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On the Cohomology Groups of Moduli Spaces of Vector Bundles on Curves

G. Harder and M. S. Narasimhan

Introduction

In this paper we take up the idea in [9] which established a connection between the cohomology groups of certain moduli spaces of vector bundles on projective non-singular curves and the Tamagawa number of SL_2. This method gained strength through the recent results of Deligne on Weil conjectures.

We now state the main results of the paper. Let k be an algebraically closed field and Y/k a non-singular projective algebraic curve. Let L_0/Y be a line bundle with $\deg(L_0)=r$ and $n \geqq 2$ an integer with $(n,r)=1$. We consider the moduli scheme $M/k = M(n, L_0)/k$ of stable vector bundles on Y of rank n and determinant isomorphic to L_0. It is known that M is a non-singular projective variety. We suppose that $(p,n)=1$, where p is the characteristic of the field k. The group T_n of n-division points of the jacobian J/k of Y/k acts on M by tensorisation.

Theorem 1. *If l is a prime number with $(l,p)=(l,n)=1$, then T_n acts trivially on the l-adic cohomology groups $H^i(M, \mathbb{Q}_l)$.*

If $k=\mathbb{C}$ then an analogous result holds for the ordinary cohomology with complex coefficients; this follows immediately from Artin's comparison theorem. This result, for $k=\mathbb{C}$, was conjectured by Narasimhan and Ramanan in [12].

Let us now assume that Y is defined over a finite field $\mathbb{F}_q$. If we choose L_0 to be in $Pic(Y)/\mathbb{F}_q$, we may assume, by going over to a finite extension of $\mathbb{F}_q$ if necessary, that M is defined over $\mathbb{F}_q$. The group scheme $T_n/\mathbb{F}_q$ acts on $M/\mathbb{F}_q$ and the quotient $N/\mathbb{F}_q = (M/T_n)/\mathbb{F}_q$ exists. Regarding the number of rational points on these schemes one has

Theorem 2. *We have*

$$|M_{\mathbb{F}_q}| = |N_{\mathbb{F}_q}|$$

where $|M_{\mathbb{F}_q}|$ and $|N_{\mathbb{F}_q}|$ denote the number of rational points over $\mathbb{F}_q$ of M and N respectively.

Theorem 2 is proved by an extension of the methods of [9]. Theorem 2, combined with the results of Deligne on the Weil conjectures and a specialisation argument, implies Theorem 1.

We shall also give some results on the Betti numbers of $\bar{M} = M \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. We mention only

Theorem 3. *The third Betti number,* $\dim H^3(\bar{M}, \mathbb{Q}_l)$, *is* $2g$, *where* g *is the genus of* $Y/\mathbb{F}_q$.

This result has been proved in [11] except when $g = 2$, $n = 3$.

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I. Vector Bundles and Projective Bundles over Curves

1.1. The Moduli Spaces of Stable Vector and Projective Bundles

Let k be a field and Y/k be a projective non-singular absolutely irreducible curve over k. Let K/k be the function field of k. If $\bar{k}$ is an algebraic closure of k, we set $\bar{Y} = Y \times_k \mathrm{Spec}\,\bar{k}$.

We shall consider locally free $\mathcal{O}_Y$ modules of finite rank and by abuse of language we shall call them vector bundles. If E/Y is a vector bundle then rkE will denote its rank. If $rkE = n$ we put $\det E = \bigwedge^n E$ and call this line bundle the determinant of E. We define $\deg E = \deg(\det E)$. For $E \neq 0$ we introduce the rational number

$$\mu(E) = \deg E / rkE\,.$$

We shall also consider projective bundles $X \to Y$ i.e., locally trivial bundles with respect to the Zariski topology whose fibre is $\mathbb{P}^{n-1}/k$. More precisely, there exists a covering $Y = \cup_i U_i$ by Zariski open subsets of Y such that for every i we have a commutative diagram

$$\begin{tikzcd}
X \times_Y U_i \arrow[rr, "\sim"] \arrow[dr] & & \mathbb{P}^{n-1} \times_k U_i \arrow[dl] \\
& U_i &
\end{tikzcd}$$

Each vector bundle E/Y gives rise to a projective bundle which is the scheme of lines in E/Y. On the other hand, it is easy to see that a projective bundle X/Y comes from a vector bundle E/Y and that this vector bundle is unique up to a tensorisation with a line bundle. Since for a line bundle L/Y and a vector bundle E/Y we have $\deg(E \otimes L) = (rkE) \cdot (\deg L) + \deg E$ we see that $\underline{\deg} X \equiv \deg E (\mathrm{mod}\, n)$ is well defined. We call $\underline{\deg} X \in \mathbb{Z}/(n)$ the degree of the projective bundle X/Y.

Let us assume for the moment that k is algebraically closed. A vector bundle E/Y is called stable (resp. semi-stable) if for all proper subbundles

$F\ (\neq 0, E)$ we have $\mu(F) < \mu(E/F)$ (resp. $\mu(F) \leqq \mu(E/F)$). One observes that these inequalities are equivalent to

$$\mu(F) < \mu(E)\ (\text{resp. } \mu(F) \leqq \mu(E)).$$

A projective bundle is called stable (resp. semi-stable) if the corresponding vector bundle is stable (resp. semi-stable). This notion is well defined since a vector bundle E is stable (resp. semi-stable) if and only if $E \otimes L$ is stable (resp. semi-stable) where L is a line bundle. It is easy to see that if E is semi-stable and $(\deg E, rkE) = 1$, then E is stable.

Let n and r be integers $n \geqq 1$ with $(n, r) = 1$ and L_0 be a fixed line bundle of degree r on Y. It is known that the set of isomorphism classes of vector bundles E/Y of rank n with $\det E \approx L_0$ is parametrised by a non-singular projective variety $M = M(n, L_0)$ over k, [17, 18]. Let us assume that the characteristic of k is prime to n. If L is a line bundle on Y such that its n^{th} power $L^{\otimes n}$ is trivial; then $\det(E \otimes L) = \det E \otimes L^{\otimes n} = \det E$, so that the group T_n of n-division points of the jacobian J/k of Y acts on M/k. The quotient of M for this action of T_n is again a projective scheme N/k and N/k provides the solution for the moduli problem for projective bundles of degree r mod n.

We now drop the assumption that k is algebraically closed. A vector bundle E/Y is called stable (resp. semi-stable) if $\bar{E}/\bar{Y}$ is stable (resp. semi-stable) where $\bar{E}/\bar{Y}$ is the extension E to $\bar{Y}/\bar{k}$, $\bar{k}$ denoting the algebraic closure of k. If the line bundle L_0 is given over Y/k we may assume, by passing to a finite extension if necessary, that the moduli spaces M and N are defined over k i.e., we have

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & N \arrow[dl] \\ & k & \end{tikzcd}$$

However M/k and N/k are only coarse moduli schemes in the sense of Mumford [10] i.e., for any field L, $k \subset L \subset \bar{k}$, if M_L and N_L denotes the set of L-rational points of M and N respectively, we have maps

$$\varphi_L : \left\{ \begin{array}{l} \text{Set of isomorphism classes of stable} \\ \text{bundles } E \to Y \times_k L \text{ with } rkE = n \text{ and} \\ \det E \approx L_0 \end{array} \right\} \to M_L$$

$$\psi_L : \left\{ \begin{array}{l} \text{Set of isomorphism classes of projective} \\ \text{bundles } X \to Y \times_k L \text{ with } rkX = n-1 \\ \text{and } \underline{\deg} X \equiv r \bmod n \end{array} \right\} \to N_L$$

which are functorial in L and which become bijections if $L = \bar{k}$; but for $L = k$ these maps are neither injective nor surjective in general.

1.2. Moduli Schemes over a Finite Field

We now assume that k is a finite field $\mathbb{F}_q$ and investigate the maps φ_k and ψ_k defined above.

Proposition 1.2.1. *The map $\varphi_{\mathbb{F}_q}$ is bijective and the map $\psi_{\mathbb{F}_q}$ is surjective.*

Proof. Let m be a point of $M_{\mathbb{F}_q}$ and let $\bar{E}\to\bar{Y}$ be a stable vector bundle in the class m. Since m is a rational point over $\mathbb{F}_q$, we have for all $\sigma\in\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ an isomorphism

$$\lambda_\sigma:\bar{E}\to\bar{E}^\sigma,$$

where $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ denotes the Galois group of the extension $\bar{\mathbb{F}}_q/\mathbb{F}_q$. Since the automorphisms of a stable bundle are given by multiplication by scalars [13, Corollary to Proposition 4.3], we see that

$$\lambda_\tau^\sigma\circ\lambda_\sigma=\lambda_{\tau\sigma}\cdot a_{\sigma\tau}$$

where $a_\sigma^\tau\in\bar{\mathbb{F}}_q^*$ and λ_τ^σ denotes the transform of λ_τ by σ. It is clear that $a_{\sigma\tau}$ is a 2-cocycle. Since the Brauer group $H^2(\mathbb{F}_q, G_m)$ is trivial [15, p. 170] we can modify the λ_σ in such a way that $a_{\sigma,\tau}=1$. But then it follows from the theory of descent that there exists a vector bundle E on Y such that $\bar{E}$ is isomorphic to $E\times_Y\bar{Y}$. Using the fact that $H^1(\mathbb{F}_q, G_m)=0$ we see that E is unique and $\det E\approx L_0$. This proves that $\varphi_{\mathbb{F}_q}$ is bijective.

To prove that $\psi_{\mathbb{F}_q}$ is surjective we apply similar arguments for projective bundles. If $\bar{X}\to\bar{Y}$ is a projective stable bundle, then the group of automorphisms $\mathrm{Aut}(\bar{X})$ of $\bar{X}/\bar{Y}$ is a subgroup of the group T_n of n-division points of $J_{\bar{\mathbb{F}}_q}$. In fact, let $\bar{E}$ be a vector bundle on $\bar{Y}$ giving rise to $\bar{X}$. Since $\bar{E}$ is stable, the group of automorphisms of $\bar{E}$ is $\bar{k}^*$ and it follows that $\mathrm{Aut}(\bar{X})$ is isomorphic to the group of isomorphism classes of line bundles $L/\bar{Y}$ with $L\otimes\bar{E}\approx\bar{E}$, L necessarily satisfying the condition that $L^{\otimes n}$ is trivial (see [6], Corollary to Proposition 2). Now if $\bar{X}\simeq\bar{X}^\sigma$ for all $\sigma\in\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ then $\mathrm{Aut}(\bar{X})$ is defined over $\mathbb{F}_q$ and the obstruction to descent is contained in $H^2(\mathbb{F}_q, \mathrm{Aut}(\bar{X}))$. But $H^2(\mathbb{F}_q, \mathrm{Aut}\,\bar{X})=0$ [16, Chapter II, 3.3] and hence $\psi_{\mathbb{F}_q}$ is surjective. This proves Proposition 1.2.1.

Next we consider how many points are mapped by $\psi_{\mathbb{F}_q}$ into the same point in $N_{\mathbb{F}_q}$, i.e., we consider how often does it happen that $X\not\approx X'$ but $\bar{X}\approx\bar{X}'$ where X and X' are projective bundles on Y. Given $X\to Y$ it is well known that the number of $\mathbb{F}_q$ forms of X is equal to the order of $H^1(\mathbb{F}_q, \mathrm{Aut}(X))$. On the other hand, it is known (and this fact is crucial for us) that

$$|H^1(\mathbb{F}_q, \mathrm{Aut}\,X)|=|\mathrm{Aut}(X)_{\mathbb{F}_q}|$$

[15, Chapter VIII, Proposition 8]. [To apply this proposition we have to take for G a sufficiently large quotient $G=\mathrm{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)$ of the Galois group. Actually we require that $\mathrm{Aut}(X)_{\mathbb{F}_{q^r}}=\mathrm{Aut}(X)_{\mathbb{F}_q}$ and that the norm mapping $\mathrm{Aut}(X)_{\mathbb{F}_{q^r}}\to\mathrm{Aut}(X)_{\mathbb{F}_q}$ is zero.] Thus we have

Proposition 1.2.2. *Let $m\in N_{\mathbb{F}_q}$. Then the number of points in $\psi_{\mathbb{F}_q}^{-1}(m)$ is equal to the order of* $\mathrm{Aut}(X)_{\mathbb{F}_q}$, *where $X\to Y$ is any projective bundle whose isomorphism class is mapped into m by $\psi_{\mathbb{F}_q}$.*

1.3. Canonical Filtrations on Non-Semistable Bundles

We first assume that k is algebraically closed and collect some results on the structure of non-semistable bundles. It will turn out at the end that the assumption that k is algebraically closed is superfluous.

Definition 1.3.1. Let E be a vector bundle on Y which is not semi-stable. A subbundle F of $E(F\neq 0, E)$ is said to be SCSS in E ("strongly contradicting semi-stability") if the following two conditions are fulfilled:

a) F is semi-stable.

b) For every subbundle F' of E with $F\subsetneqq F'\subset E$ we have $\mu(F)>\mu(F')$.

Remark 1.3.2. Condition b) is equivalent to b') for any subbundle Q of E/F with $0\neq Q\subset E/F$ we have $\mu(Q)<\mu(F)$ and b') is equivalent to b'') for any *stable* subbundle Q of E/F with $0\neq Q\subset E/F$ we have $\mu(Q)<\mu(F)$.

Clearly b) and b') are equivalent and b') implies b''). If b'') is fulfilled, let $0\neq Q\subset E/F$ be subbundle of E/F. Then there exists a *stable* subbundle $Q'\neq 0$ of Q with $\mu(Q')\geqq\mu(Q)$ [13, Proposition 4.5]. Hence $\mu(Q)\leqq\mu(Q')<\mu(F)$.

Remark 1.3.3. If E is not semi-stable and $F\neq 0$ a subbundle of F satisfying i) F and E/F are semi-stable and ii) $\mu(F)>\mu(E/F)$ then F is SCSS (in E). In fact for any bundle $0\neq Q\subset E/F$ we have $\mu(Q)\leqq\mu(E/F)<\mu(F)$ so that the Condition b') in Remark 1.3.2 is fulfilled.

Proposition 1.3.4. *If E/Y is not semi-stable then it contains a unique SCSS subbundle.*

For the proof we need two lemmas

Lemma 1.3.5. *Let F_1 and F_2 be subbundles of E such that F_1 is semi-stable and F_2 satisfies Condition* b) *of Definition 1.3.1. If F_1 is not contained in F_2 then we have $\mu(F_2)>\mu(F_1)$.*

Proof. Consider the canonical map F_1 to E/F_2 which is non-zero by assumption. Since Y is a non-singular curve we have a factorisation

$$\begin{array}{ccc} F_1 \to & F_1' & \to 0 \\ & \downarrow & \\ E/F_2 \leftarrow & F_1'' & \leftarrow 0. \end{array}$$

where $F_1' \to F_1''$ is an isomorphism on a non-empty open set [13, §4]. Since F_1 is semi-stable, $\mu(F_1) \leqq \mu(F_1')$ and since F_2 satisfies Condition b) we have $\mu(F_1'') < \mu(F_2)$. On the other hand $\mu(F_1') \leqq \mu(F_1'')$ as $\deg F_1' \leqq \deg F_1''$ and $rk F_1' = rk F_1''$. It follows that $\mu(F_1) < \mu(F_2)$.

Lemma 1.3.6. *If F_1 and F_2 are subbundles of E which are SCSS then $F_1 = F_2$.*

Proof. If F_1 is not contained in F_2 we have, by Lemma 1.3.5, $\mu(F_2) > \mu(F_1)$. Applying again the lemma, we must then have $F_2 \subsetneqq F_1$. But since F_1 is semi-stable we have $\mu(F_2) \leqq \mu(F_1)$, which is a contradiction. Thus $F_1 \subseteqq F_2$ and similarly $F_2 \subseteqq F_1$.

Proof of Proposition 1.3.4. Since the uniqueness of a SCSS bundle follows from Lemma 1.3.6 we need only to prove the existence of a SCSS bundle. Let $m = \sup_{F \subset E, F \neq 0} \mu(F)$. Since E is not semi-stable, we have $m > \mu(E)$. Among all subbundles F for which $\mu(F) = m$ (the set of such F is non-empty since the values of μ are discrete and bounded from above) we choose one, say F_0, which has maximal rank. If $0 \neq F' \subset F_0$ is subbundle we have $\mu(F') \leqq m = \mu(F_0)$ so that F_0 is semi-stable. If on the other hand we have a subbundle F' with $F_0 \subsetneqq F' \subset E$ then $rk F' > rk F_0$ and by the choice of F_0 we have $\mu(F') < \mu(F_0)$. Thus F_0 also satisfies Condition b) i.e., F_0 is SCSS. This proves Proposition 1.3.4.

Lemma 1.3.7. *If a vector bundle E is not semi-stable we have a flag*

$$0 = F_0 \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_i \subsetneqq \cdots \subsetneqq F_k = E$$

satisfying the conditions

$$\text{(A)} \begin{cases} \text{i) } F_i/F_{i-1} \text{ is semi-stable for } i = 1, \dots, k, \\ \text{ii) } F_i/F_{i-1} \text{ is SCSS in } E/F_{i-1} \text{ for } i = 1, \dots, k-1. \end{cases}$$

Moreover such a flag is uniquely determined.

Proof. The existence follows from Proposition 1.3.4. In fact, let F_1 be a subbundle of E which is SCSS. If E/F_1 is semi-stable we are through. Otherwise we find $F_2' \subset E/F_1$ which is SCSS in E/F_1 and define F_2 to be the inverse of F_2' by the map $E \to E/F_1$. By repeating this construction we find a flag satisfying A. The uniqueness is proved by induction on $\dim E$ applying Proposition 1.3.4 and noting that $\{F_i/F_{i-1}\}$, $i \geqq 2$, form a filtration of E/F_1 satisfying A.

Lemma 1.3.8. *Let*

$$0 = F_0 \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_k = E$$

be a flag. Then Conditions (A) *in Lemma 1.3.7 are equivalent to the conditions:*

$$\text{(B)}\begin{cases}\text{i}')\ F_i/F_{i-1} \text{ is semi-stable for } i=1,\dots k,\\ \text{ii}')\ \mu(F_i/F_{i-1})>\mu(F_{i+1}/F_i) \text{ for } i=1,\dots k-l.\end{cases}$$

Proof. Suppose that Conditions (A) are satisfied. We have the exact sequence

$$0\to F_i/F_{i-1}\to E/F_{i-1}\to E/F_i\to 0\,.$$

Since $F_{i+1}/F_i\subset E/F_i$ and F_i/F_{i-1} is SCSS in E/F_{i-1} we must have $\mu(F_i/F_{i-1})>\mu(F_{i+1}/F_i)$ [see Remark 1.3.2, Condition b')].

Now suppose that Conditions (B) are satisfied. We first show that F_{k-1}/F_{k-2} is SCSS in E/F_{k-2}. Consider the exact sequence

$$0\to F_{k-1}/F_{k-2}\to E/F_{k-2}\to E/F_{k-1}\to 0\,.$$

We have by Condition ii') in (B)

$$\mu(F_{k-1}/F_{k-2})>\mu(E/F_{k-1})\,;$$

since E/F_{k-1} and F_{k-1}/F_{k-2} are semi-stable, we see, by Remark 1.3.3, that F_{k-1}/F_{k-2} is SCSS in E/F_{k-2}. We proceed to prove that Condition ii) in (A) is satisfied, by downward induction on i. Consider the exact sequences

$$0\to F_i/F_{i-1}\to E/F_{i-1}\to E/F_i\to 0$$

and

$$0\to F_{i+1}/F_i\to E/F_i\to E/F_{i+1}\to 0\,.$$

To prove that F_i/F_{i-1} is SCSS in E/F_{i-1} it is sufficient to prove, by Remark 1.3.2, b''), that for any stable subbundle $Q\neq 0$ of E/F_i we have $\mu(F_i/F_{i-1})>\mu(Q)$.

Now if $Q\subset F_{i+1}/F_i$ we will have $\mu(Q)\leqq\mu(F_{i+1}/F_i)$ as F_{i+1}/F_i is semi-stable and by hypothesis we have $\mu(F_{i+1}/F_i)<\mu(F_i/F_{i-1})$ so that $\mu(Q)<\mu(F_i/F_{i-1})$.

Suppose Q is not contained F_{i+1}/F_i. By induction hypothesis we may assume that F_{i+1}/F_i is SCSS in E/F_i. Since Q is semi-stable and $Q\subset F_{i+1}/F_i$, we have, by Lemma 1.3.5, $\mu(Q)<\mu(F_{i+1}/F_i)$. Since by hypothesis $\mu(F_{i+1}/F_i)<\mu(F_i/F_{i+1})$ it follows that $\mu(Q)<\mu(F_i/F_{i-1})$.

Combining Lemmas 1.3.7 and 1.3.8 we have

Proposition 1.3.9. *Let E be a vector bundle which is not semi-stable. Then E contains a uniquely determined flag*

$$0=F_0\subsetneqq F_1\subsetneqq\cdots\subsetneqq F_k=E$$

satisfying

a) F_i/F_{i-1} *is semi-stable for* $i=1,\dots k$

and

b) $\mu(F_i/F_{i-1})>\mu(F_{i+1}/F_i)$ *for* $i=1,\dots,k-1$.

Now it is clear that the assumption that k is algebraically closed can be dropped. In fact, if E/Y is not semi-stable then by definition $\bar{E}/\bar{Y}$, where $\bar{Y}=Y\times_k\bar{k}$, is not semi-stable. Hence $\bar{E}/\bar{Y}$ has a unique flag satisfying the conditions of Proposition 1.3.9. But then it is clear, at least when k is perfect, that this flag is already defined over k i.e., it is induced by a flag in E.

Definition 1.3.10. A flag

$$0=F_0\subsetneqq F_1\cdots\subsetneqq F_i\subsetneqq\cdots\subsetneqq F_k=E$$

is said to satisfy Condition (N) ("Numerical Condition") if

$$\mu(F_i/F_{i-1})>\mu(F_{i+1}/F_i)\quad\text{for}\quad i=1,\dots k-1$$

i.e., $\mu(F_1)>\mu(F_2/F_1)>\cdots>\mu(E/F_{k-1})$.

Lemma 1.3.11. *If a flag satisfies Condition* (N) *of Definition 1.3.10, then it also satisfies the condition*

$$\mu(F_1)>\mu(F_2)>\cdots>\mu(E).$$

We first note the following lemma whose proof is trivial.

Lemma 1.3.12. *Let* $0\to E_1\to E_2\to E_3\to 0$ *be an exact sequence of vector bundles* $E_i\neq 0$. *Then the following conditions are equivalent:*

i) $\mu(E_1)>\mu(E_3)$.
ii) $\mu(E_1)>\mu(E_2)$.
iii) $\mu(E_2)>\mu(E_3)$.

Proof of Lemma 1.3.11. First consider the exact sequence

$$0\to F_1\to F_2\to F_2/F_1\to 0.$$

Since by hypothesis $\mu(F_1)>\mu(F_2/F_1)$ it follows from Lemma 1.3.12 that $\mu(F_1)>\mu(F_2)$. Let us assume by induction that $\mu(F_{i-1})>\mu(F_i)$ and consider the exact sequences

$$0\to F_{i-1}\to F_i\to F_i/F_{i-1}\to 0$$

$$0\to F_i/F_{i-1}\to F_{i+1}/F_{i-1}\to F_{i+1}/F_i\to 0.$$

Since by hypothesis $\mu(F_i/F_{i-1})>\mu(F_{i+1}/F_i)$, we have by Lemma 1.3.12

$$\mu(F_i/F_{i-1})>\mu(F_{i+1}/F_{i-1})>\mu(F_{i+1}/F_i)$$

and the assumption $\mu(F_{i-1})>\mu(F_i)$ implies $\mu(F_i)>\mu(F_i/F_{i-1})$. Hence we have $\mu(F_i)>\mu(F_{i+1}/F_i)$; this in turn implies that $\mu(F_i)>\mu(F_{i+1})$ on applying Lemma 1.3.12 to the exact sequence

$$0\to F_i\to F_{i+1}\to F_{i+1}/F_i\to 0.$$

Thus Lemma 1.3.11 is proved.

The next two definitions are motivated by considerations in § 3.

Definition 1.3.13. Let $\mathscr{F}_1 = \{0 = F_0 \subset F_1 \cdots \subset F_k = E\}$ be a flag on E and let $\mathscr{F}_2$ be a flag on E which is a refinement of $\mathscr{F}_1$. We say that the pair $(\mathscr{F}_1, \mathscr{F}_2)$ satisfies the Condition $(\mathscr{C})$ if for each i, $1 \leqq i \leqq k$, the flag on F_i/F_{i-1} induced by $\mathscr{F}_2$ satisfies Condition (N) of Definition 1.3.10.

Definition 1.3.14. Let $C = \{\mathscr{F}_1, \ldots, \mathscr{F}_l\}$ be a chain of flags on E, i.e., $\mathscr{F}_{j+1}$ is a refinement of $\mathscr{F}_j$ for $j = 1, \ldots\, l-1$. We say that the chain C satisfies the Condition (Num) if

a) the flag $\mathscr{F}_1$, satisfies Condition (N) of Definition 1.3.10 and

b) for each j, $1 \leqq j \leqq l-1$, the pair $(\mathscr{F}_j, \mathscr{F}_{j+1})$ satisfies Condition $(\mathscr{C})$ of Definition 1.3.13.

Proposition 1.3.15. *Let F be a subbundle of E occuring in a chain of flags satisfying the Condition* (Num) *of Definition 1.3.14. Then if $F \neq E$ we have $\mu(F) > \mu(E)$.*

For the proof we need

Lemma 1.3.16. *Let $0 \to E_1 \to E \to E/E_1 \to 0$ be an exact sequence of vector bundles with $\mu(E_1) > \mu(E)$. Let F be a subbundle of E containing E_1 and satisfying $\mu(F/E_1) > \mu(E/E_1)$. Then we have $\mu(F) > \mu(E)$.*

Proof. We use Lemma 1.3.12 several times in the proof. From the exact sequence

$$0 \to F/E_1 \to E/E_1 \to E/F \to 0$$

we see that the condition $\mu(F/E_1) > \mu(E/E_1)$ implies that $\mu(E/E_1) > \mu(E/F)$. Since $\mu(E_1) > \mu(E)$ we have $\mu(E) > \mu(E/E_1)$ so that $\mu(E) > \mu(E/F)$ which in turn implies that $\mu(F) > \mu(E)$.

Proof of Proposition 1.3.15. We prove the proposition by induction on rkE, the proposition being clear for $rkE = 2$. Consider the flag

$$\mathscr{F}_1 = \{0 = F_0 \subset F_1 \subset \cdots \subset F_k = E\}.$$

Since by definition, $\mathscr{F}_1$ satisfies Condition (N) of Definition 1.3.10, we have, by Lemma 1.3.11

$$\mu(F_1) > \cdots > \mu(F_i) > \cdots > \mu(F_k) = \mu(E).$$

In particular the proposition is proved for $F = F_i$ and we are through if the chain consists only of $\mathscr{F}_1$. Now suppose $F \neq F_i$ for any i. If $F \subset F_{k-1}$, we have, by the induction hypothesis, $\mu(F) > \mu(F_{k-1})$; since $\mu(F_{k-1}) > \mu(E)$ it follows that $\mu(F) > \mu(E)$. If $F \supset F_{k-1}$ we have, by the induction hypothesis, $\mu(F/F_{k-1}) > \mu(F_k/F_{k-1})$. Since we also have $\mu(F_{k-1}) > \mu(F_k)$ it follows from Lemma 1.3.16 that we must have $\mu(F) > \mu(F_k) = \mu(E)$.

II. Tamagawa Numbers and Siegel's Formula

2.1. Tamagawa Numbers of SL_n *and* PL_n

We first recall some results on Tamagawa numbers. We assume that the field of constants is a finite field $\mathbb{F}_q$ and denote the field of functions on the curve $Y/\mathbb{F}_q$ by $K/\mathbb{F}_q$. If G/K is a connected affine algebraic group, we denote the group of adeles of G/K by G_A. The group of K-rational points of G_A is a discrete subgroup of the locally compact group G_A. If ω_A is a right invariant measure on G_A then it induces a measure on G_A/G_K and we may consider the volume

$$\mathrm{Vol}_{\omega_A}(G_A/G_K) = \int_{G_A/G_K} \omega_A$$

which may be infinite.

There exists a procedure to construct a right invariant measure ω_A starting from a right invariant non-zero differential form of highest degree on G/K, defined over K [19, § 2.3]. This gives the so-called Tamagawa measure on G_A, which in some cases is uniquely determined (e.g. if G is semi-simple or unipotent) and which in some cases depends on the choice of a system of convergence factors. The volume

$$\tau(G) = \int_{G_A/G_K} \omega_A^{\tau}$$

of G_A/G_K with respect to this measure is called the Tamagawa number of G/K.

Let GL_n (resp. SL_n) denote the full (resp. special) linear group over K. Let PL_n denote the projective group, namely the quotient of GL_n by its centre which is the multiplicative group G_m/K. It has been proved in [19, Theorem 3.3.1] that

$$\begin{aligned} \tau(\mathrm{SL}_n) &= 1 \\ \tau(\mathrm{PL}_n) &= n\,. \end{aligned} \tag{2.1.1}$$

One checks easily that the canonical mapping $\pi: \mathrm{GL}_n(A) \to \mathrm{PL}_n(A)$ is surjective. If $\underline{x} \in \mathrm{PL}_n(A)$ and $\pi(\underline{y}) = \underline{x}$, then $\det(\underline{y}) \in G_m(A) = I_K$, where I_K is the idele group of K. The idele norm of $\underline{t} = \det \underline{y}$ is

$$|\det \underline{y}| = |\underline{t}| = q^{-\deg \underline{t}},$$

where $\deg \underline{t} = \Sigma\, n_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(t_{\mathfrak{p}})$, where $n_{\mathfrak{p}}$ denotes the degree over $\mathbb{F}_q$ of the of the residue field at a closed point $\mathfrak{p}$ of Y. Since for an element

$$\underline{u} = \begin{pmatrix} \underline{a} & & 0 \\ & \ddots & \\ 0 & & \underline{a} \end{pmatrix} \in \mathrm{GL}_n(A)$$

of the centre we have $\det \underline{u} = \underline{a}^n$, we see that $\deg(\det \underline{y}) \bmod n$ does not depend on the choice of $\underline{y}$ and we obtain a homomorphism

$$\underline{\deg} : \mathrm{PL}_n(A) \to \mathbb{Z}/n\mathbb{Z} .$$

For $v \in \mathbb{Z}/n\mathbb{Z}$ we put

$$\mathrm{PL}_n^v(A) = \{\underline{x} \in \mathrm{PL}_n(A) \mid \underline{\deg}(x) = v\} ;$$

this set is invariant under the right action of $\mathrm{PL}_n(K)$

Lemma 2.1.2. *For* $v \in \mathbb{Z}/n\mathbb{Z}$, *we have*

$$\int_{\mathrm{PL}_n^v(A)/\mathrm{PL}_n(K)} \omega_A^\tau = 1 .$$

Proof. The Lemma is obvious since in this case the Tamagawa measure is left invariant and since the map deg is surjective.

2.2. *Truncated Tamagawa Numbers of Parabolic Subgroups of* SL_n *and* PL_n

Let us now consider a parabolic subgroup P/K of SL_n/K. Without loss of generality we may assume that P consists of the matrices p in SL_n of the form

$$p = \begin{pmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{kk} \end{pmatrix}$$

where $a_{ii} \in \mathrm{GL}_{m_i}$ and $\Sigma\, m_i = n$. We introduce the characters

$$\gamma_v = \gamma_v^P : P \to G_m$$

$$\gamma_v : p \mapsto \prod_{i=1}^{v} \det a_{ii}$$

and $\lambda_v : p \mapsto \det a_{vv}$. Set $\chi_{ij} = \lambda_i^{m_j} \lambda_j^{-m_i}$ and put $\alpha_i = \chi_{i,i+1}$.

If $\underline{p} \in P_A$ then $\gamma_v(\underline{p}) \in I_K$ we define $\delta_v(\underline{p})$ by

$$|\gamma_v(\underline{p})| = q^{-\delta_v(\underline{p})} .$$

We then obtain a surjective homomorphism $\delta : P_A \to \mathbb{Z}^{k-1}$ by setting

$$\delta(\underline{p}) = (\delta_1(\underline{p}) \ldots, \delta_{k-1}(\underline{p})) .$$

If $\underline{n} \in \mathbb{Z}^{k-1}$ we define

$$P_A(\underline{n}) = \{\underline{p} \in P_A \mid \delta(\underline{p}) = \underline{n}\} .$$

Then $P_A(0)$ is a subgroup of P_A. We also define a map $\delta': P_A \to \mathbb{Q}^{k-1}$ by setting

$$\delta'(\underline{p}) = (\delta'_1(\underline{p}), \ldots, \delta'_{k-1}(\underline{p})),$$

where $|\alpha_i(\underline{p})| = q^{-\delta'_i(\underline{p})}$.

We now construct a Tamagawa measure on P_A; in this case we have to introduce convergence factors. The algebraic group P/K is obtained by base extension from an algebraic group $P_0/\mathbb{F}_q$, $P_0 \subset \mathrm{SL}_n/\mathbb{F}_q$. Let us choose a right invariant differential form $\omega_0 \neq 0$ of highest degree on $P_0/\mathbb{F}_q$; this gives also such a form ω on P/K. For each closed point $\mathfrak{p}$ in Y, the form ω defines a measure on $P_{\hat{K}_\mathfrak{p}}$, where $\hat{K}_\mathfrak{p}$ denotes the completion of K with respect to the valuation defined by $\mathfrak{p}$ [19, § 2.2]. If $\hat{\mathcal{O}}_\mathfrak{p}$ denotes the ring of integers in $K_\mathfrak{p}$ and $P_{\hat{\mathcal{O}}_\mathfrak{p}} = P_{\hat{K}_\mathfrak{p}} \cap \mathrm{SL}_n(\hat{\mathcal{O}}_\mathfrak{p})$ is the group of integral points in $P_{K_\mathfrak{p}}$ we then have

$$\mathrm{vol}_{\omega_\mathfrak{p}}(P_{\hat{\mathcal{O}}_\mathfrak{p}}) = |P_0(k(\mathfrak{p}))|\,(N\mathfrak{p})^{-\dim P}$$

where $k(\mathfrak{p})$ is the residue field at $\mathfrak{p}$ and $N\mathfrak{p} = |k(\mathfrak{p})|$. (Compare [19], Theorem 2.2.5 and its proof.)

Now we have obviously

$$|P_0(k(\mathfrak{p})| = (N\mathfrak{p})^{\dim U_0}\;|M_0(k(\mathfrak{p}))|\,(N\mathfrak{p}-1)^{k-1}$$

where U_0 is the unipotent radical of P and M_0 is the semi-simple group

$$M_0 = \left\{ \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{kk} \end{pmatrix} \middle| \; a_{ii} \in \mathrm{SL}_{m_i} \right\}.$$

It follows that $\lambda_\mathfrak{p} = (1 - 1/N_\mathfrak{p})^{-k+1}$ is a system of convergence factors since $\prod_\mathfrak{p} (|M_0(k(\mathfrak{p})|/(N_\mathfrak{p})^{\dim M_0})$ is well known to be convergent [19]. We define the Tamagawa measure ω_A^τ on P_A by taking $\lambda_\mathfrak{p}$ as a system of convergence factors.

Proposition 2.2.1. *We have*

a) $$\int_{P_A(0)/P_K} \omega_A^\tau = \left(\frac{|J_{\mathbb{F}_q}|}{(q-1)q^{g-1}} \right)^{k-1}$$

where $|J_{\mathbb{F}_q}|$ denotes the number of $\mathbb{F}_q$-rational points of the Jacobian $J/\mathbb{F}_q$ of Y.

b) *For $\underline{p} \in P_A$, we have*

$$\int_{\underline{p}P_A(0)/P_K} \omega_A^\tau = q^{-\sum_{i=1}^{k-1} f_i\delta_i(\underline{p})} \int_{P_A(0)/P_K} \omega_A^\tau = q^{-\sum_{i=1}^{k-1} a_i\delta_i'(\underline{p})} \int_{P_A(0)/P_K} \omega_A^\tau$$

where $f_i = m_i + m_{i+1}$ *and* $a_i = s_{i+1}(n - s_{i+1})/m_i m_{i+1}$ *with* $s_i = \sum_{i \leq l \leq k} m_l$.

Proof. Let $U \subset P$ be the unipotent radical of P; then $H = P/U$ is a reductive group over K. We apply Theorem 2.4.4 in [19] and obtain

$$\int_{P_A(0)/P_K} \omega_A^\tau = \int_{U_A/U_K} \omega_{U,A}^\tau \int_{H_A(0)/H_K} \omega_{H,A}^\tau = \tau(H_A(0)/H_K)$$

since normalisation in Tamagawa measure gives measure 1 for the unipotent group U_A. To compute $\tau(H_A(0)/H_K)$ consider the exact sequence

$$1 \to M \to H \xrightarrow{\gamma} G_m^{k-1} \to 1$$

where $\gamma = (\gamma_1 \dots \gamma_{k-1})$. We apply again Theorem 2.4.4 in [19] to this exact sequence. If $S/K \approx G_m^{k-1}/K$ then the maps $H_A \to S_A$, $H_K \to S_K$ are surjective since $H^1(K, \mathrm{SL}_n) = 0$ ([15], Chapter X, Proposition 3, Corollary). Now we take for f in Theorem 2.4.4 of [19], the characteristic function of $S_A(0)/S_K$ and get

$$\int_{H_A(0)/H_K} \omega_{H,A}^\tau = \tau(\Pi\, \mathrm{SL}_{m_i}) \int_{S_A(0)/S_K} \omega_{S,A}^\tau = \int_{S_A(0)/S_K} \omega_{S,A}^\tau$$

since $\tau(\mathrm{SL}_{m_i}) = 1$ by 2.1.1. To evaluate $\int \omega_{S,A}^\tau$, let $\mathfrak{A}$ denote the canonical maximal compact subgroup of S_A. Then the number of double cosets $\mathfrak{A}\backslash S_A(0)/S_K$ is $|J_{\mathbb{F}_q}|^{k-1}$ and, by the choice of our convergence factors, $\mathrm{vol}\, \omega_{S,A}^\tau(\mathfrak{A}) = q^{(1-g)(k-1)}$. Now the exact sequence

$$1 \to S_A(0)/\mathfrak{A}S_K \to S_A(0)/S_K \to \mathfrak{A}S_K/S_K \to 1$$

shows that

$$\int_{S_A(0)/S_K} \omega_{S,A}^\tau = (|J_{\mathbb{F}_q}|/(q-1))^{k-1} \cdot q^{(1-g)(k-1)}$$

on remarking that

$$\mathfrak{A}S_K/S_K = \mathfrak{A}/S_K \cap \mathfrak{A} = \mathfrak{A}/(\mathbb{F}_q^*)^{k-1}.$$

This proves a).

To prove b) we note that the measure ω_A^τ is not left invariant and that the modulus of the left translation by $\underline{p}$ is precisely $q^{-\Sigma f_i\delta_i(p)} = q^{-\Sigma a_i\delta_i'(p)}$. (See e.g. [3], Chapter VII, § 3, No. 3.)

Next we consider the corresponding parabolic subgroups $\underline{P}$ of PL_n/K and $\tilde{P}$ in GL_n/K. The characters $\lambda_i = \det a_{ii}$ are defined on $\tilde{P}_A$ and the roots $\chi_{ij} = \lambda_i^{m_j}\lambda_j^{-m_i}$ are defined on $\underline{P}_A$. We set $\alpha_i = \chi_{i,i+1}$. We define

$$\underline{P}_A^0(0) = \left\{ \underline{p} \in \underline{P}_A \;\middle|\; \begin{array}{ll} \deg(\underline{p}) = 0 & \text{and} \\ |\chi_{ij}(\underline{p})| = 1 & \text{for all } i,j \end{array} \right\}.$$

We see that

$$\underline{P}_A^0(0) = \{\underline{p} \in \underline{P}_A \mid \underline{\deg}(\underline{p}) = 0 \quad \text{and} \quad |\alpha_i(\underline{p})| = 1 \quad \text{for} \quad 1 \leqq i \leqq k-1\}.$$

We also note that the character

$$R = \prod_{i<j} \chi_{ij} = \prod \gamma_i^{f_i} = \prod \alpha_i^{a_i}$$

is defined on $\underline{P}_A$.

Proposition 2.2.2. *We have*

a) $$\int\limits_{\underline{P}_A^0(0)/\underline{P}_K} \omega_{\underline{P},A}^{\tau} = \left(\frac{|J_{\mathbb{F}_q}|}{(q-1)q^{g-1}} \right)^{(k-1)}.$$

b) *For* $\underline{p} \in \underline{P}_A^0$ *one has*

$$\int\limits_{\underline{p}\underline{P}_A^0(0)/\underline{P}_K} \omega_{\underline{P},A}^{\tau} = q^{-\Sigma a_i \delta_i'(\underline{p})} \int\limits_{\underline{P}_A^0(0)/\underline{P}_K} \omega_{\underline{P},A}^{\tau}$$

where the a_i have the same meaning as in Proposition *2.2.1.*

Remark. The main assertion of the proposition is that these volumes are the same as those corresponding to the parabolic subgroup P.

Proof. Referring to the proof of Proposition 2.2.1 we see easily that we have to prove that

$$\int\limits_{H_A(0)/H_K} \omega_{H,A}^{\tau} = \int\limits_{\underline{H}_A^0(0)/\underline{H}_K} \omega_{\underline{H},A}^{\tau}$$

where $\underline{H} \subset \underline{P}$ is the reductive part in $\underline{H}$ corresponding to H. We consider the diagram of algebraic groups

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & G_m & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & \mathrm{SL}_n & \to & \mathrm{GL}_n & \to & G_m & \to & 1 \\ & & & & \downarrow{\scriptstyle \pi} & & & & \\ & & & & \mathrm{PL}_n & & & & \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

Let $\tilde{H}$ be the reductive subgroup of GL_n corresponding to H and $\underline{H}$ i.e. $\tilde{H}$ is the inverse image of $\underline{H}$. Let

$$\tilde{H}_A(0) = \{\underline{h} \in \tilde{H}_A \mid |\det \underline{h}| = 1, \pi_A(\underline{h}) \in \underline{H}_A(0)\}.$$

We then have a diagram of exact sequences

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & I_k^0 & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & H_A(0) & \to & \tilde{H}_A^0(0) & \to & I_k^0 & \to & 1 \\ & & & & \downarrow & & & & \\ & & & & \underline{H}_A^0(0) & & & & \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

where $I_K^0 = G_{m,A}(0) = \{x \in I_K \mid |x| = 1\}$. It now follows by [19, Theorem 2.4.4] that

$$\int_{H_A(0)/H_K} \omega_{H,A}^{\tau} = \int_{\underline{H}_A^0(0)/\underline{H}_K} \omega_{\underline{H},A}^{\tau}.$$

2.3. *Siegel's Formula*

Let E_0/Y be a vector bundle of rank n over the curve $Y/\mathbb{F}_q$. Put $L_0 = \det E_0$. Let V/K be the generic fibre of E_0; V/K is the n-dimensional vector space of all meromorphic sections of E_0/Y. For a closed point $\mathfrak{p}$ we denote by $\mathcal{O}_\mathfrak{p}$ the ring of integers at $\mathfrak{p}$ and by $\hat{\mathcal{O}}_\mathfrak{p}$ and $\hat{K}_\mathfrak{p}$ the completions of $\mathcal{O}_\mathfrak{p}$ and K respectively with respect to the valuation defined by $\mathfrak{p}$. If $\hat{E}_{0,\mathfrak{p}}$ denotes the completion (with respect to the valuation defined by $\mathfrak{p}$) of the stalk $E_{0,\mathfrak{p}} = \varinjlim_{U \ni \mathfrak{p}} \Gamma(U, E_0)$, then $\hat{E}_{0,\mathfrak{p}}$ is an $\hat{\mathcal{O}}_\mathfrak{p}$ lattice in $V \otimes_K \hat{K}_\mathfrak{p}$. We can reconstruct E from the family of lattices $\{\hat{E}_{0,\mathfrak{p}}\}$ (see [9, § 2]).

The bundle E_0 defines a maximal compact subgroup $\mathfrak{K}$ of $\mathrm{SL}(V)_A$; namely, $\mathfrak{K} = \Pi\, \mathfrak{K}_\mathfrak{p}$ where $\mathfrak{K}_\mathfrak{p} = \mathrm{SL}(\hat{E}_{0,\mathfrak{p}}) \subset \mathrm{SL}(V \otimes \hat{K}_\mathfrak{p})$.

For any $\underline{x} \in \mathrm{SL}(V)_A$ we consider the family of lattices

$$\{\hat{E}_{0,\mathfrak{p}}^{\underline{x}}\} = \{x_\mathfrak{p}^{-1} \hat{E}_{0,\mathfrak{p}}\}.$$

Then this family defines a locally free sheaf $E_0^{\underline{x}}$ by

$$\Gamma(U, E_0^{\underline{x}}) = \{v \in V \mid v \in \underline{x}_\mathfrak{p}^{-1} \hat{E}_{0,\mathfrak{p}}\}$$

for all $\mathfrak{p} \in U$. It is easy to see that the isomorphism class of the vector bundle $\hat{E}_0^{\underline{x}}$ depends only on the double coset $\mathfrak{K}\underline{x}\mathrm{SL}(V)_K$ and that the mapping

$$\mathfrak{K}\backslash\mathrm{SL}(V)_A/\mathrm{SL}(V)_K \to \left\{\begin{matrix}\text{Set of isomorphism classes of bundles} \\ E/Y \text{ of rank } n \text{ and } \det E \approx L_0\end{matrix}\right\} \qquad (2.3.1)$$

is surjective [9, § 2]. Let $\tilde{\mathfrak{K}}$ be the maximal compact subgroup of $\mathrm{GL}(V)_A$ defined by E_0. Then the following facts are obvious

1) The group of automorphisms of $E_0^{\underline{x}}$ is $\underline{x}^{-1}\tilde{\mathfrak{K}}\underline{x}\cap \mathrm{GL}(V)_K$.
2) $E_0^{\underline{x}} \approx E_0^{\underline{y}}$ if and only if $\underline{y} \in \tilde{\mathfrak{K}}\underline{x}\,\mathrm{GL}(V)_K$.

These facts give information about the number of points in the inverse image of an isomorphism class under the mapping in 2.3.1. Let $D = \mathfrak{K}\underline{x}\,\mathrm{SL}(V)_K$ be a double coset and $D' = \mathfrak{K}\underline{y}\,\mathrm{SL}(V)_K$ a double coset defining the same isomorphism class. If $\underline{y} = \underline{k}\,\underline{x}\,a$ with $\underline{k} \in \tilde{\mathfrak{K}}$, $a \in \mathrm{GL}(V)_K$ then $\underline{x}^{-1}\underline{y} = \underline{x}^{-1}\underline{k}\,\underline{x}\,a$. Hence $\det(\underline{x}^{-1}\underline{k}\,\underline{x})$. $\det a = 1$ and $\det(a)^{-1} = \det(\underline{x}^{-1}\underline{k}\,\underline{x}) \in \mathbb{F}_q^*$. If $\underline{y} = \underline{k}_1\underline{x}\,a_1$, $\underline{k}_1 \in \tilde{\mathfrak{K}}$, $a_1 \in \mathrm{GL}(V)_K$ then $\underline{k}_1\underline{x}\,a_1 = \underline{k}\,\underline{x}\,a$ so that $\underline{x}^{-1}\underline{k}^{-1}\underline{k}_1\underline{x} = a a_1^{-1}$; this means that $a a_1^{-1} \in \underline{x}^{-1}\tilde{\mathfrak{K}}\underline{x}\cap \mathrm{GL}(V)_K = \mathrm{Aut}\,E_0^{\underline{x}}$. Hence, if we consider the map

$$\det\colon \mathrm{Aut}\,E_0^{\underline{x}} = \underline{x}^{-1}\tilde{\mathfrak{K}}\underline{x}\cap \mathrm{GL}(V)_K \to \mathbb{F}_q^*$$

we see that the pair (D, D') defines a class in $\mathbb{F}_q^*/\det \mathrm{Aut}(E_0^{\underline{x}})$. Moreover $\underline{y} \in \mathfrak{K}\underline{x}\,\mathrm{SL}(V)_K$ if and only if this class is trivial; in fact if $\underline{y} = \underline{k}\,\underline{x}\,a$ and $\det a = \det b$ for $b = \underline{x}^{-1}\underline{k}_1\underline{x} \in \mathrm{Aut}\,E_0^{\underline{x}}$, $\underline{k}_1 \in \tilde{\mathfrak{K}}$, then $\underline{y} = \underline{k}\,\underline{k}_1\underline{x}\,b^{-1}a$; since $\det b^{-1}a = \det \underline{x} = \det \underline{y} = 1$ we have $\det(\underline{k}\,\underline{k}_1) = 1$ so that $\underline{y} \in \mathfrak{K}\underline{x}\,\mathrm{SL}(V)_K$. On the other hand it is clear that given $\underline{x} \in \mathrm{SL}(V)_A$ we can find $\underline{y} \in \mathrm{SL}(V)_K$ such that $\underline{y} = \underline{k}\,\underline{x}\,a$, $\underline{k} \in \tilde{\mathfrak{K}}$, $a \in \mathrm{GL}(V)_K$ and $\det(a)$ is a given element in $\mathbb{F}_q^*$. Therefore we see that for a given $\underline{x} \in \mathrm{SL}(V)_A$ there are exactly $|\mathbb{F}_q^*/\mathrm{Aut}\,\underline{E}_0^{\underline{x}}|$ double cosets in $\mathfrak{K}\backslash\mathrm{SL}(V)_A/\mathrm{SL}(V)_K$ which map into the same isomorphism class of vector bundles. Since the automorphisms of a stable bundle consists only of scalars, we have, from the above considerations, the following

Lemma 2.3.2. *The number of double cosets in* $\mathfrak{K}\backslash\mathrm{SL}(V)_A/\mathrm{SL}(V)_K$ *which are mapped into the same isomorphism class of stable bundles is* $|\mathbb{F}_q^*/\mathbb{F}_q^{*n}|$.

We now exploit the fact that the Tamagawa number of $\mathrm{SL}(V)_K$ is 1:

$$\int_{\mathrm{SL}(V)_A/\mathrm{SL}(V)_K} \omega_A^\tau = 1\,.$$

Decomposing $\mathrm{SL}(V)_A$ into double cosets $\mathfrak{K}\underline{x}\,\mathrm{SL}(V)_A$ we have

$$1 = \int_{\mathrm{SL}(V)_A/\mathrm{SL}(V)_K} \omega_A^\tau = \sum_{\underline{x}} \int_{\mathfrak{K}\cdot\underline{x}\mathrm{SL}(V)_K/\mathrm{SL}(V)_K} \omega_A^\tau$$

where $\underline{x}$ runs through representatives of double cosets. Since

$$(\mathrm{vol}(\mathfrak{K}\underline{x}\,\mathrm{SL}(V)_K/\mathrm{SL}(V)_K)) = (\mathrm{vol}_{\omega_A^\tau}(\mathfrak{K}))\cdot|\underline{x}^{-1}\mathfrak{K}\underline{x}\cap\mathrm{SL}(V)_K|^{-1}$$

we obtain

$$(2.3.3)\quad 1/\mathrm{vol}_{\omega_A^\tau}(\mathfrak{K}) = \sum_{\underline{x}}\left(\frac{1}{|\underline{x}^{-1}\mathfrak{K}\underline{x}\cap\mathrm{SL}(V)_K|}\right)$$

where $\underline{x}$ runs through representatives of double cosets.

We now assume that $\deg(L_0) = r$ is coprime to n. In this case all semi-stable bundles are stable. Moreover, if $\underline{x} \in \mathrm{SL}(V)_A$ is such that $E_0^{\underline{x}}$ is stable, we have $\mathrm{Aut}\, E_0^{\underline{x}} = \mathbb{F}_q^*$ and hence

$$\begin{aligned} |\underline{x}^{-1}\mathfrak{K}\underline{x} \cap \mathrm{SL}(V)_K| &= \text{number of } n\text{th roots of unity in } \mathbb{F}_q \\ &= |\mathbb{F}_q^*/\mathbb{F}_q^{*n}| \,. \end{aligned}$$

It then follows from Lemma 2.3.2, that the contribution to the sum in (2.3.3) from the double cosets for which $E_0^{\underline{x}}$ is stable is exactly the number of isomorphism classes of stable bundles E/Y with $\det E \approx L_0$. But by Proposition 1.2.1 this number is the same as the number $|M_{\mathbb{F}_q}|$ of $\mathbb{F}_q$ rational points of the corresponding moduli scheme $M/\mathbb{F}_q$. On the other hand

$$\begin{aligned} \mathrm{vol}_{\omega_A^\tau}(\mathfrak{K}) &= q^{-(n^2-1)(g-1)} \left(\prod_{\mathfrak{p}} \mathrm{vol}(\mathfrak{K}_{\mathfrak{p}})\right) \\ &\equiv q^{-(n^2-1)(g-1)} \prod_{\mathfrak{p}} \left(1 - \frac{1}{(N\mathfrak{p})^2}\right) \cdots \left(1 - \frac{1}{(N\mathfrak{p})^n}\right) \\ &\qquad \text{(see [19], pp. 22, 33)} \\ &= q^{-(n^2-1)(g-1)} \zeta(2)^{-1} \ldots \zeta(n)^{-1} \end{aligned}$$

where ζ denotes the Zeta function of K. Thus we obtain

Proposition 2.3.4. *We have*

$$|M_{\mathbb{F}_q}| = q^{(n^2-1)(g-1)} \zeta(2) \ldots \zeta(n) - \sum_{\underline{x}} \frac{1}{|\underline{x}^{-1}\mathfrak{K}\underline{x} \cap \mathrm{SL}(V)_K|}$$

where $\underline{x}$ runs through representatives of double cosets (in $\mathfrak{K}\backslash \mathrm{SL}(V)_A/\mathrm{SL}(V)_K$) such that $E_0^{\underline{x}}$ is not stable.

Next we consider the projective bundle X_0/Y where $X_0 = \mathbb{P}(E_0)$. Then the scheme of automorphisms of X_0/Y is locally isomorphic to PL_n/Y and the generic fibre is $\mathrm{PL}(V)$.

Let

$$\underline{\mathfrak{K}} = \prod_{\mathfrak{p}} (\mathrm{Aut}\, X_0)_{\hat{\mathcal{O}}_{\mathfrak{p}}} \subset \mathrm{PL}(V)_A \,.$$

We easily see that

$$\underline{\mathfrak{K}}\backslash \mathrm{PL}^0(V)_A/\mathrm{PL}(V)_K \simeq \left\{\begin{array}{l} \text{Isomorphism classes of projective} \\ \text{bundles } X/Y \text{ with fibre dimension} \\ (n-1) \text{ and } \underline{\deg}(X) = (\deg L_0) \bmod n\,. \end{array}\right\}$$

(This is proved by the same kind of arguments as used for $\mathrm{SL}(V)_A$. In contrast to that case we have bijectivity here since $\mathrm{PL}_n = \mathrm{Aut}\,\mathbb{P}(V)$ while in the case SL, $\mathrm{Aut}\, V = \mathrm{GL}$ but we take double cosets in SL.)

We get, as above,

$$\frac{1}{\mathrm{vol}_{\omega_{\mathfrak{T}_A}}(\mathfrak{K})} = \sum_{\underline{x} \in \mathfrak{K} \backslash \mathrm{PL}^0(V)_A/\mathrm{PL}(V)_K} \frac{1}{|\underline{x}^{-1} \mathfrak{K} \underline{x} \cap \mathrm{PL}(V)_K|}.$$

Now $\underline{x}^{-1}\underline{\mathfrak{K}}\underline{x} \cap \mathrm{PL}(V)_K = \mathrm{Aut}(X/Y)_{\mathbb{F}_q}$, where $X = X_0^{\underline{x}}$.

It may happen that a stable X/Y has non-trivial automorphisms: but in that case there are exactly $|\mathrm{Aut}(X)_{\mathbb{F}_q}|$ isomorphism classes of projective bundles which are mapped into the same point as X on the moduli scheme $N/\mathbb{F}_q$ (Proposition 1.2.2).

Hence

$$\sum_{\underline{x} \in \underline{\mathfrak{K}} \backslash \mathrm{PL}^0(V)_A/\mathrm{PL}(V)_K} 1/|\underline{x}^{-1} \underline{\mathfrak{K}} \underline{x} \cap \mathrm{PL}(V)_K| = |N_{\mathbb{F}_q}|$$
$$+ \sum_{\substack{\underline{x} \in \underline{\mathfrak{K}} \backslash \mathrm{PL}^0(V)_A/\mathrm{PL}(V)_K \\ X_0^{\underline{x}} \text{ not stable}}} 1/|\underline{x}^{-1} \underline{\mathfrak{K}} \underline{x} \cap \mathrm{PL}(V)_K|$$

On the other hand

$$1/\mathrm{vol}_{\underline{\omega}_{\mathfrak{T}_A}}(\underline{\mathfrak{K}}) = q^{(n^2-1)(g-1)} \zeta(2) \dots \zeta(n).$$

Thus we obtain

Proposition 2.3.5.

$$|N_{\mathbb{F}_q}| = q^{(n^2-1)(g-1)} \zeta(2) \dots \zeta(n) - \sum_{\underline{x}} \frac{1}{|\underline{x}^{-1} \underline{\mathfrak{K}} \underline{x} \cap \mathrm{PL}(V)_K|}$$

where $\underline{x}$ runs through representatives of double cosets in $\underline{\mathfrak{K}} \backslash \mathrm{PL}^0(V)_A/\mathrm{PL}(V)_K$ such that $X_0^{\underline{x}}$ is not stable.

III. Proof of the Theorems

3.1. The Summation over the Unstable Part in Siegel's Formula. Proof of Theorem 2

We prove Theorem 2 by showing that the summation over unstable bundles in Proposition 2.3.4 and 2.3.5 are the same.

To prove this let us assume that

$$E_0 = L_0 \oplus \underbrace{\mathcal{O}_Y \oplus \cdots \oplus \mathcal{O}_Y}_{(n-1) \text{ summands}}.$$

We define a (complete) flag $0 \subset F_{0,1} \cdots \subset F_{0,n-1} \subset E_0$ on E_0 by setting $F_{0,i} = L_0 \oplus \underbrace{\mathcal{O}_Y \oplus \cdots \oplus \mathcal{O}_Y}_{(i-1) \text{ summands}}$ and call it the standard flag on E_0. If $m_i \geqq 1$

are integers with $\sum_{i=1}^{k} m_i = n$, we call the stabilizer of the flag $0 \subset F_{0,m_1} \subset F_{0,m_1+m_2} \subset \cdots \subset E_0$ a standard parabolic subgroup (of type $(m_1, \ldots, m_k)$).

Suppose that $\underline{x} \in \mathrm{SL}(V)_A$ is such that $E_0^{\underline{x}}$ is not stable. Then by Proposition 1.2.9 we have a uniquely determined flag

$$0 \subset F_1 \subset F_2 \cdots \subset F_{k-1} \subset F_k = E_0^{\underline{x}}$$

such that F_i/F_{i-1} is semi-stable and $\mu(F_1) > \mu(F_2/F_1) \cdots > \mu(E_0^{\underline{x}}/F_{k-1})$. If $\dim(F_1) = m_1$, $\dim F_2 = m_1 + m_2, \ldots$, then consider the flag on E_0 defined by $0 \subset F_{0,m_1} \subset F_{0,m_1+m_2} \cdots \subset E_0$ and the stabilizer of this flag in $\mathrm{SL}(V)_A$, which is a (standard) parabolic subgroup. Thus to each $\underline{x}$ with $E_0^{\underline{x}}$ not stable there corresponds a standard parabolic subgroup and this parabolic group depends only the double coset containing $\underline{x}$. Thus if $\mathrm{Inst} \subset \mathfrak{K} \backslash \mathrm{SL}(V)_A / \mathrm{SL}(V)_K$ is the set of double cosets which give rise to vector bundles which are not stable, then we have a decomposition

$$\mathrm{Inst} = \bigcup_P \mathrm{Inst}^P$$

where P runs through the different parabolic subgroups fixing subflags of the standard flag. A similar decomposition holds for the set of double cosets $\underline{\mathrm{Inst}} \subset \underline{\mathfrak{K}} \backslash \mathrm{PL}^0(V)_A / \mathrm{PL}(V)_K$ giving rise to projective bundles which are not stable.

If P is a standard parabolic subgroup of type $(m_1, \ldots, m_k)$ and $\underline{p} \in P_A$, then $E_0^{\underline{p}}$ has a canonical flag

$$0 \subset F_{0,m}^{\underline{p}} \subset F_{0,m_1+m_2}^{\underline{p}} \cdots \subset E_0^{\underline{p}}$$

and an analogous assertion is clear for $\underline{P} \subset \mathrm{PL}(V)/K$. Now we see easily that, writing $v_i = m_1 + \cdots + m_i$,

$$\begin{aligned}\deg(F_{0,v_i}^{\underline{p}}/F_{0,v_{i-1}}^{\underline{p}}) &= \deg(F_{0,v_i}/F_{0,v_{i+1}}) + (n_i - n_{i-1}) \\ &= (n_i - n_{i-1}) + d_i\end{aligned}$$

where $d_1 = r$ and $d_i = 0$ for $i > 1$ and $n_i = \delta_i(\underline{p})$ with the notation of § 2.2.

We then define a subset $P_{A,\mathrm{num}}$ of P_A by

$$P_{A,\mathrm{num}} = \left\{ \underline{p} \in P_A \;\middle|\; \frac{l_1(\underline{p})}{m_1} > \cdots > \frac{l_k(\underline{p})}{m_k} \right\}$$

where $l_i(\underline{p}) = d_i + n_i - n_{i-1}$. We easily verify that

$$P_{A,\mathrm{num}} = \{\underline{p} \in P_A \mid |\alpha_i(\underline{p})| < C_i \quad \text{for} \quad 1 \leqq i \leqq k-1\}$$

where $\alpha_i = \lambda_i^{m_{i+1}} \lambda_{i+1}^{-m_i}$ and $C_i = q^{(d_i m_{i+1} - d_{i+1} m_i)}$.

We also define $\underline{P}^0_{A,\mathrm{num}}$ similarly:

$$\underline{P}^0_{A,\mathrm{num}} = \{\underline{p} \in \underline{P}^0_A \mid |\alpha_i(\underline{p})| < C_i \quad \text{for} \quad 1 \leqq i \leqq k-1\}$$

noting that α_i are defined on $\underline{P}$. Moreover we define

$$P^{sst}_{A,\mathrm{num}} = \{\underline{p} \mid \underline{p} \in P_{A,\mathrm{num}} \quad \text{and} \quad F^{\underline{p}}_{0,\nu_i}/F^{\underline{p}}_{0,\nu_{i-1}} \quad \text{is semi-stable}\}$$

and

$$\underline{P}^{0,sst}_{A,\mathrm{num}} = \{\underline{p} \mid \underline{p} \in \underline{P}^0_{A,\mathrm{num}} \quad \text{and} \quad P(F^{\underline{p}}_{0,\nu_i}/F^{\underline{p}}_{0,\nu_{i-1}}) \quad \text{is semi-stable}\}.$$

It is now clear that we have mappings

$$\mathfrak{K} \cap P_A \backslash P^{sst}_{A,\mathrm{num}}/P_K \to \mathrm{Inst}^P$$

and

$$\underline{\mathfrak{K}} \cap \underline{P}_A \backslash \underline{P}^{0,sst}_{A,\mathrm{num}}/\underline{P}_K \to \mathrm{Inst}^P$$

and these mappings are bijective by Proposition 1.3.9. Moreover we have

$$\underline{p}^{-1}\mathfrak{K}\underline{p} \cap \mathrm{SL}(V)_K = \underline{p}^{-1}\mathfrak{K}\underline{p} \cap P_K \quad \text{for} \quad \underline{p} \in P^{sst}_{A,\mathrm{num}}$$

and

$$\underline{p}^{-1}\underline{\mathfrak{K}}\underline{p} \cap \mathrm{PL}(V)_K = \underline{p}^{-1}\underline{\mathfrak{K}}\underline{p} \cap P_K \quad \text{for} \quad \underline{p} \in P^{0,sst}_{A,\mathrm{num}},$$

since the flag structure is unique. Thus we get

$$\sum_{\underline{x} \in \mathrm{Inst}^P} \frac{1}{|\underline{x}^{-1}\mathfrak{K}\underline{x} \cap \mathrm{SL}(V)_K|} = \sum_{P_A \cap \mathfrak{K} \backslash P^{sst}_{A,\mathrm{num}}/P_K} \frac{1}{|\underline{p}^{-1}\mathfrak{K}\underline{p} \cap P_K|}$$

and an analogous formula for $\mathrm{PL}(V)/K$.

If we introduce right invariant measures $\omega_{P,A}$ and $\omega_{\underline{P},A}$ on P and $\underline{P}$ such that the maximal compact subgroups $\mathfrak{K} \cap P_A$ and $\underline{\mathfrak{K}} \cap \underline{P}_A$ have volume 1 we see that the values of the summation over the unstable part are

$$\sum_P \int_{P^{sst}_{A,\mathrm{num}}/P_K} \omega_{P,A} \quad \text{for} \quad \mathrm{SL}(V)_K$$

and

$$\sum_{\underline{P}} \int_{P^{0,sst}_{A,\mathrm{num}}/P_K} \omega_{\underline{P},A} \quad \text{for} \quad \mathrm{PL}(V)_K$$

where P (resp. $\underline{P}$) runs through standard parabolic subgroups. But still it is not clear that these two expressions are equal; the difficulty is due to the fact that the condition for $\underline{p}$ to be in P^{sst}_A is difficult to handle. Therefore we write

$$\int_{P^{sst}_{A,\mathrm{num}}/P_K} \omega_{P,A} = \int_{P_{A,\mathrm{num}}/P_K} \omega_{P,A} - \int_{(P_{A,\mathrm{num}}/P_K)-(P^{sst}_{A,\mathrm{num}}/P_K)} \omega_{P,A}\,.$$

(It will be seen later that the integrals are convergent.) To understand the second ("error") term on the right, we observe that an element $\underline{p} \in P_{A,\mathrm{num}} - P^{sst}_{A,\mathrm{num}}$ gives rise to a bundle $E^{\underline{p}}_0$ with a flag

$$0 \subset F^{\underline{p}}_{0,\nu_1} \subset F^{\underline{p}}_{1,\nu_2} \subset \cdots \subset F^{\underline{p}}_{0,\nu_k} = E^{\underline{p}}_0$$

for which at least one of the quotients $F^{\underline{p}}_{0,\nu_i}/F^{\underline{p}}_{0,\nu_{i-1}}$ is not semi-stable. Therefore we can find, by Proposition 1.3.9 unique flags in the non-semi-stable quotients satisfying the conditions of that proposition. Thus we see that

$$\int\limits_{P_{A,\mathrm{num}}/P_K} \omega_{P,A} - \int\limits_{P^{sst}_{A,\mathrm{num}}/P_K} \omega_{P,A} = \sum_{Q \subsetneqq P} \int\limits_{Q^*_A/Q_K} \omega_{Q,A}$$

where Q runs through proper standard parabolic subgroups of P and Q^*_A is the subset of Q_A consisting of elements $\underline{q} \in Q_A$ satisfying the following two conditions:

i) $\underline{q} \in P_{A,\mathrm{num}}$,

ii) if $0 \subset V_l \cdots \subset V_k = V$ is the flag defined by P then the image $\bar{Q}^{(i)}$ of Q in $\mathrm{GL}(V_i/V_{i-1})$ by the canonical homomorphism $P \to \mathrm{GL}(V_i/V_{i-1})$ is a parabolic subgroup of $\mathrm{GL}(V_i/V_{i-1})$ and the second condition is $\bar{q}_i \in \bar{Q}^{(i)\,sst}_{A,\mathrm{num}}$ where $\bar{q}_i$ denotes the image of q.

These considerations lead to the following definitions. (Compare Definitions 1.3.12 and 1.3.13.)

Definition 3.1.1. Let Q and P be standard parabolic subgroups with $Q \subset P$. We say that $\underline{q} \in Q_A$ satisfies Condition $\mathscr{C}$ with respect to P_A if the following holds: If $0 \subset V_1 \subsetneqq \cdots \subsetneqq V_k = V$ is the flag defined by P, $\bar{Q}_i$ is the image Q in $\mathrm{GL}(V_i/V_{i-1})$ and $\bar{\underline{q}}_i \in \bar{Q}^{(i)}_A$ is the image of $\underline{q} \in \bar{Q}_{i,A}$ then we have $\bar{\underline{q}}_i \in Q^{(i)}_{A,\mathrm{num}}$ for $i = 1, \ldots, k$.

Definition 3.1.2. Let C be a chain of parabolic subgroups: $Q = Q^1 \subsetneqq \cdots \cdots \subsetneqq Q^\lambda \subsetneqq Q^{\lambda+1} \cdots \subsetneqq Q^l$. Then we denote by $Q^C_{A,\mathrm{num}}$ the subset of Q_A defined by

$$Q^C_{A,\mathrm{num}} = \left\{ \underline{q} \in Q_A \;\middle|\; \begin{array}{l} \underline{q} \in Q^l_{A,\mathrm{num}} \text{ and } \underline{q} \in Q^\lambda_A \text{ satisfies} \\ \text{Condition } \mathscr{C} \text{ (of Definition 3.1.1) with} \\ \text{respect to } Q^{\lambda+1}_A \text{ for } \lambda = 1, \ldots\, l-1\,. \end{array} \right\}$$

We then have

Proposition 3.1.3. *Let C be a chain of parabolic subgroups of $\mathrm{SL}(V)_A$ and let us denote also by C the corresponding chain of parabolic subgroups in PL^0_A. We then have*

1) $$\int\limits_{Q^C_{A,\mathrm{num}}/Q_K} \omega_{Q,A} = \int\limits_{\underline{Q}^{0,C}_{A,\mathrm{num}}/\underline{Q}_K} \omega_{\underline{Q},A} < \infty\,.$$

2) $$\sum_{\underline{x} \in \mathrm{Inst}} \frac{1}{|\underline{x}^{-1}\mathfrak{K}\underline{x} \cap \mathrm{SL}(V)_K|} = \sum_C (-1)^{|C|+1} \int\limits_{Q^C_{A,\mathrm{num}}/Q_K} \omega_{Q,A}$$

$$= \sum_{\underline{x} \in \underline{\mathrm{Inst}}} \frac{1}{|\underline{x}^{-1}\underline{\mathfrak{K}}\underline{x} \cap \mathrm{PL}(V)_K|}$$

where C runs through chains of (standard) parabolic subgroups and $|C|$ denotes the length of the chain.

Proof. From the considerations above we see that

$$\sum_{\underline{x}\in\mathrm{Inst}} \frac{1}{|\underline{x}^{-1}\mathfrak{K}\underline{x}\cap\mathrm{SL}(V)_K|} = \sum_C (-1)^{|C|+1} \int_{Q^C_{A,\mathrm{num}}/Q_K} \omega_{Q,A}$$

and

$$\sum_{\underline{x}\in\underline{\mathrm{Inst}}} \frac{1}{|\underline{x}^{-1}\underline{\mathfrak{K}}\underline{x}\cap\mathrm{PL}(V)_K|} = \sum_C (-1)^{|C|+1} \int_{\underline{Q}^{0,C}_{A,\mathrm{num}}/\underline{Q}_K} \omega_{\underline{Q},A}$$

and (2) follows from (1).

We first show that $\int_{Q^C_{A,\mathrm{num}}/Q_K} \omega_{Q,A}$ is finite. We decompose $Q^C_{A,\mathrm{num}}$ into the fibres of the map $\underline{q}\mapsto(\delta_1(\underline{q}),\dots\delta_{k-1}(\underline{q}))\in\mathbb{Z}^{k-1}$. We have, by Proposition 2.2.1 b

$$\int_{\underline{q}Q_A(0)/Q_K} \omega_{Q,A} = q^{-\Sigma f_i n_i} \int_{Q_A(0)/Q_K} \omega_{Q,A}$$

where $n_i=\delta_i(\underline{q})$ and $f_i=m_i+m_{i+1}$. Each n_i is, up to an additive constant, equal to the degree of the bundle $F^{\underline{q}}_{0,i}\supset E^{\underline{q}}_0$ and as $\underline{q}\in Q^C_{A,\mathrm{num}}$ these degrees are bounded below by Proposition 1.3.15. It follows that $\int_{Q^C_{A,\mathrm{num}/Q_K}} \omega_{Q,A}$ is finite.

Now we have $Q^C_{A,\mathrm{num}}/Q_A(0)\approx Q^{0,C}_{A,\mathrm{num}}/Q^0_A(0)$. In fact if we consider the map $Q_A\to\mathbb{Q}^{k-1}$ given by $\underline{q}\mapsto(\delta'_1(\underline{q}),\dots\delta'_{k-1}(\underline{q}))$ where $|\alpha_i(\underline{q})|=q_i^{-\delta'(\underline{q})}$, then $Q^C_{A,\mathrm{num}}$ is the inverse image of a certain subset X^C_Q of $\mathbb{Q}^{k-1}$ (defined by certain inequalities) and we see that $Q^{0,C}_{A,\mathrm{num}}$ is also the inverse image of X^C_Q by the corresponding map $Q^0_A\to\mathbb{Q}^{k-1}$ given by the roots α_i of Q^0_A. Moreover we have, for $\underline{q}\in Q^0_{A,\mathrm{num}}$

$$\int_{\underline{q}Q_A(0)/Q_K} \omega_{Q,A} = q^{-\Sigma a_i\delta_i(\underline{q})} \int_{Q_A(0)/Q_K} \omega_{Q,A} \text{ by Proposition 2.2.1},$$

and for $\underline{q}\in Q^{0,C}_{A,\mathrm{num}}$ we have

$$\int_{\underline{q}Q^0_A(0)/Q_K} \omega_{Q,A} = q^{-\Sigma a_i\delta'_i(\underline{q})} \int_{\underline{Q}^0_A(0)/\underline{Q}_K} \omega_{Q,A} \text{ by Proposition 2.2.2}.$$

Thus to complete the proof of the proposition it is enough to show that

$$\int_{Q_A(0)/Q_K} \omega_{Q,A} = \int_{\underline{Q}^0_A(0)/\underline{Q}_K} \omega_{\underline{Q},A}\,.$$

By Propositions 2.2.1 and 2.2.2 we have equality if we take the Tamagawa measures instead of the measures $\omega_{Q,A}$ and $\omega_{\underline{Q},A}$.

So we have to prove that

$$\int_{\mathfrak{K} \cap Q_A} \omega^{\tau}_{Q,A} = \int_{\mathfrak{K} \cap \underline{Q}_A} \omega^{\tau}_{\underline{Q},A} .$$

To prove this we observe that

$$\int_{\mathfrak{K} \cap U_A} \omega^{\tau}_{U,A} = \int_{\mathfrak{K} \cap \underline{U}_A} \omega^{\tau}_{\underline{U},A} = q^{(1-g)\dim U + p(Q)}$$

where U denotes the unipotent radical and $p(Q)$ is the numerical invariant of the parabolic group scheme Q/Y defined by the flag in E_0 (see [8, § 1, 3]). Thus it is enough to show that

$$\int_H \omega^{\tau}_{H,A} = \int_{\underline{H}} \omega^{\tau}_{\underline{H},A}$$

where $\mathfrak{K}_H$ and $\mathfrak{K}_{\underline{H}}$ are the obvious maximal compact subgroups in $H_A = Q_A/U_A$ and $\underline{H}_A = \underline{Q}_A/\underline{U}_A$ defined by E_0 and the flag. But this equality is clear in view of [19, Theorem 2.2.5]. This completes the proof of Proposition 3.1.3.

Propositions 2.3.4, 2.3.5, and 3.1.3 together yield Theorem 2.

3.2. *The Action of T_n on the Etale Cohomology of M*

We first recall briefly some results on the l-adic cohomology and zeta functions of algebraic varieties. Let $X/\mathbb{F}_q$ be a projective variety over the finite field $\mathbb{F}_q$. Let $\bar{X} = X_{\mathbb{F}_q} \times \bar{\mathbb{F}}_q$. If l is a prime coprime to q and $\mathbb{Q}_l$ is the field of l-adic numbers then the l-adic cohomology groups

$$H^i(\bar{X}, \mathbb{Q}_l) = \left(\varprojlim_{\nu} H^i(\bar{X}; \mathbb{Z}/l^{\nu}\mathbb{Z}) \right) \otimes \mathbb{Q}_l$$

of $\bar{X}$ are defined by means of etale cohomology [1, 7] [5, Exposé III]. The Frobenius map $\varphi: X \to X$ defines an endomorphism

$$\varphi_i^*: H^i(\bar{X}, \mathbb{Q}_l) \to H^i(\bar{X}, \mathbb{Q}_l) .$$

The spaces $H^i(\bar{X}, \mathbb{Q}_l)$ are finite dimensional and vanish for $i > 2 \dim X$. We define

$$Z_X(t) = \prod_{i=0}^{2N} \det(\mathrm{Id} - \varphi_i^* t)^{(-1)_{i+1}} = \frac{P_1(t) \dots P_{2N-1}(t)}{P_0(t) \dots P_{2N}(t)}$$

where $N = \dim X$. By a theorem of Grothendieck we have

$$\frac{Z'_X(t)}{Z_X(t)} = \sum_{n=1}^{\infty} |X_{\mathbb{F}_{q^n}}| \, t^{n-1}$$

where $|X_{\mathbb{F}_{q^n}}|$ denotes the numbers of $\mathbb{F}_{q^n}$ rational points [9].

Recently Deligne [4] has proved the *Weil Conjecture: If* $X/\mathbb{F}_q$ *is projective and smooth then the eigenvalues of*

$$\varphi_i^*: H^i(\bar{X}, \mathbb{Q}_l) \to H^i(\bar{X}, \mathbb{Q}_l)$$

are algebraic integers of absolute value $q^{i/2}$. (This means each conjugate has absolute value $q^{i/2}$.)

This has the following consequence. If $X/\mathbb{F}_q$ is smooth, then the polynomials $P_i(t) = \det(\mathrm{Id} - \varphi_i^* t)$ are pairwise coprime and this implies that the nuberator $\prod_{i \text{ odd}} P_i(t)$ and the denominator $\prod_{i \text{ even}} P_i(t)$ are determined by $Z_X(t)$ i.e., by the number of rational points $|X_{\mathbb{F}_{q^n}}|$ for all n.

We need the following proposition for which we could not find a reference.

Proposition 3.2.1. *Let* $X/\bar{\mathbb{F}}_q$ *be a projective variety and let* G *be a finite group of automorphisms of* $\bar{X}$. *Then* $\bar{X}/G$ *exists as a projective variety. If* $(|G|, l) = 1$, *the mapping*

$$H^i(\bar{X}/G, \mathbb{Q}_l) \to H^i(\bar{X}, \mathbb{Q}_l)^G$$

is an isomorphism, where $H^i(\bar{X}, \mathbb{Q}_l)^G$ *denotes the invariants of* G *in* $H^i(\bar{X}, \mathbb{Q}_l)$.

Proof. The existence of $\bar{X}/G$ is well known [14, Chapter III, No. 12]. To prove the second part, it is sufficient to prove that

$$H^i(\bar{X}/G, \mathbb{Z}/l^\nu\mathbb{Z}) \xrightarrow{\sim} H^i(X, \mathbb{Z}/l^\nu\mathbb{Z})^G .$$

Let $f: \bar{X} \to \bar{X}/G$ denote the projection. It is known [1, Exposé VIII, Proposition 5.5] that

$$R^q f_*(\mathbb{Z}/l^\nu\mathbb{Z}) = 0 \quad \text{for} \quad q > 0 .$$

The sheaf $F = R^0 f_*(\mathbb{Z}/l^\nu\mathbb{Z})$ is constructible and the group G acts on F. Moreover it is clear that the constant sheaf $\mathbb{Z}/l^\nu\mathbb{Z}$ on $\bar{X}/G$ injects into F. We claim that $\mathbb{Z}/l^\nu\mathbb{Z} \xrightarrow{\sim} F^G$. To see this we consider the fibre at any geometric point $\bar{y} \in \bar{X}/G$ (Compare [1], Exposé VIII). Then we have to show that $(\mathbb{Z}/l^\nu\mathbb{Z})_{\bar{y}} = \mathbb{Z}/l^\nu\mathbb{Z} = F_{\bar{y}}^G$.

Let $\bar{X}_y = f^{-1}(\bar{y})$, then we have, for $f_{\bar{y}}: \bar{X}_{\bar{y}} \to \bar{y}$,

$$F_{\bar{y}} = R^0 f_{\bar{y}}(\mathbb{Z}/l^\nu\mathbb{Z}) = \mathfrak{C}(\bar{X}_y, \mathbb{Z}/l^\nu\mathbb{Z})$$

where the last term denotes the set of mappings from the underlying set $X_{\bar{y}}$ into $\mathbb{Z}/l^\nu\mathbb{Z}$. The group G acts on $X_{\bar{y}}$ and $\mathfrak{C}$; the only invariants in $\mathfrak{C}$ are the constants.

Since $(|G|, l) = 1$, we see that we get a decomposition

$$F = F^G \oplus R = \mathbb{Z}/l^\nu\mathbb{Z} \oplus R$$

where R is the sheaf of elements of trace zero. By the Leray spectral sequence [1, Exposé VII, 1.5] we have

$$H^i(\bar{X}, \mathbb{Z}/l^\nu\mathbb{Z}) = H^i(\bar{X}/G, R^0 f_*(\mathbb{Z}/l^\nu\mathbb{Z})):$$

since

$$H^i(\bar{X}/G, R^0 f_*(\mathbb{Z}/l^\nu\mathbb{Z})) = H^i(\bar{X}/G, \mathbb{Z}/l^\nu\mathbb{Z}) \oplus H^i(\bar{X}/G, R)$$

the proposition is proved.

We now proceed to prove Theorem 1 in the case Y is defined over a finite field $\mathbb{F}_q$. We apply Proposition 3.2.1 to the projection $M/\mathbb{F}_q \to (M/T_n)/\mathbb{F}_q = N/\mathbb{F}_q$ and obtain

$$H^i(\bar{M}, \mathbb{Q}_l)^{T_n} = H^i(\bar{N}, \mathbb{Q}_l).$$

Then, if

$$\psi_i^* = \mathrm{res}_{H^i(\bar{N}, \mathbb{Q}_l)}(\varphi_i : H^i(\bar{M}, \mathbb{Q}_l) \to H^i(\bar{M}, \mathbb{Q}_l)),$$

we see that $\det(\mathrm{Id} - \psi_i^* t)$ divides $\det(\mathrm{Id} - \varphi_i^* t)$. On the other hand, we have, by Theorem 2, $Z_M(t) = Z_N(t)$ and we know from Weil conjectures that there are no cancellations in the expression for $Z_M(t)$. It follows that $\det(\mathrm{Id} - \psi_i^* t) = \det(\mathrm{Id} - \varphi_i^* t)$.

This proves Theorem 1 when Y is defined over a finite field.

We now remove the restriction of the field of constants to be a finite field. Let k by any field with $\mathrm{char}(k) \nmid n$ and let Y/k be a smooth projective curve. We shall derive the validity of Theorem 1 for $\bar{Y} = Y \times_k \bar{k}$ — where $\bar{k}$ is an algebraic closure of k — by means of a specialisation argument. One tool will be the proper base change theorem of Artin ([2], Exposé XVI, Corollary 2.2). The other tool needed is the theory of Mumford and Seshadri on the action of reductive groups on projective schemes, which is essential for the construction of our moduli schemes. At this place it would be very convenient if we knew that Seshadri's theory [18] works also for families of curves, especially for curves over valuation rings with unequal characteristics. Unfortunately we do not know whether this is actually true; in any case it does not seem to be obvious. To avoid this difficulty we shall use a rather crude argument which shows that over a Dedekind ring the construction of the moduli scheme is almost everywhere compatible with specialisation.

Lemma 3.2.2[1]. *Let A be a Dedekindring with quotient field k. Let $Y \to Spec(A)$ be a smooth projective curve. Let L_0' be a line bundle on $Y \times_{Spec(A)} k$ of degree r prime to n. This line bundle has a unique extension to a line bundle L_0 over $Y/Spec(A)$. Then there exists a non empty open*

[1] We are thankful to C. S. Seshadri for pointing out an inaccuracy in an earlier version of the proof of this lemma.

subset $U \supset Spec(A)$ and a projective smooth scheme $M \to U$, such that for all points $\mathfrak{p} \in U$ with residue field $k(\mathfrak{p})$ the scheme $M \underset{U}{\times} Spec(k(\mathfrak{p})) = M_{\mathfrak{p}}$ is isomorphic to the moduli scheme $M(n, L_0)/Spec(k(\mathfrak{p}))$ (Compare 1.1).

If moreover the residue characteristics for all $\mathfrak{p} \in U$ are prime to n, we have an action of T_n on $M \to U$, which induces on all fibres the standard action.

Proof. We have to analyse Seshadri's construction of the moduli scheme $M(n, L_0)$ over an algebraically closed field (Compare [17, 18]). We choose first of all an ample line bundle L on Y/A. Moreover we choose N points $P_1, \ldots, P_N$ on $Y_{\bar{k}}$ such that the divisor $\sum_{i=1}^{N} P_i$ is rational over k and such that these points are pairwise distinct and remain pairwise distinct after reduction mod a prime $\mathfrak{p} \in U$, where $U \subset Spec(A)$ is suitably chosen. We also choose an integer $m > 0$ such that for this choice of m and $P_1 \ldots P_N$, the Corollary 7.1 in [17] will be true for any specialisation of our curve induced by a homomorphism $A \to \bar{k}$ or $A \to \bar{k}(\mathfrak{p})$ for $G \in U$. Then we get a functorial mapping τ

$$\left\{\begin{array}{l}\text{isomorphism classes of}\\ \text{stable vector bundles}\\ \text{with determinant } L_0 +\\ \text{basis on } H^0(\ ,\ \otimes L^m)\end{array}\right\} \xrightarrow{\tau} \mathscr{G}$$

where $\mathscr{G} \to U$ is a twisted form of the projective scheme $\mathscr{G}^N_{d,n} \to U$ and where the twisting comes in since the P_i are not necessarily defined over k. Here $d = \dim H^0(Y, \otimes L^m)$.

The group GL_d/U acts on $\mathscr{G}/U$ in the usual way (Compare [17, § 4]) and we choose on $\mathscr{G}/U$ the standard ample line bundle which has a GL_d-linearisation [10].

We claim that there exists a closed subscheme $\mathscr{G}_{ns} \subset \mathscr{G}/U$ whose geometric points are exactly the non-stable geometrie points of $\mathscr{G}/U$. To see this we refer to the proof of Theorem 3.1 in [18]. We first choose a split maximal torus $T \subset \mathrm{GL}_d/U$ (for example the standard diagonal torus) and a Borel subgroup $B \subset T$. Then we know that there exists a finite set of one parameter subgroups in T such that a geometric point which is stable with respect to this finite set of one parameter groups is also stable with respect to all other one parameter subgroups of B. Therefore these finitely many one parameter subgroups define a closed subscheme $\mathscr{W} \subset \mathscr{G}/U$ of non-stable points. Then we proceed as in the proof of Theorem 3.1 in [18] by using the action of GL_d and the completeness of GL_d/B.

We set $\mathscr{G}_s/U = (\mathscr{G} - \mathscr{G}_{ns})/U$. This is a quasi-projective smooth scheme over U which is GL_d invariant. Then we know that $\mathscr{G}_s \underset{U}{\times} k$ and $\mathscr{G}_s \underset{U}{\times} k(\mathfrak{p})$ are the schemes of stable points respectively. It follows from [17], that the image of τ is a closed smooth subscheme

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathscr{G}_s \arrow[dl] \\ & U & \end{tikzcd}$$

Moreover it is clear that the quotient $X \underset{U}{\times} k/\mathrm{GL}_d \underset{U}{\times} k = M'$ exists. (A priori M' is defined only over $\bar{k}$; but by a standard argument of descent we see that it is defined over k.) We know that in this special case M'/k is a smooth projective scheme and that

$$p' : X \underset{U}{\times} k \to M'$$

is a principal fibre space with structure group PGL_d. Making U smaller if necessary we can extend M' to a smooth projective scheme M/U and we can extend

$$X \xrightarrow{p} M$$

such that X is still a PGL_d principal fibre space over M. This follows by standard arguments from the fact that p' is locally trivial with respect to the etale topology. But then it is clear that $M \underset{U}{\times} k(\mathfrak{p})$ is the quotient of $X \underset{U}{\times} k(\mathfrak{p})$ by the action of $\mathrm{GL}_d \underset{U}{\times} k(\mathfrak{p})$ for $\mathfrak{p} \in U$ because

$$X \underset{U}{\times} k(\mathfrak{p}) \to M \underset{U}{\times} k(\mathfrak{p})$$

is still a $\mathrm{PGL}_d \times k(\mathfrak{p})$ principal fibre space. Therefore

$$M \times k(\mathfrak{p}) \xrightarrow{\sim} M(n, L_{0,\mathfrak{p}})/Spec(k(\mathfrak{p}))\,.$$

The action of T_n on $M \underset{U}{\times} k = M'$ extends to an action on M if we make U smaller if necessary. Then it has to induce on $M \underset{U}{\times} k(\mathfrak{p})$ the action we want, since we know what happens on the bundles, which are the points on M. This completes the proof of Lemma 3.2.2.

Now it is more or less clear how we can deduce Theorem 1 for an arbitrary algebraically closed ground field. If $\bar{Y}/\bar{k}$ is any projective nonsingular curve we find a field $k \subset \bar{k}$ which is finitely generated over the prime field in $\bar{k}$ and such that $\bar{Y} = Y' \underset{k}{\times} \bar{k}$. We may find a Dedekind ring

$A \subset k$ with quotient field k and infinitely many closed prime ideals and such that Y'/k extends to a smooth curve $Y/Spec(A)$. We apply Lemma 3.2.2 and get from Artin's proper base change theorem ([2], Exposé XVI, Corollary 2.2) that Theorem 1 is true for $\bar{Y}/\bar{k}$ if it is true for $Y \underset{Spec(A)}{\times} \overline{k(\mathfrak{p})}$ where $\mathfrak{p}$ runs over the closed prime ideals in A and $\overline{k(\mathfrak{p})}$ is an algebraic closure of $A/\mathfrak{p} = k(\mathfrak{p})$. But the transcendence degree of $k(\mathfrak{p})$ over the prime field is less than the trancendence degree of k over the prime field if the characteristic does not change. Therefore we can reduce Theorem 1 by induction to the case that $k = \mathbb{Q}$ or $k = \mathbb{F}_q$. Only the first case is still of interest for us. Now we apply again the method above but this time we will have a change of the characteristic; our Dedekind ring will be the ring of integers in an algebraic number field and the residue field is finite. Therefore we can reduce the first case to the second one.

3.3. *The Computation of Some Betti Numbers*

We shall now derive some explicit formulas for Betti numbers of $\bar{M}/\bar{\mathbb{F}}_q$ in low and high dimensions. This will be done by estimating the sum over the instable part in our formula for $|M_{\mathbb{F}_q}|$ (Proposition 2.3.3). It turns out that this part has a lower order of magnitude than the term $q^{(n^2-1)(g-1)}\zeta(2)\ldots\zeta(n)$ and therefore we can read off the Betti numbers in a certain range from the expansion of that term. To obtain these estimates we have to consider the expressions (Proposition 3.1.3)

$$\int_{Q^C_{A,\mathrm{num}}/Q_K} \omega_{Q,A}\,.$$

Let us assume that Q is of type $(m_1, \ldots, m_k)$.

We consider the map

$$\delta : Q_A \to \mathbb{Z}^{k-1},$$

$$\delta : \underline{q} \mapsto (\delta_1(\underline{q}) \ldots \delta_{k-1}(\underline{q})) = (n_1, \ldots, n_{k-1}) = \underline{n}\,.$$

We know that $Q^C_{A,\mathrm{num}}$ is the inverse image of a certain subset $Y^C_Q \subset \mathbb{Z}^{k-1}$ which is described by inequalities derived from the Definition 3.1.2; the actual shape of Y^C_Q is not of interest for us at the moment. It follows from the considerations in the proof of Proposition 3.1.3 that

$$\int_{Q^C_{A,\mathrm{num}}/Q_K} \omega_{Q,A} = \sum_{\underline{n} \in Y^C_Q} q^{-f_1 n_1 - \cdots - f_{k-1} n_{k-1}} \int_{Q_A(0)/Q_K} \omega_{Q,A}$$

where $f_i = m_i + m_{i+1}$. If $H = Q/U$ where U is the unipotent radical of Q then we see as in the proof of Proposition 3.1.3 that

$$\int_{Q_A(0)/Q_K} \omega_{Q,A} = \frac{1}{\mathrm{vol}_{\omega^{\tau}_{Q,A}}(\mathfrak{R}_Q)} \int_{Q_A(0)/Q_K} \omega^{\tau}_{Q,A}$$

where $\omega^{\tau}_{Q,A}$ is the Tamagawa measure (Compare 2.2) and $\mathfrak{K}_Q = \mathfrak{K} \cap Q_A$ is the compact subgroup defined by means of E_0 (Compare 2.3). We use the formula a) in Proposition 2.2.1 and get

$$\int_{Q_A(0)/Q_K} \omega_{Q,A} = \frac{1}{\mathrm{vol}_{\omega^{\tau}_{Q,A}}(\mathfrak{K}_Q)} \left(\frac{|J_{\mathbb{F}_q}|}{(q-1)\,q^{(g-1)}}\right)^{(k-1)}.$$

If U is the unipotent radical of Q we find

$$\mathrm{vol}_{\omega^{\tau}_{Q,A}}(\mathfrak{K}_Q) = q^{(1-g)\dim U + p(Q)}\, \mathrm{vol}_{\omega^{\tau}_{H,A}}(\mathfrak{K}_H)$$

(Compare proof of Proposition 3.1.3) and for the last factor we obtain

$$\mathrm{vol}_{\omega^{\tau}_{H,A}}(\mathfrak{K}_H) = q^{(1-g)\dim H} \times \begin{array}{l}\text{product of values of } \zeta_K \text{ at some}\\ \text{of the arguments } 2, 3, \ldots, n\,,\end{array}$$

since we have only to take into account the semi simple part of H which is isogeneous to a product $\Pi\,\mathrm{SL}_{m_i}$. The central part of H gives contribution 1 because of the choice of the convergence factors. This shows that the order of magnitude is

$$\int_{Q_A(0)/Q_K} \omega_{Q,A} = q^{(\dim Q)(g-1) - p(Q)}\,(1 + 0(q^{-\frac{1}{2}}))\,.$$

We shall abbreviate this and just write

$$\int_{Q_A(0)/Q_K} \omega_{Q,A} \sim q^{(\dim Q)(g-1) - p(Q)}$$

and we shall say that the integral on the left hand side has the order of magnitude $q^{(\dim Q)(g-1) - p(Q)}$.

Therefore we get in total

$$\int_{Q^G_{A,\mathrm{num}}/Q_K} \omega_{Q,A} = \left(\sum_{\underline{n} \in Y^C_Q} q^{-f_1 n_1 - \cdots - f_{k-1} n_{k-1}}\right) q^{\dim Q(g-1) - p(Q)}\,.$$

To estimate the infinite sum in the bracket we use a very crude majorisation of Y^C_Q. Let us put $d_i = m_1 + m_2 + \cdots + m_i$; then $f_i = d_{i+1} - d_{i-1}$. We started from a flag

$$0 \subset F_{0,1} \subset F_{0,2} \subset \cdots \subset F_{0,k} = E_0$$

where

$$F_{0,i} = L_0 \oplus \mathcal{O}_Y \cdots \oplus \mathcal{O}_Y \quad \text{(Compare 3.1)}.$$

First of all we observe that for $\underline{q} \in Q_A$ we have

$$\deg F^{\underline{q}}_{0,i} = \deg L_{0,i} + n_i = r + n_i$$

if $n_i = \delta_i(\underline{q})$. Then it follows from Proposition 1.3.15 that $\underline{q} \in Q^C_{A,\mathrm{num}}$ implies

$$\mu(F^{\underline{q}}_{0,i}) = \frac{r+n_i}{d_i} > \mu(E_0) = \frac{r}{n}$$

and this is equivalent to

$$n_i > d_i \frac{r}{n} - r\,.$$

This is a necessary condition for a point $\underline{n}$ to be in Y^C_Q. We remark that for a maximal parabolic subgroup this condition is also sufficient (Lemma 1.3.12). Therefore we see that the order of magnitude of

$$\sum_{\underline{n} \in Y^C_Q} q^{-n_1 f_1 - \cdots - n_{k-1} f_{k-1}}$$

is less than or equal to

$$q^{-\sum_{i=1}^{k-1} f_i \left(\left\lceil \frac{d_i r}{n} \right\rceil - r\right)}$$

where $\left\lceil \frac{d_i r}{n} \right\rceil$ is the smallest integer which is greater than or equal to $\frac{d_i r}{n}$. On the other hand one checks easily from the definition that

$$p(Q) = \sum_{i=1}^{k-1} f_i \deg F_{0,i} - d_{k-1} \deg E_0 = \left(\sum_{i=1}^{k-1} f_i\right) r - d_{k-1} r\,.$$

This altogether gives that the order of magnitude of

$$\int_{Q^C_{A,\mathrm{num}}/Q_K} \omega_{Q,A}$$

is less than or equal to

$$q^{\dim P(g-1) - \sum_{i-1}^{k-1} f_i \left\lceil \frac{d_i r}{n} \right\rceil + d_{k-1} r}\,.$$

This gives the exact order of magnitude if Q is maximal parabolic as we see from the above remark.
We have

$$|M_{\mathbb{F}_q}| = q^{(n^2-1)(g-1)} \zeta(2) \ldots \zeta(n) - \sum_{\underline{x} \in \mathrm{Inst}} \frac{1}{|\underline{x}^{-1} \mathfrak{K} \underline{x} \cap \mathrm{SL}(n,k)|}$$

and we obtained above an estimate for the second term on the right hand side. The difference in the exponents in $q^{\cdots}$ for both terms is

$$\geqq (\mathrm{codim}_G Q)(g-1) + \sum_{i=1}^{k-1} f_i \left\lceil \frac{d_i r}{n} \right\rceil - r d_{k-1}$$

and therefore we have to look for the minimum of this expression if Q runs over all parabolic subgroups.

We have

Proposition 3.3.1. *We assume $g \geqq 2$ and $0<r<n$. Then the minimum of the expression*

$$(\operatorname{codim}_G Q)(g-1)+\sum_{i=1}^{k-1} f_i\left[\frac{d_i r}{n}\right]-r d_{k-1},$$

where Q runs over the parabolic subgroups, is obtained only once. The parabolic subgroup for which the minimum is obtained is maximal and defined by a line bundle or a hyperplane bundle. The value of that minimum is

$$\max(r, n-r)+(n-1)(g-1).$$

Proof. Without loss of generality we may assume that $r<\frac{n}{2}$, otherwise we pass to the dual situation. We perform some easy calculations:

$$\begin{aligned}
&\sum_{i=1}^{k-1} f_i\left[\frac{d_i r}{n}\right]-d_{k-1} r \\
&=\sum_{i=1}^{k-1} d_{i+1}\left[\frac{d_i r}{n}\right]-\sum_{i=1}^{k-1} d_{i-1}\left[\frac{d_i r}{n}\right]-d_{k-1} r \\
&=\sum_{i=1}^{k-1} d_{i+1}\left[\frac{d_i r}{n}\right]-\sum_{i=1}^{k-2} d_i\left[\frac{d_{i+1} r}{n}\right]-d_{k-1} r \\
&=\sum_{i=1}^{k-2}\left(d_{i+1}\left[\frac{d_i r}{n}\right]-d_i\left[\frac{d_{i+1} r}{n}\right]\right)+d_k\left[\frac{d_{k-1} r}{n}\right]-d_{k-1} r \\
&=\sum_{i=1}^{k-1}\left(d_{i+1}\left\{\frac{d_i r}{n}\right\}-d_i\left\{\frac{d_{i+1} r}{n}\right\}\right)
\end{aligned}$$

where $\{x\}=[x]-x$. Then our expression above becomes

$$d_2\left\{\frac{d_1 r}{n}\right\}+\sum_{i=2}^{k-1} f_i\left\{\frac{d_i r}{n}\right\}.$$

This last expression is strictly positive and since it is an integer it is $\geqq 1$.

We check first what happens if Q is maximal and defined by a line bundle or a hyperplane bundle. In that case we have $d_2=n$ and

$$\begin{aligned}
&(\operatorname{codim}_G Q)(g-1)+n\left\{\frac{d_i r}{n}\right\} \\
&=\begin{cases}(\operatorname{codim}_G Q)(g-1)+r & \text{if } d_1=1 \\ (\operatorname{codim}_G Q)(g-1)+n-r & \text{if } d_1=n-1 .\end{cases}
\end{aligned}$$

Therefore our assumption $r < \frac{n}{2}$ implies that the minimum is obtained in the case $d_1 = 1$. It is now sufficient to show that for the parabolic subgroups that are different from these two maximal parabolic subgroups we have

$$(\operatorname{codim}_G Q)(g-1)+1 > (n-1)(g-1)+r .$$

Case 1. Q itself is maximal and therefore of type (m) where $1 < m < n-1$. Then we have to check that

$$(n-m)\,m(g-1)+1 > (n-1)(g-1)+r .$$

It is clear that the left hand side takes its minimum value at $m=2$, therefore we must check that

$$2(n-2)(g-1)+1 > (n-1)(g-1)+r$$

or

$$(n-3)(g-1) > r-1 .$$

This is certainly allright if $n \geqq 6$. For $n=5$ and $n=4$ it is easily checked and for $n=3$, 2 this first case cannot occur.

Case 2. Q is the intersection of the two maximal parabolic groups defined by $d_1 = 1$ and $d_1 = n-1$. In this case we have to prove that

$$(2n-3)(g-1)+1 > (n-1)(g-1)+r .$$

But it is clear that this is covered by our considerations in the first case since $2n-4 < 2n-3$.

Case 3. Q is arbitrary but different from the two maximal ones defined by $d_1 = 1$, $d_1 = n-1$. In this case it is clear that we can find a parabolic subgroup $Q' \supset Q$, which is covered by one of the first two cases; since the codimension decreases we are through.

Now Proposition 3.3.1 yields that for $r < \frac{n}{2}$

$$|M_{\mathbb{F}_q}| = q^{(n^2-1)(g-1)}\,\zeta(2)\ldots\zeta(n) - q^{(n^2-n)(g-1)-r} + O\!\left(q^{(n^2-n)(g-1)-r-\frac{1}{2}}\right).$$

We express the ζ-function in the usual form

$$\zeta(s) = \frac{\Pi(1-\omega_i q^{-s})}{(1-q^{-s})(1-q\cdot q^{-s})} .$$

In the expression for $\zeta(m)$, $m=2,\ldots n$, we substitute $(-T)$ for each of the ω_i and T^2 for q. Then we get rational functions

$$Z_m(T) = \frac{(1+T^{1-2m})^{2g}}{(1-T^{-2m})(1-T^{2(1-m)})} .$$

Let us define

$$P(T) = T^{2(n^2-1)(g-1)} Z_2(T) \dots Z_n(T).$$

This is a Laurent series in the variable T^{-1} and define $b_\nu (\nu = 0, 1, \dots)$ by

$$P(T) = T^{2(n^2-1)(g-1)} \left(\sum_{\nu=0}^{\infty} b_\nu \cdot T^{-\nu} \right).$$

It is now obvious that for $0 \leqq \nu < 2(n-1)(g-1) + r$ the number b_ν is equal to the number of terms in our formula for $|M_{\mathbb{F}_q}|$ which have absolute value $q^{(n^2-1)(g-1) - \frac{\nu}{2}}$.

For $\nu = 2((n-1)(g-1) + r)$ this number of terms is equal to $b_\nu - 1$. Using the Weil conjectures and Poincaré duality this gives us

Theorem 3.3.2. $\dim H^\nu(\bar{M}\ \mathbb{Q}_l) = b_\nu$ for $0 \leqq \nu < 2((n-1)(g-1) + r)$.

$$\dim H^\nu(\bar{M}, \mathbb{Q}_l) = b_\nu - 1 \quad \text{for} \quad \nu = 2((n-1)(g-1) + r).$$

Corollary 3.3.3. *The third Betti number of* $\bar{M}/\bar{F}_q$ *is always equal to* $2g$.

Corollary 3.3.4. *If* $r \not\equiv \pm r' \bmod n$ *then the corresponding moduli spaces (of vector bundles of* rank n*) are not topologically equivalent.*

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G. Harder
Sonderforschungsbereich 40
Mathem. Institut
D-5300 Bonn
Wegelerstr. 10
Federal Republic of Germany

M. S. Narasimhan
Tata Institute for Fundamental Research
Bombay 5, India

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