open set is a sum of closed sets and every set $G_\alpha \cdot H_\alpha$ is of form F_σ. The reducible set N_β is enumerable and also of form F_σ. The set A is the sum of the sets $G_\alpha \cdot H_\alpha$ and the reducible set N_β. Therefore, A is of form F_σ. By the same argument the complement B is also of form F_σ. Therefore, A_σ is of

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[6]) Annali di Matematica (3) **3**, p. 37.

[7]) Math. Annalen **17** (1880), S. 357.

form F_σ and G_δ and of type $(1,1)$, by definition. Therefore, the condition is sufficient.

From this theorem it is evident that a reducible set, which is nowhere dense in every perfect set is not of higher type than $(1,1)$. That is, a reducible set must be of type $(1,0)$ or $(1,1)$.

The Decomposition of Sets of Type (1, 1).

Theorem 2 furnishes a procedure for the decomposition of any set of type $(1,1)$ into a sum of disjoined sets of form $F \cdot G$. Let $A_0 = A$ be any set of type $(1,1)$ with its complement B. The set $A_0 \cdot H_0$ is every where dense in the perfect subset H_0 of A_0^0, and $B \cdot H_0$ is nowhere dense in H_0. In every portion of H_0 there is a portion containing no point of $B \cdot H_0$. By definition, G_0 is the open set determined by all those portions of H_0 containing no point of $B \cdot H_0$ and the set $G_0 \cdot H_0$ is a subset of A_0. Let $A_1 = A - G_0 \cdot H_0$ and repeat the same process on A_1. The set $A_1 \cdot H_1$ is everywhere dense in the perfect subset H_1 of A_1^0, and $B \cdot H_1$ is nowhere dense in H_1. Let G_1 be the open set determined by all those portions of H_1 containing no point of $B \cdot H_1$. Then the set $G_1 \cdot H_1$ is a subset of A_1 and of A, and the sets $G_0 \cdot H_0$ and $G_1 \cdot H_1$ are disjoined. Let $A_2 = A_1 - G_1 \cdot H_1$ and continue the decomposition with A_2. As was shown in Theorem 2 this decomposition must end, since it gives rise to a decreasing sequence of closed sets $A_0^0, A_1^0, A_2^0, \ldots$. The set A_0 is thus decomposed into the sum of disjoined sets $G_\alpha \cdot H_\alpha$ and a reducible set N_β, where $\alpha < \beta < \Omega$. Therefore,

$$A_0 = A = G_0 \cdot H_0 + G_1 \cdot H_1 + G_2 \cdot H_2 + \ldots + G_\alpha \cdot H_\alpha + \ldots + N_\beta .$$

The Structure of Sets of Type (1, 2) and (2, 1).

It is evident from the condition of Theorem 2 that sets of higher type than $(1,1)$ must all possess the property of being everywhere dense with their complements in some perfect set.

The following proposition characterizes the sets of type $(1,2)$:

The necessary and sufficient condition that any set A of form F_σ, with the complementary set B, be of type $(1,2)$ is that there exist a perfect set H in which both the sets $A \cdot H$ and $B \cdot H$ are everywhere dense.

If no such perfect set exists the set is of type $(1,1)$, at most. Therefore, the condition is necessary.

If there exists such a perfect set, A cannot be open or closed, or of type $(1,1)$, by Theorem 2. Any set of form F_σ is of type $(1,2)$, at most. Therefore, the condition is sufficient.

Since the complement of a set of type $(1,2)$ is a set of type $(2,1)$, there is the analogous proposition for sets of type $(2,1)$:

The necessary and sufficient condition that any set A of form G_δ, with the complementary set B, be of type $(2,1)$ is that there exist a perfect set H in which both the sets $A \cdot H$ and $B \cdot H$ are everywhere dense.

The characterization of sets of type $(1,2)$ at the same time exposes the character of their complementary sets of type $(2,1)$.

The Decomposition of Sets of Type (1, 2).

Let A_0 be any set of type $(1,2)$ with its complement B. If B is everywhere dense in the perfect subset H_0 of A_0^0, then A_0 cannot fill any portion of H_0. Hence, A_0 is the sum of closed sets nowhere dense in H_0 and a reducible set. If B is not everywhere dense in H_0, there is at least one portion of H_0 containing no point of B. As in the case of a set of type $(1,1)$ a subset $G_0 \cdot H_0$ may be removed from A_0 leaving a subset A_1. The decomposition may then be repeated with A_1. It is evident that B must be everywhere dense in some perfect set H_α. Then A_0 is the sum of closed sets nowhere dense in H_α, and the sets of points common to the open sets G_α and the perfect sets H_α and a reducible set. The open sets will be null sets if B is everywhere dense in H_0.

For the decomposition of sets of type $(1,2)$ the following proposition may then be stated:

Any set of type $(1,2)$ is the sum of closed sets nowhere dense in a perfect set, a reducible set, and the sum of non-overlapping sets $G_\alpha \cdot H_\alpha$, where $\alpha < \beta < \Omega$.

The Structure of Sets of Form $F_{\sigma\delta}$.

Theorem 3. *If any set A of form $F_{\sigma\delta}$ has a subset $A \cdot H$ of the second category relative to a perfect set H, then $A \cdot H$ is the sum of sets $A_1 \cdot H$ and G_δ of the first and second categories, respectively, relative to H.*

The set $A \cdot H$ is the common subset of sets F_σ which must also be of the second category relative to H and everywhere of the second category relative to at least one portion of H. For each set F_σ, in a portion H_2 of H_1 at least one of the closed sets F is dense and fills a portion of H_2. It follows that each set F_σ is a sum of closed sets nowhere dense in H_1 and an open set G which is the sum of all those open sets determined by the portions filled by the closed sets F. These open sets are everywhere dense in H_1. Hence, $A \cdot H$ is the sum of the subset $A_1 \cdot H$ of the first category relative to H and the subset G_δ everywhere of the second category relative to H_1.

Let A be any set of form $F_{\sigma\delta}$. If A is of the second category relative to the fundamental set P it is the sum of a subset A_1 of the first category relative to P and a set G_δ, by Theorem 3.

If one of the sets F_σ of which A is the common subset is of the first category relative to P then A is also and the procedure is the same as for the subset A_1 in the following argument.

Suppose A_1 is not of form F or F_σ, it is then of form $F_{\sigma\delta}$. Every closed set is the sum of a perfect set H and a reducible set. Hence, taking the product of such sets, $A_1 = \Pi(\Sigma H_{n_1} + N) = \Pi \Sigma H_{n_1} + N_1$. Except for the enumerable subset N_1, A_1 is a subset of the sum of non-overlapping perfect sets H_{n_1}, of diameter less than a given positive number ε_1, each nowhere dense in P. Consider the category of $A_1 \cdot H_{n_1}$ relative to the perfect set H_{n_1}. By Theorem 3, we may remove a set G_1 from A_1 if it is of the second category relative to H_{n_1}. Therefore, A_1 is the sum of an enumerable set N_1, a subset A_2 of the first category relative to each set H_{n_1} and a set of form $G_{1\delta}$ or $G_{1\delta\sigma}$, respectively, when A_1 is of the second category relative to a finite number or an enumerable infinity of the perfect sets H_{n_1}.

Suppose A_2 is not of form F or F_σ, it is then of form $F_{\sigma\delta}$. Except for an enumerable subset N_2, A_2 is a subset of the sum of non-overlapping perfect sets $H_{n_1 n_2}$ each nowhere dense in H_{n_1} and of diameter less than a given positive number ε_2. Consider the category of $A_2 \cdot H_{n_1 n_2}$ relative to each perfect set $H_{n_1 n_2}$. As before, A_2 is the sum of an enumerable set N_2, a subset A_3 of the first category relative to each set $H_{n_1 n_2}$ and a set of form $G_{2\delta}$ or $G_{2\delta\sigma}$, respectively, when A_2 is of the second category relative to a finite or an enumerable infinity of sets $H_{n_1 n_2}$.

Suppose there has been chosen a monotonic decreasing sequence of positive numbers ε_r with the limit zero. The continuation of the foregoing process gives rise to sequences of perfect sets such as the following:

$$H_{n_1},\ H_{n_1 n_2},\ H_{n_1 n_2 n_3},\ \ldots,\ H_{n_1 n_2 n_3 \ldots n_r},\ \ldots,$$

where each set is a subset of its predecessor and is of diameter less than the corresponding ε_r. Each of these sequences must end with a null set, when one of the sets contains no point and the corresponding closed set is finite or enumerable, or the sequence must determine a point of a residual set A_ω.

Consider the sequence of subsets of $A, A_1, A_2, A_3, \ldots, A_\alpha, \ldots$, where $\alpha < \omega + 1$. Suppose any set A_α of this sequence is of form F, F_σ or G_δ. Then the process of the decomposition of A should stop with that set A_α. The set A is then the sum of an enumerable set of form F or F_σ, a set of form $G_{\delta\sigma}$, that is, the sum of sets G_δ removed at each step, and a set A_α of form F, F_σ or G_δ. The enumerable set is open of class one, the set of form $G_{\delta\sigma}$ is open of class two, at most, and the set A_α is open of class two, at most. Therefore, A is open of class two, at most. By hypothesis, A is of form $F_{\sigma\delta}$ and is, therefore, closed of class two, at most. Therefore, A is of type $(2, 2)$, at most.

The Structure of the Residual Set A_ω.

Suppose there is no set A_α of the sequence of form F, F_σ, or G_δ. Then the residual set A_ω exists and is of form $F_{\sigma\delta}$.

Theorem 4. *If the residual set A_ω of form $F_{\sigma\delta}$ is dense in the fundamental set P it is of type* $(3, 2)$.

Assume the contrary, that the complementary set B is also of form $F_{\sigma\delta}$ or $\prod_m B_m$ where $B_m = \sum_n F_{mn}$ and B_{m+1} is a subset of B_m. Now A is of the first category relative to P. It follows that B, and consequently each set B_m, is everywhere of the second category relative to P. Hence, for each set B_m, in a portion of P at least one of the closed sets F_{mn} is dense and fills a portion P_m of P.

By hypothesis, A is dense in P, that is, A is everywhere dense in a portion K of P. The set B_1 fills a portion P_1 of K and has a perfect subset $H_{n_1 n_2 \ldots n_{r_1}}$. For this it is sufficient to choose r_1 so that the diameter ε_{r_1} of $H_{n_1 n_2 \ldots n_{r_1}}$ is less than one half the diameter of P_1. In the portion of P_1 of diameter ε_{r_1} determined by $H_{n_1 n_2 \ldots n_{r_1}}$, the set B_2 fills a portion of P_2. There exists a number r_2, such that the perfect set $H_{n_1 n_2 \ldots n_{r_1} \ldots n_{r_2}}$ is a subset of B_2. In general, for a set B_m there exists a number r_m such that the perfect set $H_{n_1 n_2 \ldots n_{r_1} \ldots n_{r_2} \ldots n_{r_m}}$ is a subset of B_m and of every $B_{m'}$ preceding B_m.

But the sequence of perfect sets

$$H_{n_1 n_2 \ldots n_{r_1} \ldots n_{r_2} \ldots n_{r_m}} \qquad (m = 1, 2, \ldots)$$

determines a point of A_ω. This point is a point of B since every set B_m contains a perfect set of this sequence. But B is the complement of A_ω. Therefore, the assumption that B is also of the form $F_{\sigma\delta}$ is false, and A_ω cannot be of type $(2, 2)$.

Since any set of the form $F_{\sigma\delta}$ is (by construction) of the type $(3, 2)$ at most, the theorem follows.

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