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Uniqueness of (Δ) Bases and Isometries of Banach Spaces

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0. Introduction

A basis $B=\{b_i\}_{i\geqq 1}$ in a Banach space E is called 1-uc (1-unconditional) if for every $x=\Sigma x_i b_i \in E$ and every choice of $\varepsilon_i=\pm 1$, with $\varepsilon_i=-1$ for finitely many i, $\|x\| = \|\Sigma \varepsilon_i x_i b_i\|$. B is called a *transitive basis* if for every two indices i,j there exists an "onto" isometry g of E such that $g(b_i)=b_j$. Let $G(E)$ denote the group of all "onto" isometries of E. A subgroup G of $G(E)$ is called a *permutation transitive group* (with respect to B) if G consists of operators which act as finite permutations on the set B and if G is transitive on B, that is, given any i,j there exists $g\in G$ such that $g(b_i)=b_j$; in this case B will be called a *permutation transitive basis*. A 1-uc basis B is called *1-symmetric* if $G(E)$ contains all the operators induced by the finite permutations of the set B.

One of the main results proved here is Theorem 2.12 which states that if $B=\{b_i\}_{i\geqq 1}$ and B_1 are two (Δ) bases for the same Banach space, then $B_1=\{\varepsilon_i b_i\}_{i\geqq 1}$, where $\varepsilon_i=\pm 1$, that is a (Δ) basis is essentially unique. This generalizes the result proved in [5] about the uniqueness of a 1-symmetric basis in a Banach space which is not isometric to a Hilbert space. The definition of a (Δ) basis is very general and appears in Sect. 2, roughly, this is a 1-uc normalized basis for which subsets of the basis are permutation transitive, and each of the subsets satisfies any one of two other conditions (a), (b). For example, a 1-symmetric basis for a space not isometric to a Hilbert space is a (Δ) basis if the dimension of the space is not 2 or 4. A 1-uc normalized permutation transitive basis for an n-dimensional space which has a finite group of isometries is a (Δ) basis if n is odd and indivisible by 7. If E_i, $i=1,2,\ldots$, has a (Δ) basis then $\left(\sum_{i\geqq 1} \oplus E_i\right)_{l_p}$ also has a (Δ) basis $(1\leqq p<\infty)$.

To prove the uniqueness of a (Δ) basis all the results developed in Sects. (1) and (2) will be needed. Section 1 is concerned with the case when $\dim(E)<\infty$ and $G(E)$ is an infinite group. Theorem 1.5 proves that under certain conditions E has a proper subspace F such that $G(F)$ is also infinite. This result is used in the proof of

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Theorem 2.12. The case when $\dim(E)<\infty$ and $G(E)$ is a finite group is discussed in §2. We prove various results on algebraic number fields and use combinatorial arguments to establish in Lemmas 2.7 and 2.8 the form of every reflection point in an n-dimensional Banach space which has a normalized 1-uc permutation transitive basis. It is then shown in Lemma 2.10 that if E has (Δ) basis B every reflection point must have one of two possible forms with respect to this basis, and this fact is used to prove the uniqueness of a (Δ) basis; it follows that every onto isometry of E must act as a permutation and changes of signs on B.

In Sect. 3 we consider the following problem: If G is a group of linear operators on R^n $(1\leqq n\leqq\omega)$, is it possible to construct a Banach space $E=(R^n, \|\cdot\|)$ such that $G(E)=G$? The answer is affirmative (Theorem 3.1) if $n<\omega$ and G is a finite group which contains $-I$, where I is the identity operator on R^n. This answers a question raised by Lindenstrauss. However, if $n=\omega$, the situation is more complicated because the algebraic conditions alone on G may make $G(E)=G$ impossible to achieve. For example, if G contains the group of operators induced by the even finite permutations of the linear basis $\{e_i\}_{i=1}^{\infty}$ of R^{ω}, then whenever $E=(R^{\omega}, \|\cdot\|)$ is a normed space such that $G\subsetneqq G(E)$, $G(E)$ must contain also all the finite odd permutations of the basis. Because of such complications, Theorem 3.3 does not, and indeed cannot, yield the existence of a solution E to the equation $G(E)=G$, but rather shows that in a certain sense an approximate best solution does exist when G is a subgroup of the group induced by changes of signs and permutations of the basis $\{e_i\}_{i=1}^{\infty}$ of R^{ω}.

Other aspects concerning spaces having τ symmetric basis, or isometrically unconditional transitive bases, notions similar to some of our definitions, appeared in [6] Sect. 3. Results on finite-dimensional spaces such that $G(E)$ contains the group of all permutations of a given basis may be found in [3].

1. E is a Finite-Dimensional Space and $G(E)$ is an Infinite Group

Let E be an n-dimensional real Banach space. Let $(.,.)$ be the scalar product generated by the ellipsoid $\mathscr{E}$ of least volume containing the unit ball $B(E)=\{x\in E; \|x\|\leqq 1\}$. Let $\|x\|_2=\sqrt{(x,x)}$ for all $x\in E$. Since $\mathscr{E}$ is unique ([5]), each $g\in G(E)$ is a $(.,.)$ orthogonal transformation, that is $g(\mathscr{E})=\mathscr{E}$ for all $g\in G(E)$. Denote by $\Sigma_n=\left\{x=\sum_1^n x_i e_i; \sum_1^n x_i^2\leqq 1\right\}$ and by $S_n=\left\{x; \sum_1^n x_i^2=1\right\}$. Then there exists an invertible operator T such than $\mathscr{E}=T(\Sigma_n)$, hence $T^{-1}G(E)T=\{T^{-1}gT; g\in G(E)\}$ is a subgroup of the standard orthogonal group O_n on Σ_n.

Lemma 1.1. *If $B=\{b_i\}_{i=1}^n$ is a* 1-uc *basis for E, then $(b_i,b_j)=0$ for all $1\leqq i\neq j\leqq n$. If B is a transitive basis, then $\|b_i\|_2=\|b_1\|_2$ for every i.*

Proof. If $i\neq j$, there exists $g\in G(E)$ such that $g(b_i)=b_i$ and $g(b_j)=-b_j$, hence $(b_i,b_j)=(g(b_i), g(b_j))=(b_i, -b_j)$.

If B is transitive, given i there is $g\in G(E)$ s.t. $g(b_i)=b_1$, then $\|b_i\|_2=\|g(b_i)\|_2=\|b_1\|_2$. □

Theorem 1.2. *Let $B=\{b_i\}_{i=1}^n$ be a normalized* 1-uc *transitive basis for a Banach space E and suppose G(E) is infinite. Then each of the following conditions implies* $E=l_2^n$ *and* $\|x\|=\left(\sum_1^n x_i^2\right)^{1/2}$ *for every* $x=\sum_1^n x_i b_i \in E$:

(1) *G(E) contains the shift operator* τ_n, *defined by* $\tau_n(b_i)=b_{i+1}$ *if* $1\leqq i<n$, *and* $\tau_n(b_n)=b_1$.

(2) *n is a prime number and G(E) contains a permutation transitive subgroup.*

(3) $n\leqq 3$.

For the proof of the theorem we need two lemmas. First, observe that by Lemma 1.1 if B is a normalized 1-uc transitive basis there exists $\lambda>0$ s.t. $(b_i,b_j)=\lambda\delta_{i,j}$ $(i,j=1,2,\ldots,n)$. Let $e_i=\lambda^{-1/2}b_i$, then $\{e_i\}_1^n$ is also a 1-uc transitive basis which is $(.,.)$-orthonormal as well. In this case the ellipsoid of least volume which contains $B(E)$ is

$$\mathscr{E}=\left\{x=\sum_{i=1}^n x_i e_i;\ \|x\|_2\leqq 1\right\}=\Sigma_n, \quad \text{so} \quad G(E)\subseteqq O_n.$$

If $G(E)$ is infinite then by a well known theorem on Lie groups it has a one parameter subgroup, that is there exists a linear operater $u\neq 0$ such that $\{e^{tu};\ -\infty<t<\infty\}$ is a subgroup of $G(E)$, and $G(E)\subseteqq O_n$ implies $u+u'=0$.

Lemma 1.3. *Suppose E is an n-dimensional Banach space and u is a linear operator such that* $\{e^{tu};\ -\infty<t<\infty\}\subseteqq G(E)\subseteqq O_n$. *If* $x\in E$ *is such that* $\operatorname{span}\{gug^{-1}(x);g\in G(E)\}$ *is* $(n-1)$*-dimensional, then* $E=l_2^n$.

Proof. Note that for any $y\in E$ and $g\in G(E)$ $(gug^{-1}(y),y)=(y,(g^{-1})'u'g'(y))=-(y,gug^{-1}(y))$, hence $\dim\operatorname{span}\{gug^{-1}(y);\ g\in G(E)\}\leqq n-1$.

Let $\{g_i\}_1^{n-1}\subset G(E)$ be such that $\dim\operatorname{span}\{g_iug_i^{-1}(x);\ i=1,\ldots,n-1\}=n-1$, and $\varphi: R^{n-1}\to S_n$ be the map

$$\varphi(t_1,\ldots,t_{n-1})=g_1e^{t_1u}g_1'g_2e^{t_2u}g_2'\ldots g_{n-1}e^{t_{n-1}u}g_{n-1}'(z)$$

where $z=x/\|x\|_2$. Then $\varphi(0)=z$, and $\left(\frac{\partial\varphi}{\partial t_k}\right)_{t=0}=g_kug_k'(z)$ $(k=1,\ldots,n-1)$ are $(n-1)$ independent vectors, hence φ maps an open neighbourhood of the origin in R^{n-1} onto a neighborhood of z in S_n.

Let $N=\{y\in E;\text{ there exists } g\in G(E) \text{ such that } g(z)=y\}$. Since $G(E)$ is compact, N is a closed subset of S_n. However N is open because if $y\in N$ then the map $g\in G(E)$ for which $g(z)=y$ maps any neighbourhood of z onto a neighborhood of y (in the relative topology of S_n). Therefore $S_n=N$, which implies $\|y\|=\|z\|$ for every $y\in S_n$. □

Let $\mathcal{N}=\{1,2,\ldots,n\}$ and H be a transitive subgroup of the *symmetric group* on $\mathcal{N}$ (i.e., the group of all permutations). For $i,j\in\mathcal{N}$, $i\neq j$, we say iTj iff there exists $\sigma\in H$ such that $\{\sigma(1),\sigma(2)\}=\{i,j\}$. Note that iTj iff jTi. Define the following relation R on $\mathcal{N}$:

(a) iRi for all $i\in\mathcal{N}$

(b) If $i\neq j$ then iRj iff there exist

$$i_1=i,\quad i_2,\ldots,i_k,\quad i_{k+1}=j \quad \text{such that} \quad i_rTi_{r+1} \quad \text{for all} \quad r=1,2,\ldots,k.$$

Lemma 1.4. *R is an equivalence relation on $\mathcal{N}$, and the cardinality of $C_i=\{j\in\mathcal{N};\ iRj\}$, $|C_i|$, is not dependent on the choice of $i\in\mathcal{N}$.*

Proof. It follows immediately from the definition of R and the fact that T is a symmetric relation that R is an equivalence relation.

Let $s\in\mathcal{N}$, we shall prove $|C_s|=|C_1|$. Since H is transitive, there exists $\theta\in H$ such that $\theta(1)=s$. Consider $\theta(C_1)$, suppose that $j\in C_1$ and $j\neq 1$, then there exist $i_1=1, i_2, \ldots, i_k, i_{k+1}=j$ in $\mathcal{N}$ such that $i_r T i_{r+1}$ for all $r=1,\ldots,k$. We claim $\theta(i_r)\in C_s$ for all $r=1,\ldots,k+1$. Clearly $\theta(i_1)=s\in C_s$, and assume by induction that $\theta(i_{l-1})\in C_s$. Since $i_{l-1}Ti_l$, then either (i) $\sigma(1)=i_{l-1}$, $\sigma(2)=i_l$ for some $\sigma\in H$; or, (ii) $\sigma(1)=i_l$, $\sigma(2)=i_{l-1}$ for some $\sigma\in H$.

Without loss of generality assume (i) holds. Then $\theta(i_{l-1})=\theta(\sigma(1))=(\theta\circ\sigma)(1)$ and $\theta(i_l)=(\theta\circ\sigma)(2)$, whence $\theta(i_{l-1})T\theta(i_l)$, and since by assumption $\theta(i_{l-1})\in C_s$, it follows that $\theta(i_l)\in C_s$. Hence $\theta(C_1)\subseteq C_s$, whence $|C_1|\leqq|C_s|$. But the converse inequality is also true since H is transitive, completing the proof. □

Proof of Theorem 1.2. Let $\{e_i\}_1^n$ be the 1-uc transitive and $(.,.)$-orthonormal basis for E. Let $x=\sum_1^n e_i$ and let $0\neq y$ be any vector orthogonal to $\operatorname{span}\{gug^{-1}(x);\ g\in G(E)\}$. Since $u+u'=0$, u can be written as $u=\sum_{i<j}u_{ij}(e_i\otimes e_j-e_j\otimes e_i)$, hence $gug'=\sum_{i<j}u_{ij}(g(e_i)\otimes g(e_j)-g(e_j)\otimes g(e_i))$, therefore for all $g\in G(E)$

$$0=(gug'(x),y)=\sum_{i<j}u_{ij}[(g(e_i),x)(g(e_j),y)-(g(e_j),x)(g(e_i),y)].$$

Since $u\neq 0$ at least one $u_{ij}\neq 0$, suppose that $u_{12}\neq 0$. Let $A=\{\varepsilon=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n);\ \varepsilon_1=\varepsilon_2=1,\ \varepsilon_i=\pm 1 \text{ for } 3\leqq i\leqq n\}$, then setting $g_\varepsilon(e_i)=\varepsilon_i e_i (1\leqq i\leqq n)$

$$\begin{aligned}0&=2^{-(n-2)}\sum_{\varepsilon\in A}(gg_\varepsilon ug'_\varepsilon g'(x),y)\\&=2^{-(n-2)}\sum_{\varepsilon\in A}\sum_{i<j}u_{ij}\varepsilon_i\varepsilon_j[(g(e_i),x)(g(e_j),y)-(g(e_j),x)(g(e_i),y)]\\&=u_{12}[(g(e_1),x)(g(e_2),y)-(g(e_2),x)(g(e_1),y)],\end{aligned}$$

that is for all $g\in G(E)$

$$(g(e_1),x)(g(e_2),y)=(g(e_2),x)(g(e_1),y). \tag{*}$$

In case (1) taking $g=\tau_n^{i-1}$, we see that (*) implies $y_{i+1}=y_i$ for all $i=1,2,\ldots,n-1$, implying $y=\lambda x$, this however means $\dim\operatorname{span}\{gug^{-1}(x);\ g\in G(E)\}=n-1$ and Lemma 1.3 implies $E=l_2^n$.

In case (2) if π is a permutation of $\mathcal{N}$ denote by g_π the operator defined by: $g_\pi(e_i)=e_{\pi(i)}$, and let $H=\{\pi;\ g_\pi\in G(E)\}$. Since B is a permutation transitive basis, H is a transitive subgroup of the symmetric group. Equation (*) implies $y_{\pi(1)}=y_{\pi(2)}$ for every $\pi\in H$. We shall see that $y_1=y_j$ for all j.

Let C_i be as in Lemma 1.4. Since the identity permutation is in H hence $\{1,2\}\in C_1$, and since all the C_i's have the same cardinality, and every pair C_i and

C_j are disjoint or equal sets, it follows from the fact that n is a prime number that $\mathcal{N}=C_1$. This implies that for any $1\neq j\in\mathcal{N}$, $1Rj$, that is there exist $i_1=1, i_2, \ldots, i_k$, $i_{k+1}=j$ such that $i_r T i_{r+1}$ for all $r=1,\ldots,k$, that is $y_{i_r}=y_{i_{r+1}}$, in particular $y_1=y_j$. As in (1) we conclude that $E=l_2^n$.

In case (3), if $n=2$, taking $g=I$ the identity on E, we obtain $y_1=y_2$ from (*), so $E=l_2^2$. If $n=3$, replacing g in (*) by $g_\varepsilon g$ we obtain

$$\begin{aligned}
0&=(g(e_1),g_\varepsilon(x))(g(e_2),g_\varepsilon(y))-(g(e_2),g_\varepsilon(x))(g(e_1),g_\varepsilon(y))\\
&=\sum_{i,j=1}^{3}(g(e_1),e_i)(g(e_2),e_j)\varepsilon_i\varepsilon_j y_j-\sum_{i,j=1}^{3}(g(e_2),e_j)(g(e_1),e_i)\varepsilon_i\varepsilon_j y_i\\
&=\sum_{i\neq j}(g(e_1),e_i)(g(e_2),e_j)\varepsilon_i\varepsilon_j(y_j-y_i)\\
&=\sum_{i<j}\varepsilon_i\varepsilon_j(y_j-y_i)\,[(g(e_1),e_i)(g(e_2),e_j)-(g(e_1),e_j)(g(e_2),e_i)]\,.
\end{aligned}$$

Fixing the indices i and j, $1\leqq i\neq j\leqq 3$, and averaging the last equality over all $\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ with $\varepsilon_i=\varepsilon_j=1$, $\varepsilon_l=\pm 1$, then for any $g\in G(E)$

$$(y_j-y_i)\,[(g(e_1),e_i)(g(e_2),e_j)-(g(e_1),e_j)(g(e_2),e_i)]=0\,.$$

Taking $g=I$, $i=1$, $j=2$ it follows that $y_1=y_2$. If $y_3\neq y_1$ then taking $j=3$ and $i=1$ or 2, let $h\in G(E)$ be such that $h(e_1)=e_3$, then $(h(e_1),e_i)=0$, so $(h(e_2),e_i)=0$, and since $\|h(e_2)\|_2=\|e_2\|_2=1$, $h(e_2)=\pm e_3$, contradicting $h(e_1)=e_3$. The conclusion $y_1=y_2=y_3$ implies $E=l_2^3$. The fact that $\|b_i\|=1$ and $(b_i,b_j)=\lambda\delta_{ij}$ implies that $\left\|\sum_{i=1}^{n}\xi_i b_i\right\|=\left(\sum_{i=1}^{n}\xi_i^2\right)^{1/2}$ for all $\sum\xi_i b_i\in E$. $\square$

The following theorem will be needed in the proof of Theorem 2.12.

Theorem 1.5. *Let $B=\{b_\nu\}_{\nu=1}^n$ be a 1-uc basis for E, $n\geqq 3$, and assume $G(E)$ is infinite. Let $k\geqq 2$, and I_j $(j=1,2,\ldots,k)$ be non-empty disjoint sets satisfying $\bigcup_{j=1}^{k} I_j=\{1,2,\ldots,n\}$. Let $E_j=[b_\nu;\nu\in I_j]$ and suppose $G(E_j)$ contains a permutation transitive subgroup G_j such that for every $g\in G_j$ the extension operator $\tilde g$ defined by $\tilde g(b_\nu)=\begin{cases} g(b_\nu); & \nu\in I_j\\ b_\nu\ ; & \nu\notin I_j\end{cases}$ is in $G(E)$. Then there exists a proper subset $\alpha\subsetneqq\{1,2,\ldots,k\}$, such that the subspace $\left[b_\nu;\nu\in\bigcup_{j\in\alpha}I_j\right]$ has an infinite group of isometries.*

Proof. As in Lemma 1.1, $(b_\nu,b_\mu)=0$ if $\mu\neq\nu$, and since G_j is transitive $\|b_\nu\|_2=\|b_\mu\|_2$ for $\mu,\nu\in I_j$. Let $e_\mu=b_\mu/\|b_\mu\|_2$ $(\mu=1,2,\ldots,n)$, then $\{e_\mu\}$ is a 1-uc and $(.,.)$ orthonormal basis for E and satisfies the same conditions as the basis B.

Since $G(E)$ is infinite, there exists $u\neq 0$ as in Theorem 1.2 such that $\{e^{tu};-\infty<t<\infty\}\subseteqq G(E)\subseteqq O_n$. Let $u_{\mu,\nu}=-u_{\nu,\mu}$ for $1\leqq\nu\leqq\mu\leqq$ n, so that $u=\sum_{\mu,\nu=1}^{n}u_{\mu,\nu}e_\mu\otimes e_\nu$.

Define now the following relation on $\{I_j\}_{j=1}^k$: I_j and I_i will be called *friends* if $i\neq j$ and there exists $\mu\in I_j$ and $\nu\in I_i$ such that $u_{\nu,\mu}\neq 0$.

If I_j has no friends, then $u=u_1+u_2$, where $u_1=\sum_{\mu,\nu\in I_j}u_{\mu,\nu}e_\mu\otimes e_\nu$, $u_2=\sum_{\mu,\nu\notin I_j}u_{\mu,\nu}e_\mu\otimes e_\nu$. If $u_1\neq 0$, then $e^{tu}|_{E_j}=e^{tu_1}|_{E_j}$ is an infinite subgroup of $G(E_j)$. If $u_1=0$, then $u_2\neq 0$ and $e^{tu}|_{\tilde E_j}=e^{tu_2}|_{\tilde E_j}$ is an infinite subgroup of the isometries of $\tilde E_j=[b_\mu;\mu\in\{1,2,\ldots,n\}\sim I_j]$.

We can therefore assume I_1 has a friend, then let P be the maximal subset having the following property: $1 \in P$ and for each $p \in P$ there exist $q_0, q_1, \ldots, q_m \in P$, $q_0 = 1$, $q_m = p$, such that I_{q_i} and $I_{q_{i-1}}$ are friends $(1 \leqq i \leqq m)$.

Without loss of generality assume $P = \{1, 2, \ldots, p_0\}$, $2 \leqq p_0 \leqq k$. Let $I = \bigcup_{p=1}^{p_0} I_p$, then $u = v_1 + v_2$, where $v_1 = \sum_{\mu, \nu \in I} u_{\mu,\nu} e_\mu \otimes e_\nu$, $v_2 = \sum_{\mu, \nu \notin I} u_{\mu,\nu} e_\mu \otimes e_\nu$.

If $p_0 < k$, then the same argument as above shows that e^{tv_1} or e^{tv_2} restricted to the proper subspace is an infinite group of isometries of that subspace. If $p_0 = k$, for every p there exists q such that I_p and I_q are friends. Let $x = \sum_{i=1}^{n} e_i$, and $y \neq 0$ be orthogonal to $\operatorname{span}\{gug^{-1}(x); g \in G(E)\}$. Given any p, choose $\mu \in I_p$, $\nu \in I_q$ such that $u_{\mu,\nu} \neq 0$, then as in the proof of Theorem 1.2, we get an equation similar to (*) with μ, ν instead of $1, 2$, from which $y_{\pi(\mu)} = y_{\pi(\nu)}$ for all permutations π such that $g_\pi \in G(E)$. Since $I_p \cap I_q = \emptyset$ and G_p are permutation transitive and extend to isometries of $G(E)$, it follows that $y_s = y_t$ for all $s \in I_p$, $t \in I_q$, and since $P = \{1, 2, \ldots, k\}$, we get $y = \lambda x$. Hence, $\dim \operatorname{span}\{gug^{-1}(x); g \in G(E)\} = n - 1$, therefore $E = l_2^n$ by Lemma 1.3, proving the theorem. □

Corollary 1.6. *Suppose E_j $(j = 1, 2, \ldots, n)$ is a finite-dimensional normed space, $\dim(E_j) \geqq 2$, $G(E_j)$ is finite, and E_j possesses a* 1-uc *permutation transitive basis. Then the space $\left(\sum_{j=1}^{n} \oplus E_j\right)_{l_2}$ has a finite group of isometries.*

Proof. Otherwise, applying Theorem 1.5 several times, each time reducing the size of the set α, we shall eventually obtain $\alpha = \{j_0\}$ for some j_0, implying $G(E_{j_0})$ is infinite, but this ontradicts the assumption. □

Remark. The Corollary is false if $\dim(E_j) = 1$, since l_2^n is a counter example. Since l_p^n for $p \neq 2$ has $G(l_p^n)$ finite, the Corollary applies for sums of these spaces.

2. Uniqueness of a (Δ) Basis

The main result of this section is the uniqueness of a (Δ) basis for a finite or infinite-dimensional real Banach space E which is proved in Theorem 2.12. The Lemmas preceding the theorem are all essential for the proof. *We shall assume throughout this section that $G(E)$ is a finite group and* $\dim(E) < \infty$ *unless it is mentioned otherwise.*

$\|\cdot\|_2$ will always denote the ellipsoid of least volume which contains $B(E)$. A point $x \in E$ is called a *reflection point* (cf. [1]) if $\|x\|_2 = 1$ and $I - 2x \otimes x \in G(E)$, where I is the identity operator and $x \otimes x$ the rankone operator defined by $x \otimes x(y) = (x, y)x$. Let R be the set of all reflection points. Observe that $g(R) = R$ for all $g \in G(E)$, since if $x \in R$ and $g \in G(E)$, then $\|g(x)\|_2 = \|x\|_2 = 1$ and $I - 2g(x) \otimes g(x) = g(I - 2x \otimes x)g^{-1} \in G(E)$, since $G(E) \subset O_n$.

Lemma 2.1. *If $x, y \in R$, then $(x, y) = \cos(r\pi)$ where r is a rational number.*

Proof. By choosing the coordinate system we may assume $x=e_1$ and $y=\cos\left(\frac{\theta}{2}\right)e_1 + \sin\left(\frac{\theta}{2}\right)e_2$. Then, if $g=I-2x\otimes x$ and $h=I-2y\otimes y$ we get in matrix form

$$(gh)^m=\begin{pmatrix} \cos m\theta & \sin m\theta & 0\ \dots & 0 \\ -\sin m\theta & \cos m\theta & 0\ \dots & 0 \\ 0 & 0 & 1 & 0\dots 0 \\ 0 & 0 & 0 & 1\dots 0 \\ & \dots\dots\dots & & \end{pmatrix}$$

and since $G(E)$ is finite there exists an integer m such that $\cos(m\theta)=1$, completing the proof. $\square$

The next is Lemma 2.2 [3]:

Lemma 2.2. *Assume $z\in E$ and $x\in R$ satisfies*

$$0<|(x,z)|=\min\{|(y,z)|;\ y\in R \text{ and } (z,y)\neq 0\}.$$

(1) *If $y\in R$ and $(z,y)=0$, then $|(x,y)|=\cos\left(\frac{\pi}{m}\right)$ for some integer $m\geqq 2$.*

(2) *If $z\in R$, then $|(x,z)|=\sin\left(\frac{\pi}{m}\right)$ for some even interger $m\geqq 2$.*

Lemma 2.3. (i) *If $0<r<1/2$ and $0<s\leqq 1/2$ are two rational numbers, then $\frac{1}{2}\cos(s\pi)\neq\cos(r\pi)$.*

(ii) *If $m\geqq 5$ is an integer, then for every rational number r, $2^{-1/2}\sin\left(\frac{\pi}{m}\right)\neq\cos(r\pi)$.*

Proof. We use some well known facts about algebraic number fields, in particular automorphisms of such fields (cf. [4]). Given any positive integer l, let $\xi_l=e^{2\pi i/l}$, then ξ_l is an l-th primitive root of 1, and $Q(\xi_l)$, the extension field obtained by adjoining ξ_l to the rational field Q, is of degree $\varphi(l)$ over Q, where $\varphi(l)$ denotes the Euler function of l, defined by: $\varphi(l)=\prod_{i=1}^{j}(p_i-1)p_i^{k_i-1}$, where $l=p_1^{k_1}p_2^{k_2}\dots p_j^{k_j}$ and the p_i's are distinct primes.

If β is any automorphism of $Q(\xi_l)$, then there exists a positive integer k such that $(k,l)=1$ and $\beta(\xi_l)=\xi_l^k$. Conversely, if k is a positive integer and $(k,l)=1$, then there exists an automorphism β of $Q(\xi_l)$ such that $\beta(\xi_l)=\xi_l^k$.

Let a and b be relatively prime positive integers. Let $[Q(\cos(2\pi a/b));Q]$ and $[Q(\xi_b);Q]\,(=\varphi(b))$ denote the degrees of $Q(\cos(2\pi a/b))$ and $Q(\xi_b)$, respectively, over Q. We shall show

$$2[Q(\cos(2\pi a/b));Q]=[Q(\xi_b);Q]. \tag{$*$}$$

Clearly, $\cos(2\pi a/b)=2^{-1}(\xi_b^a+\xi_b^{-a})$, so $Q(\cos(2\pi a/b))\subset Q(\xi_b^a)=Q(\xi_b)$. Since $(i\sin(2\pi a/b))^2=(\cos(2\pi a/b))^2-1$, it follows that $[Q(\xi_b^a);Q(\cos(2\pi a/b))]\leqq 2$. But the identity automorphism of $Q(\xi_b)$ as well as the automorphism that sends ξ_b to ξ_b^{-1} leave $\cos\left(\frac{2\pi a}{b}\right)$ fixed, whence $[Q(\xi_b^a);Q(\cos(2\pi a/b))]=2$ and ($*$) follows.

We now turn to the proof of (i) and (ii). Suppose that $r=p/q$, where p and q are positive relatively prime integers. If p is odd, then $\cos(p\pi/q)=\cos(2p\pi/2q)$, while if p is even, then q is odd and for $\bar{p}=p/2$ we have $\cos(p\pi/q)=\cos(\bar{p}\cdot 2\pi/q)$. Hence there exists an automorphism β (of an appropriate algebraic number field containing $\cos(r\pi)$) such that

$$\beta(\cos(r\pi))=\begin{cases}\cos(\pi/q) & \text{if } \ p \text{ is odd}\\ \cos(2\pi/q) & \text{if } \ p \text{ is even}\end{cases}. \tag{I}$$

To prove (i), suppose that equality holds in (i) for $s=p_1/q_1$, where p_1 and q_1 are positive relatively prime integers. For the automorphism β given by (I) one can show in a similar way that there exists p_2 such that

$$\beta(\cos(p_1\pi/q_1))=\cos(p_2\pi/q_1), \qquad (p_2,q_1)=1 \tag{II}$$

while clearly $\beta(\frac{1}{2})=\frac{1}{2}$. Therefore, $\cos(\pi/q)=\frac{1}{2}\cos(p_2\pi/q_1)$ if p is odd, and $\cos(2\pi/q)=\frac{1}{2}\cos(p_2\pi/q_1)$ if p is even. Thus, if p is odd, $\cos(\pi/q)\leqq\frac{1}{2}$, so $q=3$, and then $p=1$, so $\frac{1}{2}\cos(s\pi)=\cos(\pi/3)=\frac{1}{2}$ implies $s=0$ which contradicts $s>0$. If p is even, $|\cos(2\pi/q)|\leqq\frac{1}{2}$ implies $q=3,4,5$ or 6 but since q is odd, $q=3$ or 5, but p is even and $p/q<\frac{1}{2}$, hence $p=2$ and $q=5$. We then obtain

$$2^{-1}\cos(p_2\pi/q_1)=\cos(2\pi/5)=\frac{\sqrt{5}-1}{4} \tag{III}$$

hence $[Q(\cos(2\pi/5));Q]=2$. If p_2 is odd then by $(\substack{*\\*})$ and (III) $\varphi(2q_1)=4$, but the only integers l such that $\varphi(l)=4$ are 5, 8, 10, 12, whence $q_1=4,5$ or 6. But in these cases (III) cannot hold. If p_2 is even, then q_1 is odd, and so $\varphi(q_1)=4$, hence $q_1=5$, and again (III) cannot hold.

To prove (ii), let r be as in (i) and β be the automorphism given by (I), and assume equality holds in (ii). Since $(\sqrt{2})^2=2$, $\beta(2^{-1/2})=\pm 2^{-1/2}$, whence

$$\beta(2^{-1/2}\sin(\pi/m))=\beta(2^{-1/2})\,\beta(\cos((m-2)\pi/2m))=2^{-1/2}\cos(p_3\pi/2m) \tag{IV}$$

for some positive integer p_3. Suppose first that p is odd. Since $m\geqq 5$ it is clear by (IV) that equality in (ii) implies $q\leqq 4$, so $r=1/4$ or $1/3$ and obviously $2^{-1/2}\sin(\pi/m)\neq\cos(r\pi)$ in these cases. If p is even then q is odd and since $2^{-1/2}\cos(p_3\pi/2m)\neq\cos(2\pi/q)$ whenever $q\geqq 9$, it remains to consider the values $q=5$ or 7, in which cases $r=2/5$ or $2/7$. It is easily seen that $2^{-1/2}\sin(\pi/m)\neq\cos(r\pi)$ in these cases. □

Lemma 2.4. *Assume $\{b_i\}_{i=1}^n$ is* 1-uc *basis for E, and let $e_i=b_i/\|b_i\|_2$ $(i=1,2,\ldots,n)$. If $\cos(k\pi/m)e_i+\sin(k\pi/m)e_j$ is in R where $(k,m)=1$, then $\cos(t\pi/m)e_i+\sin(t\pi/m)e_j\in R$ for all $t=1,2,\ldots,2m$.*

Proof. Without loss of generality assume $i=1, j=2$, and write $\theta/2=\pi k/m$. Let $g=I-2e_1\otimes e_1$ and $h=I-2y\otimes y$, where $y=\cos(k\pi/m)e_1+\sin(k\pi/m)e_2$. Since $\{e_i\}_1^n\subset R$, then

$$g(gh)^l=\begin{pmatrix}-\cos(l\theta) & -\sin(l\theta) & 0\ldots 0\\ -\sin(l\theta) & \cos(l\theta) & 0\ldots 0\\ 0 & 0 & 1\ldots 0\\ & \cdots & \end{pmatrix}$$

is in $G(E)$ and equal to $I-2z\otimes z$ where $z=\cos(kl\pi/m)e_1+\sin(kl\pi/m)e_2$, so $z\in R$. Since $(k,m)=1$, $ku=mv+1$ for some integers u,v. Taking $l=ut$, implies $\cos(t\pi/m)e_1+\sin(t\pi/m)e_2\in R$. □

Lemma 2.5. *Let $\{b_i\}_1^n$ and $\{e_i\}_1^n$ be as in Lemma 2.4. If $\cos\alpha e_i+\sin\alpha e_j$ and $\cos\beta e_i+\sin\beta e_j$ are in R, then $\cos(k\alpha+l\beta)e_i+\sin(k\alpha+l\beta)e_j\in R$ for every integers k, l.*

Proof. Similar to the proof of Lemma 2.4. □

Lemma 2.6. *Let π_n denote the group of all permutations of $\{1,2,\dots,n\}$ and let G be a transitive subgroup of π_n. If n is* odd, *then given any $i\neq j$, $1\leq i,j\leq n$, there exists $\pi\in G$ such that $\pi(i)=j$ and $\pi(j)\neq i$.*

Proof. We assume for simplicity that $i=1$, $j=2$. Suppose to the contrary that $\pi(1)=2$ implies $\pi(2)=1$ for all $\pi\in G$. Let $F_i=\{\sigma\in G;\sigma(1)=i\}$, $i=1,\dots,n$. We show first that for any i, $i=1,\dots,n$, there exists an integer a_i such that $\sigma(2)=a_i$ for all $\sigma\in F_i$. Indeed, let $\theta\in G$ be such that $\theta(i)=2$, then for any $\sigma\in F_i$, we have $(\theta\circ\sigma)(1)=2$, so by our assumption $(\theta\circ\sigma)(2)=1$, therefore $\sigma(2)=\theta^{-1}(1)$ for any $\sigma\in F_i$, hence $\sigma(2)=a_i$ independent on the choice of $\sigma\in F_i$.

We claim that $a_i\neq a_j$ if $i\neq j$. Suppose $a_i=a_j$, choose $\sigma\in F_i$, $\pi\in F_j$ and let $\theta\in G$ be such that $\theta(i)=2$. Then $(\theta\circ\sigma)(1)=\theta(i)=2$, so $\theta(a_i)=(\theta\circ\sigma)(2)=1$. Now, $\pi(2)=a_j=a_i=\theta^{-1}(1)$, therefore $\pi^{-1}\theta^{-1}(1)=2$ hence $\pi^{-1}\theta^{-1}(2)=1$, that is $i=\theta^{-1}(2)=\pi(1)=j$.

We shall now see that $a_{a_i}=i$. Let $\sigma\in F_i$ and $\theta\in G$ be such that $\theta(i)=2$. Then $(\theta\circ\sigma)(1)=2$, so $(\theta\circ\sigma)(2)=1$, that is $\theta^{-1}(1)=\sigma(2)=a_i$, therefore $\theta^{-1}(2)=a_{a_i}$, but also $\theta^{-1}(2)=i$.

The map $\varphi\in\pi_n$ defined by $\varphi(k)=a_k(k=1,2,\dots,n)$ satisfies $\varphi^2=$identity and $\varphi(k)\neq k$ for all k, hence n must be even, contradicting the assumption. □

Lemma 2.7. *Assume $B=\{b_i\}_{i=1}^n$ is a* 1-uc *permutation transitive basis for E, and let $e_i=b_i/\|b_i\|_2$ $(1\leq i\leq n)$. Assume $m\neq 2,4,6$ is the integer such that $\sin(\pi/m)=\min\{|(\xi,e_1)|;(\xi,e_1)\neq 0,\ \xi\in R\}$. Then m and n are even integers and there exists a subset $S\subseteq\{1,2,\dots,n\}\times\{1,2,\dots,n\}$ such that*

$$R=\{\varepsilon\sin(k\pi/m)e_i+\delta\cos(k\pi/m)e_j;|\varepsilon|=|\delta|=1,\ 1\leq k\leq m,(i,j)\in S\}.$$

Proof. First recall that by Lemma 1.1 $\|b_i\|_2=\|b_1\|_2$ and $(b_i,b_j)=0$ for all $1\leq i\neq j\leq n$. $\{e_i\}_1^n$ is then a 1-uc permutation transitive and $(.,.)$ – orthonormal basis for E, hence $R\supseteq\{e_i\}_1^n$. By (2) of Lemma 2.2 m is an even integer. Let $\xi=\sum_1^n\xi_ie_i\in R$ be such that $\xi_1=\sin\frac{\pi}{m}$ and $\xi_i\geq 0$ for all i. By (1) of Lemma 2.2 $\xi_i=\cos(\pi/m_i)$ since $(e_i,e_1)=0$ for all $i>1$.

Since $\sum_1^n\xi_i^2=1$ and $\xi_1\neq 0$, ξ cannot contain more than four non-zero coefficients. There are exactly four cases where ξ has three or more non-zero entries, namely the set of non-zero entries is $\{1/2,\ 1/2,\ 1/2,\ \sin(\pi/m)\}$, $\{1/2,1/2,\sin(\pi/m)\}$, $\{1/2,1/\sqrt{2},\sin(\pi/m)\}$ or $\{1/2,\cos(\pi/5),\sin(\pi/m)\}$, but the assumption on m and the fact that $\sum_1^n\xi_i^2=1$ clearly exclude these cases.

Hence it remains to consider the case $\xi=\sin(\pi/m)e_1+\cos(\pi/m)e_j$, where $1<j\leqq n$. Let $x=\sum_1^n x_ie_i$ be any point in R. Without loss of generality assume $t=\sqrt{x_1^2+x_j^2}>0$. We'll prove $t=1$. Let $x_1=t\sin\theta$, $x_j=t\cos\theta$. Let $z_k=\cos(k\pi/m)e_1+\sin(k\pi/m)e_j$, then $z_k\in R$ and $((I-2z_k\otimes z_k)(x),e_1)=-t\sin(\theta+2k\pi/m)$. There exists l such that $|\theta+2l\pi/m|\leqq\pi/m$, so replacing x by $y^l=(I-2z_l\otimes z_l)(x)$ we may assume $|\theta|\leqq\pi/m$. Note that $y_i^l=x_i$ for all $i, i\neq 1, j$.

If $\theta\neq 0$ then $t=1$, since otherwise $0<|x_1|=|t\sin\theta|<\sin(\pi/m)$. It remains to check the case $\theta=0$. We have $x=\sum_{i\neq 1,j} x_ie_i+te_j$. Let $g\in G(E)$ be a permutation on $\{e_i\}_1^n$ such that $g(e_j)=e_1$. Let $y=g(x)$. There are two cases:

(a) $g(e_1)\neq e_j$. Here $(g(\xi),\xi)=2^{-1}\sin(2\pi/m)$, which is not $\cos(r\pi)$ for rational r by Lemma 2.3 (*i*), hence this case cannot occur.

(b) $g(e_1)=e_j$. Hence $y_j=0$, define $h=I-2y\otimes y$. Then $h(\xi)\in R$ and $(h(\xi),e_1)=(1-2t^2)\sin(\pi/m)$.

If $t<1$ then $|(1-2t^2)\sin(\pi/m)|<\sin(\pi/m)$, so by assumption of the Lemma $1-2t^2=0$, so $t=2^{-1/2}$. By Lemma 2.3 (ii), $(y,\xi)=2^{-1/2}\sin(\pi/m)$ cannot be $\cos(r\pi)$ for rational r. This contradiction implies $t=1$.

We conclude that $x=\pm\sin(k\pi/p)e_1\pm\cos(k\pi/p)e_j$ for some k,p such that $(k,p)=1$. If p does not divide m, then by the right choice of integers a,b, we get $0<|\sin(ak\pi/p+b\pi/m)|<\sin(\pi/m)$, and since by Lemma 2.5 $\sin(ak\pi/p+b\pi/m)e_1+\cos(ak\pi/p+b\pi/m)e_j\in R$, this is a contradiction. Hence p divides m and the representation for R follows. It remains to show n is even. Assume n is odd, let $\xi=\sin(\pi/m)e_1+\cos(\pi/m)e_j\in R$. By Lemma 2.6 there exists $g\in G(E)$, a permutation on $\{e_i\}_{i=1}^n$, such that $g(e_1)=e_j$ and $g(e_j)\neq e_1$. Hence $(g(\xi),\xi)=2^{-1}\sin(2\pi/m)$ which is impossible. Therefore n is even. □

Lemma 2.8. *Assume $\{b_i\}_{i=1}^n$ is a* 1-uc *permutation transitive basis for E, and let $e_i=b_i/\|b_i\|_2$, $i=1,2,\ldots,n$. Assume $\min\{|(\xi,e_1)|;\ (\xi,e_1)\neq 0,\ \xi\in R\}\geqq 1/2$. Then every $x\in R$ has one of the following forms:*

(1) $x=\pm e_i$, $1\leqq i\leqq n$.

(2) $x=2^{-1/2}(\pm e_i\pm e_j)$, $1\leqq i\neq j\leqq n$.

(3) $x=2^{-1}(\pm e_{i_1}\pm e_{i_2}\pm e_{i_3}\pm e_{i_4})$, $1\leqq i_1<i_2<i_3<i_4\leqq n$.

(4) $x=2^{-1}(\pm e_i\pm\sqrt{3}e_j)$, $1\leqq i\neq j\leqq n$.

(5) $x=2^{-1}(\pm e_{i_1}\pm e_{i_2})\pm 2^{-1/2}e_{i_3}$, $1\leqq i_1\neq i_2\neq i_3\neq i_1\leqq n$.

Proof. Write $x=\sum_1^n x_ie_i$, then $x_i=0$ or $|x_i|\geqq 1/2$ for each i, we may assume $x_i\geqq 0$. There are several distinct possibilities:

(a) Only one of the x_i's is not zero. Then $x=e_i$.

(b) At least four x_i's are not zero, then x is as in (3).

(c) Exactly three of the x_i's are not zero, say for simplicity $1/2\leqq x_1\leqq x_2\leqq x_3$. Then, $1=\sum_1^3 x_i^2\geqq 3x_1^2$ implies $2^{-1}\leqq x_1\leqq 3^{-1/2}$, similarly $1=\sum_1^3 x_i^2\geqq 2x_2^2+4^{-1}$ implies $x_1\leqq x_2\leqq\sqrt{3}/2\sqrt{2}$, and $1=\sum_1^3 x_i^2\geqq x_3^2+1/2$ implies $x_2\leqq x_3\leqq 2^{-1/2}$.

Let $g=I-2x\otimes x$, $y=x_1e_1-x_2e_2+x_3e_3\in R$. Then, $(g(y),e_1)=x_1(1-2x_1^2+2x_2^2-2x_3^2)=x_1(4x_2^2-1)$. If $4x_2^2=1$, then (5) follows. If $4x_2^2\neq 1$, then $2^{-1}\leqq x_1(4x_2^2-1)\leqq x_1(\frac{3}{8}\cdot 4-1)=x_1/2$, implying $1\leqq x_1$, which is impossible.

(d) Exactly two of the x_i's are not zero, say x_1 and x_2. Then $x=\sin(k\pi/m)e_1+\cos(k\pi/m)e_2$, $(k,m)=1$. By Lemma 2.4, $y=\sin(\pi/m)e_1+\cos(\pi/m)e_2\in R$. So $\sin(\pi/m)\geqq 1/2$ hence $m=3,4,5$, or 6. If $m=3,4$ or 6 then Lemma 2.8 is proved. So assume $m=5$, and consider $\xi=\sum_{i=1}^{n}\xi_i e_i\in R$ such that ξ_1 is the least possible positive value which is attained by R. By Lemma 2.2, $\xi_1=\sin(\pi/k)$ where k is even, since $\sin(\pi/k)$ must be less than $\sin(\pi/5)$, hence $k=6$, so $\xi_1=1/2$. Conrider two possibilities for ξ:

(i) There exists $\xi\in R$ with $|\xi_1|=1/2$ such that $\xi=2^{-1}(\pm e_{i_1}\pm e_{i_2}\pm e_{i_3}\pm e_{i_4})$.
(ii) No $\xi\in R$ with $|\xi_1|=1/2$ has the form in (i).

In case (i), $(\xi,y)=\pm 2^{-1}\sin(\pi/5)$ or $2^{-1}(\pm\sin(\pi/5)\pm\cos(\pi/5))$, but by arguments similar to those in Lemma 2.3, these values cannot be $\cos(r\pi)$ for any rational r. Hence (ii) holds. Then arguing for ξ as for x, $\xi=\pm 2^{-1}e_1\pm(\sqrt{3}/2)e_k$ or $\pm 2^{-1}e_1\pm 2^{-1}e_{i_2}\pm 2^{-1/2}e_{i_3}$. In the first case, if $k\neq 2$, $(\xi,y)=2^{-1}\sin(\pi/m)$ which is impossible, and if $k=2$ then by Lemma 2.5 $\sin(\pi/5-\pi/6)e_1+\cos(\pi/5-\pi/6)e_2\in R$, and since $\sin(\pi/5-\pi/6)<1/2$ this is also impossible. If $\xi=\pm 2^{-1}e_1\pm 2^{-1}e_{i_2}\pm 2^{-1/2}e_{i_3}$, then $(\xi,y)=\pm 2^{-1}\sin(\pi/5), 2^{-1}(\pm\sin(\pi/5)\pm\cos(\pi/5))$ or $\pm 2^{-1}\sin(\pi/5)\pm 2^{-1/2}\cos(\pi/5)$, and arguing as in Lemma 2.3 shows that this is never $\cos(\pi r)$ for any rational number r, completing the proof. □

Remark. Lemmas 2.7 and 2.8 determine the form of every point in R, and by Theorem 3.1 of § 3 each of the cases listed can be realized in some n-dimensional Banach space.

Lemma 2.9. *Under the assumptions of Lemma* 2.8, *if R contains a point having the form in* (3), *then either n is even, or n is* odd *and divisible by* 7.

Proof. First observe that if $x=2^{-1}\sum_{k=1}^{4} e_{i_k}\in R$ then the operator $(i_1i_2)(i_3i_4)$ (defined to be the operator which maps $e_{i_1}\to e_{i_2}$, $e_{i_2}\to e_{i_1}$, $e_{i_3}\to e_{i_4}$, $e_{i_4}\to e_{i_3}$ and $e_k\to e_k$ for $k\neq i_1,i_2,i_3,i_4$) is in $G(E)$. This follows easily from the equality

$$(i_1i_2)(i_3i_4)=h_1h_2(I-2x\otimes x)h_3h_4(I-2x\otimes x)$$

where $h_k=I-2e_{i_k}\otimes e_{i_k}$.

Let $M=\left\{x=\sum_{k=1}^{4} e_{i_k};2^{-1}x\in R\right\}$, and denote by $(\xi_1,\xi_2,\ldots,\xi_n)$ the points $\sum_{i=1}^{n}\xi_ie_i$. Obviously, if $u,v\in M$ then $(u,v)=0,2$ or 4, because 1/4 and 3/4 cannot be the scalar products between points of R.

We shall split the set $\{1,2,\ldots,n\}$ into connected components by defining the following equivalence relation: Say that $i,j(1\leqq i,j\leqq n)$ not necessarily distinct integers, are in relation Q, iQj, iff there exist $u_1,u_2,\ldots,u_k\in M$ such that $(u_1,e_i)=(u_k,e_j)=1$ and $(u_r,u_{r+1})>0$ for all $r=1,2,\ldots,k-1$.

Claim. Every connected component (i.e. equivalence class of Q) must be of the following form:

(a) A set consisting of an even number $2m(\geqq 4)$ of integers.

(b) A set consisting of exactly 7 distinct integers.

Obviously, every equivalence class contains at least 4 distinct integers, since every $u \in M$ has 4 non-zero coordinates. To prove the claim, let l be the number of integers in some connected component C. Clearly $l \neq 5$. Henceforth when we talk about a connected component we shall consider the coordinates of the points in M which are "in" this component (we shall say a point $u \in M$ is "in" a given component iff all the indices i for which $(u, e_i) = 1$ are in this component).

We have to consider the case $l \geqq 9$. By relabeling the integers, using the fact that $G(E)$ contains a permutation transitive subgroup, we may suppose $u_1 = (1,1,1,1,0,\ldots)$ and $u_2 = (1,1,0,0,1,1,0,\ldots)$ are "in" C, applying (13) (24) on u_2, also $u_3 = (0,0,1,1,1,1,0,\ldots)$ is "in" C. Since C is connected, there exists u "in" C with 1 entries appearing in the $1,2,\ldots,6$ places and in the $7,8,\ldots$ places.

If two 1 entries of u appear in the $7,8,\ldots$ places, without loss of generality say in the 7 and 8th places, then the other two 1 entries must appear in the 1 and 2, or, 3 and 4, or, 5 and 6 places. But since (13)(24), (15)(26) are in $G(E)$, applying them on u shows that $u_4 = (1,1,0,0,0,0,1,1,0,\ldots)$, $u_5 = (0,0,1,1,0,0,1,1,0,\ldots)$, $u_6 = (0,0,0,0,1,1,1,1,0,\ldots)$ are "in" C, that is $\{u_i\}_{i=1}^6$ are "in" C.

If only one 1 entry of u appears in the $7,8,\ldots$ places, say in the 7th place, then the three other 1 entries must appear in the 1 or 2, and 3 or 4, and 5 or 6 places. Since the permutations (12) (34), (12) (56), (34) (56) are in $G(E)$, if we apply them on u we get that $u'_4 = (1, 0, 1, 0, 1, 0, 1, 0,\ldots)$, $u'_5 = (0, 1, 0, 1, 1, 0, 1, 0,\ldots)$, $u'_6 = (0,1,1,0,0,1,1,0,\ldots)$, $u'_7 = (1,0,0,1,0,1,1,0,\ldots)$ are "in" C (one must keep in mind the fact that the scalar product of any two distinct points "in" the same component must be equal to 2 or zero). Since C is connected there is a u'_8 with 1 entries appearing in the $1,2,\ldots,7$ places, and in the $8,9,\ldots$ places. Checking scalar product of u'_8 with u_i, $u'_j (1 \leqq i \leqq 3 < j \leqq 7)$, u'_8 has three 1 entries in the $1,2,\ldots,7$ places, and we may suppose u'_8 has 1 entry in the 8th place. Checking the possibilities the three 1 entries appear in the 127 or 136 or 145 or 235 or 246 or 347 or 567 places.

The cases 127 or 347 or 567 immediately show that $\{u_i\}_{i=1}^6$ are "in" C. The case 235: $u'_8 = (0,1,1,0,1,0,0,1,0,\ldots)$, and applying on u'_8 the permutation $(13)(57) \in G(E)$, we get $(1,1,0,0,0,0,1,1,0,\ldots)$, and again $\{u_i\}_{i=1}^6$ are "in" C. By similar arguments the other cases also imply $\{u_i\}_1^6$ is "in" C.

Since $l \geqq 9$, there exists a v "in" C with 1 entries appearing in the $1,2,\ldots,8$ places, and in the $9,10,\ldots$ places. Checking scalar products with elements of $\{u_i\}_{i=1}^6$, v must have two 1 entries in $1,2,\ldots,8$ places, and two 1 entries in, say, the 9 and 10th places. Continuing, in this manner, it is now obvious that l must be an even number, proving the claim.

We shall prove now that if n is an odd number then $7|n$. Let k be the number of connected components of length 7. $k \geqq 1$, since n is odd. Let $S_1, S_2, \ldots, S_k$ be the sets of indices corresponding to such components, and $S_{k+1}, \ldots, S_m$ the sets corresponding to components of even length. We may assume $S_i = \{7i-6, 7i-5, \ldots, 7i\}$ $(1 \leqq i \leqq k)$. Let $g \in G(E)$ be a permutation on $\{e_j\}$ and P be the permutation induced by g on $\{1,2,\ldots,n\}$.

We claim that if $P(S_i)\cap S_j \neq \emptyset$ $(1 \leqq i,j \leqq k)$ then $P(S_i)=S_j$. Assume $i=1$. Observe that if $u\in M$ has some of its support on S_1 then all its support is in S_1 since S_1 is connected, so without loss of generality we can assume $u_1=(1,1,1,1,0,\ldots)$ and $u_2=(1,1,0,0,1,1,0,\ldots)$ are "in" S_1. Following earlier arguments also u_3, u'_4, u'_5, u'_6, u'_7 are "in" S_1. It is easy to check that no other $u\in M$ can be "in" S_1, so only these seven points form S_1.

If $P(S_1)\neq S_j$ and $P(S_1)\cap S_j\neq\emptyset$, pick $i_1,i_2\in S_1$ so that $P(i_1)\in S_j$ $P(i_2)\notin S_j$. Considering the seven points of M "in" S_1, there is $u\in M$ "in" S_1 with $(u,e_{i_1})=(u,e_{i_2})=1$, but then $g(u)\in M$ has only a part of its support in S_j, which is impossible because S_j is connected. This proves the claim.

Let $S=\bigcup_{t=k+1}^{m} S_t$. It now follows that if $P(S_i)\cap P'(S_j)\cap S\neq\emptyset$ $(1\leqq i,j\leqq k)$ for some permutations P, P' corresponding to elements in $G(E)$, then $P(S_i)=P'(S_j)\subsetneqq S$. Since $|S_i|=7$ $(1\leqq i\leqq k)$ and $G(E)$ contains a permutation transitive subgroup, it is now obvious that $|S|$ is divisible by 7, hence $7|n$. $\square$

Definition. Let $B=\{b_j\}_{j\geqq 1}$ be a normalized 1-uc basis for a finite or infinite-dimensional Banach space E. We shall say B is a (Δ) basis if in addition the following conditions hold:

(I) There exists a sequence $0=n_0<n_1<n_2<\ldots$ [with $n_k=\dim(E)$ if it is finite] such that $n_i-n_{i-1}\geqq 3$ for each i. If we denote by $B_i=\{b_j\}_{j=n_{i-1}+1}^{n_i}$ and $E_i=\operatorname{span}(B_i)$, then $G(E_i)$ is a finite group, contains a permutation transitive subgroup G_i (with respect to B_i), and every $g\in G_i$ extends to an isometry $\tilde{g}\in G(E)$ by defining:

$$\tilde{g}(b_j)=\begin{cases} g(b_j)\,; b_j\in B_i \\ b_j \;; b_j\notin B_i \end{cases}.$$

(II) For each $i\geqq 1$, one of the following two conditions holds:

(a) n_i-n_{i-1} is odd and indivisible by 7.

(b) For each $p\in\{2,4\}$ and every p distinct integers $\{k_j\}_{j=1}^{p}$, $n_{i-1}<k_j\leqq n_i$, there exists $g\in G_i$ such that $\{g(b_{k_j})\}_{j=1}^{p}\cap\{b_{k_j}\}_{j=1}^{p}$ has cardinality equal to one or three.

Remarks. (1) A 1-symmetric basis for a space not isometric to a Hilbert space is obviously a (Δ) basis if the dimension of the space is not 1, 2 or 4.

(2) If $B_i=\{b_\mu^{(i)}\}_{\mu\geqq 1}$ is a (Δ) basis for $F_i (i=1,2,\ldots)$, then under the proper ordering $\bigcup_{i\geqq 1} B_i$ would be a (Δ) basis for $\left(\sum_{i\geqq 1}\oplus F_i\right)_{l_p}$ $(1\leqq p<\infty)$. The same statement would be usually false if "(Δ) basis" is replaced by "1-symmetric basis"; for example $(l_1^n\oplus l_\infty^n)_{l_2}$ is "highly" non-symmetric by [2], but it has a (Δ) basis.

(3) If B is a (Δ) basis for a finite-dimensional space E, then $G(E)$ is finite. This follows from Theorem 1.5, since we can select $I_i=\{v\}_{v=n_{i-1}+1}^{n_i}$ $(i=1,2,\ldots,k)$, so $|I_i|\geqq 3$, and if $G(E)$ were infinite, by reducing the size of α in Theorem 1.5 sufficiently many times, we would eventually obtain that $G(E_{i_0})$ is also infinite for some i_0.

Lemma 2.10. *Assume E has a (Δ) basis $B=\{b_j\}_{j=1}^{n}$. Then, if $e_j=\|b_j\|_2^{-1}b_j$ $(1\leqq j\leqq n)$, every $x\in R$ has the form* (1) *or* (2) *of Lemma* 2.8.

Proof. By (2) of Lemma 2.2, $\min\{|(\xi,e_j)|;\ \xi\in R, (\xi,e_j)\neq 0\}=\sin(\pi/m_j)$, where m_j is an even integer ($j=1,2,\ldots,n$). Suppose $m_j>6$, then exactly as in the first part of Lemma 2.7, the minimum is attained for $\xi\in R$ of the form $\xi=\sin(\pi/m_j)e_j+\cos(\pi/m_j)e_l$, where $1\leqq j,\ l\leqq n$.

Claim. There exists i, $1\leqq i\leqq k$, such that $\{e_j,\ e_l\}\subset\tilde{B}_i$, where $\tilde{B}_i=\{e_j;\ n_{i-1}<j\leqq n_i\}$.

Otherwise, for example, $1\leqq j\leqq n_1<l$, then since m_j is even, by Lemma 2.4 also $\eta=\cos(\pi/m_j)e_j+\sin(\pi/m_j)e_l\in R$. Since G_1 is permutation transitive on $\tilde{B}_1$, there exists $g\in G_1$ such that $g(e_j)\neq e_j$, then $(\tilde{g}(\xi),\eta)=2^{-1}\sin(2\pi/m_j)$ which is an impossible scalar product by Lemmas 2.1 and 2.3(i).

Having proved the claim, if case (a) applies for n_i-n_{i-1} then by Lemma 2.6, there exists $g\in G_i$ such that $g(e_j)=e_l$ and $g(e_l)\neq e_j$; if case (b) applies, there exists $g\in G_i$ such that $\{g(e_j),\ g(e_l)\}\cap\{e_j,e_l\}$ contains only one element, so in all cases $(\tilde{g}(\xi),\xi)$ or $(\tilde{g}(\xi),\eta)$ is $2^{-1}\sin(2\pi/m_j)$ which is impossible.

We therefore proved that $m_j\leqq 6$. We can now repeat the proof of Lemma 2.8 step by step to show that every $x\in R$ must have one of the forms listed in Lemma 2.8. We claim that only forms (1) or (2) are possible.

Indeed, if $x=2^{-1}e_j+\sqrt{3}/2e_l\in R$, then exactly as above there exists i, $1\leqq i\leqq k$, such that $\{e_j,e_l\}\subset\tilde{B}_i$, and in each of cases (a), (b) there exists $g\in G_i$ so that $\{g(e_j),g(e_l)\}\cap\{e_j,e_l\}$ contains only one element, so $(\tilde{g}(x),x)$ is 1/4 or 3/4 or $\sqrt{3}/4$, but none of these values is $\cos(r\pi)$ for any rational r, so (4) of Lemma 2.8 is impossible.

If $x=2^{-1}(e_{i_1}+e_{i_2}+e_{i_3}+e_{i_4})\in R$ we have to consider several possibilities. If $\{e_{i_j}\}_{j=1}^4\subseteq\tilde{B}_i$ for some i, $1\leqq i\leqq k$, then case (a) is not possible here because of Lemma 2.9, but case (b) is also impossible, because otherwise there would exist $g\in G_i$ such that $\{g(e_{i_j})\}_{j=1}^4\cap\{e_{i_j}\}_{j=1}^4$ consists of one or three elements, hence $(\tilde{g}(x),x)=1/4$ or 3/4, and both values are impossible.

Therefore, there exists i, $1\leqq i\leqq k$, such that $\tilde{B}_i\cap\{e_{i_j}\}_{j=1}^4$ contains one or two elements only. If the intersection contains only one element, say e_{i_1}, since B_i is permutation transitive, there exists $g\in G_i$ so that $g(e_{i_1})\neq e_{i_1}$ hence $(\tilde{g}(x),x)=3/4$ which is impossible. If the intersection contains two elements, say e_{i_1} and e_{i_2}, then as above we can find in each of cases (a), (b) a $g\in G_i$ so that $\{g(e_{i_1}),\ g(e_{i_2})\}\cap\{e_{i_1},e_{i_2}\}$ consists of one element, therefore again $(\tilde{g}(x),x)=3/4$, impossible. Hence (3) is excluded.

If $x=2^{-1}(e_{i_1}+e_{i_2})+2^{-1/2}e_{i_3}\in R$, then repeating previous arguments it follows that there can be no i, $1\leqq i\leqq k$, so that $\{e_{i_j}\}_{j=1}^3\cap\tilde{B}_i$ contains one or two elements. Hence, $\{e_{i_j}\}_{j=1}^3\subseteq\tilde{B}_i$ for some i, $1\leqq i\leqq k$. For simplicity, assume $i_j=j$ ($j=1,2,3$) and $i=1$. We shall show this means $4|n_1$. Since G_1 is permutation transitive on B_1 there exists $g\in G_1$ such that $g(e_3)=e_2$, it is then easy to check that $y=g(x)$ must have the form $y=2^{-1/2}e_2+2^{-1}(e_3+e_{i_4})$ where $i_4\geqq 4$, because any other arrangement on the coordinates of y would give an impossible scalar product (x,y). For simplicity, assume $i_4=4$, and select $h\in G_1$ such that $h(e_3)=e_1$ and let $z=h(x)$, then $z=2^{-1/2}e_1+2^{-1}(e_3+e_4)$, because otherwise (x,z) or (x,y) is an impossible scalar product. Now choose $f\in G_1$ such that $f(e_3)=e_4$, then $f(x)=2^{-1}(e_1+e_2)+2^{-1/2}e_4$ $(=u)$, because otherwise (x,u), (y,u), or (z,u) is impossible. Consider now the points $\{x,y,z,u\}$ of R. Any $w=2^{-1}(e_{k_1}+e_{k_2})+2^{-1/2}e_{k_3}$ in R must belong to $\{x,y,z,u\}$ or must satisfy $k_j \geqq 5$ $(j=1,2,3)$, because otherwise it is easy to see that w will give an impossible scalar product with one of x, y, z, u. Since B_1 is permutation transitive, this implies that $4|n_1$, hence case (b) applies for B_1.

By (b), choose $g_1 \in G_1$ such that $\{e_1, e_2\} \cap \{g_1(e_1), g_1(e_2)\}$ consists of one element only. Then $g_1(x)$ does not belong to the set $\{x, y, z, u\}$, and this is a contradiction. We have shown that only cases (1) or (2) of Lemma 2.8 are possible. $\square$

As usual $\|\cdot\|_2$ denotes the norm defined by the ellipsoid $\mathscr{E}$ of least colume containing the unit ball of E, and $(.,.)$ is the scalar product defined by $\mathscr{E}$. If $x \in E$, $x \otimes x$ will denote the operator defined by $(x \otimes x)(y) = (x, y)x$; if however $x' \in E'$ and $x \in E$, $x' \otimes x$ will denote the operator defined by $(x' \otimes x)(y) = \langle y, x' \rangle x$, where $\langle .,. \rangle$ is the scalar product on $E \times E'$.

If E is any Banach space and $e \in E$, $e' \in E'$, we shall say $e' \times e$ is a *reflection pair*, and write $e' \times e \in \mathbb{R}(E)$ iff $\|e\| = \|e'\| = \langle e, e' \rangle = 1$ and $I - 2e' \otimes e \in G(E)$. Let $\mathbb{R}_1(E) = \{e;\, e' \times e \in \mathbb{R}(E)$ for some $e' \in E'\}$.

Lemma 2.11. *If* $\dim(E) < \infty$ *and* $e \in E$, $e' \in E'$ *are such that* $\langle e, e' \rangle = 1$ *and* $g = I - 2e' \otimes e \in G(E)$, *then* $g = I - 2\|e\|_2^{-2}\, e \otimes e$.

Proof. Let $H = \{x \in E;\, \langle x, e' \rangle = 0\}$, then $g(x) = x$ for all $x \in H$ and $g(e) = -e$. To prove the Lemma it is sufficient to observe that $\dim(E/H) = 1$ and to prove e is $(.,.)$-orthogonal to H. Suppose $e = z + y$ where $z \in H$ and y is $(.,.)$-orthogonal to H. Since g is an isometry, g is an orthogonal transformation, so

$$(z, z) = (e, z) = (g(e), g(z)) = (-e, z) = -(z, z),$$

hence $z = 0$. $\square$

We can now prove the main result of § 2.

Theorem 2.12. *Let E be a real finite or infinite-dimensional Banach space which has two (Δ) bases $B = \{b_i\}_{i \geqq 1}$ and $B_1 = \{d_i\}_{i \geqq 1}$. Then, there exist $\varepsilon_i = \pm 1$ such that $B_1 = \{\varepsilon_i b_i\}_{i \geqq 1}$.*

Proof. Let $\{n_i\}_{i=0}^{\infty}$ and $\{m_i\}_{i=0}^{\infty}$ be the sequences of integers which appear in the definitions of the (Δ) property of B and B_1 respectively. We shall break the proof into two parts: (i) $\dim(E) = n$ is finite, (ii) $\dim(E) = \infty$.

In (i), by Remark (3) following the definition of a (Δ) basis, $G(E)$ is a finite group.

By Lemma 2.11, if $e' \times e \in \mathbb{R}(E)$, $I - 2e' \otimes e = I - 2\|e\|_2^{-2} e \otimes e$, meaning $\langle x, e' \rangle = \|e\|_2^{-2}(x, e)$ for all $x = \sum_{j=1}^{n} x_j b_j \in E$. By Lemma 2.10, since $\|e\| = 1$, either (a) $e = \varepsilon_i b_i$, or (b) $e = b_{i,k}\left(\varepsilon_i \frac{b_i}{\|b_i\|_2} + \varepsilon_k \frac{b_k}{\|b_k\|_2}\right)$, where $|\varepsilon_i| = |\varepsilon_k| = 1$, and $b_{i,k} = \left\| \frac{b_i}{\|b_i\|_2} + \frac{b_k}{\|b_k\|_2} \right\|^{-1}$. Let $\{b_i'\}$ be the coefficient functionals of B.

In case (a), $\langle x, e' \rangle = \|e\|_2^{-2}(x, e) = \varepsilon_i x_i = \langle x, \varepsilon_i b_i' \rangle$, so $e' = \varepsilon_i b_i'$. In case (b), $\langle x, e' \rangle = \|e\|_2^{-2}(x, e) = (x, e)/2b_{i,k}^2 = (x_i \varepsilon_i \|b_i\|_2 + x_k \varepsilon_k \|b_k\|_2)/2b_{i,k}$, which implies $e' = (\varepsilon_i \|b_i\|_2 b_i' + \varepsilon_k \|b_k\|_2 b_k')/2b_{i,k}$.

Let $\{d_j'\}$ be the coefficient functionals of the second basis $B_1=\{d_j\}_{j=1}^n$, then $d_j'\times d_j\in\mathbb{R}(E)$, therefore d_j must have form (a) or (b). We shall see that form (b) is impossible. Assume to the contrary that $d_j=b_{i,k}\left(\frac{b_i}{\|b_i\|_2}+\frac{b_k}{\|b_k\|_2}\right)$, where we take $\varepsilon_i=\varepsilon_k=1$ for simplicity of notation.

Observe that if $e'\times e\in\mathbb{R}(E)$ and $g\in G(E)$, then $\|g(e)\|=\|(g^{-1})'(e')\|=\langle g(e), (g^{-1})'e'\rangle$ and $g(I-2e'\otimes e)g^{-1}\in G(E)$, so $g(\mathbb{R}_1(E))=\mathbb{R}_1(E)$. Since changes of signs on $\{b_\nu\}$ are isometries, also $z=b_{i,k}\left(\frac{b_i}{\|b_i\|_2}-\frac{b_k}{\|b_k\|_2}\right)\in\mathbb{R}_1(E)$. There are then two possibilities for the same reasons given above:

$$z=\varepsilon_l d_l, \tag{1}$$

$$z=d_{l,m}\left(\varepsilon_l\frac{d_l}{\|d_l\|_2}+\varepsilon_m\frac{d_m}{\|d_m\|_2}\right), \tag{2}$$

where

$$d_{l,m}=\left\|\frac{d_l}{\|d_l\|_2}+\frac{d_m}{\|d_m\|_2}\right\|^{-1}, \qquad |\varepsilon_l|=|\varepsilon_m|=1.$$

In case (2), $(2b_{i,k})^{-1}(\|b_i\|_2b_i'-\|b_k\|_2b_k')=(2d_{l,m})^{-1}(\varepsilon_l\|d_l\|_2d_l'+\varepsilon_m\|d_m\|_2d_m')$. Since

$$\left\langle\frac{b_i}{\|b_i\|_2}+\frac{b_k}{\|b_k\|_2}, \|b_i\|_2b_i'-\|b_k\|_2b_k'\right\rangle=0, \quad \langle d_j, \varepsilon_l\|d_l\|_2d_l'+\varepsilon_m\|d_m\|_2d_m'\rangle=0,$$

hence l, m, j are distinct integers. But then

$$2b_{i,k}\frac{b_i}{\|b_i\|_2}=d_{l,m}\left(\varepsilon_l\frac{d_l}{\|d_l\|_2}+\varepsilon_m\frac{d_m}{\|d_m\|_2}\right)+d_j,$$

which is impossible because $b_i\in\mathbb{R}_1(E)$ so must have the form (a) or (b) with respect to $\{d_\nu\}$. It follows from the (Δ) property of the basis B that there exists $p\neq i, k$ such that $b_{k,p}\left(\frac{b_k}{\|b_k\|_2}+\frac{b_p}{\|b_p\|_2}\right)$ or $b_{i,p}\left(\frac{b_i}{\|b_i\|_2}+\frac{b_p}{\|b_p\|_2}\right)$ is in $\mathbb{R}_1(E)$. Assume $w=b_{k,p}\left(\frac{b_k}{\|b_k\|_2}+\frac{b_p}{\|b_p\|_2}\right)\in\mathbb{R}_1(E)$. There are two cases to consider:

$$w=\varepsilon_m d_m, \tag{3}$$

or

$$w=d_{m,q}\left(\varepsilon_m\frac{d_m}{\|d_m\|_2}+\varepsilon_q\frac{d_q}{\|d_q\|_2}\right). \tag{4}$$

If (3) holds, $\varepsilon_m d_m'=(\|b_k\|_2b_k'+\|b_p\|_2b_p')/2b_{k,p}$, but

$$\left\langle\frac{b_i}{\|b_i\|_2}\pm\frac{b_k}{\|b_k\|_2}, \|b_k\|_2b_k'+\|b_p\|_2b_p'\right\rangle\neq 0$$

for both $\pm$ signs, so $\langle d_j, d_m'\rangle$ and $\langle d_l, d_m'\rangle$ are both non-zero, which is impossible.

If (4) holds, since $\left\langle\frac{b_i}{\|b_i\|_2}\pm\frac{b_k}{\|b_k\|_2}, \|b_k\|_2b_k'+\|b_p\|_2b_p'\right\rangle\neq 0$ for both $\pm$ signs, it follows that $\langle d_t, \varepsilon_m\|d_m\|_2d_m'+\varepsilon_q\|d_q\|_2d_q'\rangle\neq 0$ for $t=j$ and $t=l$, and since $j\neq l$, it follows that $\{m,q\}=\{j,l\}$, so $w=d_{j,l}\left(\delta_j\frac{d_j}{\|d_j\|_2}+\delta_l\frac{d_l}{\|d_l\|_2}\right)$ where $|\delta_j|=|\delta_l|=1$, therefore

$$d_{j,l}^{-1}w=2^{-1/2}\left\{\delta_j\left(\frac{b_i}{\|b_i\|_2}+\frac{b_k}{\|b_k\|_2}\right)+\varepsilon_l\delta_l\left(\frac{b_i}{\|b_i\|_2}-\frac{b_k}{\|b_k\|_2}\right)\right\}$$

$$=\pm 2^{1/2}\|b_i\|_2^{-1}b_i \quad \text{or} \quad \pm 2^{1/2}\|b_k\|_2^{-1}b_k$$

which are both impossible. We have shown $B_1\subseteqq\{\pm b_i\}_{i=1}^n$, completing the proof of case (i).

(ii) Let $e'\otimes e\in \mathbb{R}(E)$, we shall first determine the form of e. There are two cases to consider:

(a) $\langle e,b_i'\rangle\neq 0$ for only a finite number of indices i.

(b) $\langle e,b_i'\rangle\neq 0$ for an infinite number of indices i.

In case (a), let $\langle e,b_n'\rangle$ be the last non-zero coordinate of e. Choose $n_k\geqq n$, then $e\in E_{n_k}=[b_1,b_2,\dots,b_{n_k}]$, so $g=I-2e'\otimes e\in G(E_{n_k})$, that is g is an isometry on E_{n_k}. By Lemma 2.11, $g=I-2x\otimes x$ where $x=e/\|e\|_2$ and $\|\cdot\|_2$ is the appropriate Euclidean norm on E_{n_k}. Hence x is a reflection point of E_{n_k} and by Lemma 2.10 must have the form (1) or (2) of Lemma 2.8.

In case (b), choose $N=n_{k-1}$ so that if $e=\sum_{i=1}^{\infty}y_ib_i$ then $\{y_i\}_1^N$ contains at least five non-zero elements. Let $b=\sum_{i>N}y_ib_i$, then $b\neq 0$ and $z=I-2e'\otimes e$ is an isometry of the $N+1$ dimensional space $F=[b_1,b_2,\dots,b_N,b]$. In particular $e\in F$, and the basis $\{b_1,\dots,b_N,b\}$ for F is 1-uc and therefore $(.,.)$ orthogonal with respect to the scalar product on F.

We claim that $G(F)$ is finite. Indeed all the assumptions of Theorem 1.5 are satisfied for this basis if we choose $I_j=\{n_{j-1}+1,\ n_{j-1}+2,\dots,n_j\}$ $(1\leqq j<k)$ and $I_k=\{N+1\}$. Hence if $G(F)$ were infinite, there must be an i_0, $1\leqq i_0<k$, such that $G(E_{i_0})$ is infinite; this follows from applying Theorem 1.5 several times, each time reducing the size of the set α, until $\alpha=\{i_0\}$.

Let $e_j=b_j/\|b_j\|_2$ $(1\leqq j\leqq N)$, $e_{N+1}=b/\|b\|_2$, and let R be the set of reflection points of F. By Lemma 2.11, $x=e/\|e\|_2\in R$ and from assumption (b) $x=\sum_{j=1}^{N+1}x_je_j$ has a coordinate x_i with $0<|x_i|<1/2$, we may assume $i=1$.

Let $\xi=\sum_{j=1}^{N+1}\xi_je_j\in R$ satisfy $\xi_j\geqq 0$ and $\xi_1=\min\{|(\eta,e_1)|;\ \eta\in R,\ (\eta,e_1)\neq 0\}$. Then $0<\xi_1\leqq|x_1|<1/2$, so $\xi_1=\sin(\pi/m)$, $\xi_j=\cos(\pi/p_j)$ for $1<j\leqq N+1$. Since $\cos(\pi/p_j)=0$ or $\geqq 1/2$, it follows that $\xi=\sin(\pi/m)e_1+\cos(\pi/m)e_l$ for some l, $1<l\leqq N+1$, and where $m>6$.

If $l=N+1$, then $\xi^{(p)}=\sin(p\pi/m)e_1+\cos(p\pi/m)e_{N+1}\in R$ for every integer p, and by Lemma 2.2 m is even, so taking $p=m/2-1$, $\tilde{\xi}=\cos(\pi/m)e_1+\sin(\pi/m)e_{N+1}\in R$, and applying on $\tilde{\xi}$ an isometry g which is a permutation on the basis elements and which satisfies $g(e_1)=e_2$, $g(e_{N+1})=e_{N+1}$, we get $(g(\tilde{\xi}),\xi)=2^{-1}\sin(2\pi/m)$, which is never $\cos(r\pi)$ for any rational r.

Hence $1<l\leqq N$, then $\xi\in[e_1,e_2,\ldots,e_N]$, and since $m>6$, similarly as in the proof of Lemma 2.10 we easily get a contradiction. Therefore case (b) cannot occur.

Hence,

$$e=\varepsilon_i b_i \quad\text{or}\quad b_{i,j}\left(\varepsilon_i\frac{b_i}{\|b_i\|_2}+\varepsilon_j\frac{b_j}{\|b_j\|_2}\right), \qquad |\varepsilon_i|=|\varepsilon_j|=1,$$

in which cases as in part (i) of the proof $e'=\varepsilon_i b_i'$ or $(\varepsilon_i\|b_i\|_2 b_i'+\varepsilon_j\|b_j\|_2 b_j')/2b_{i,j}$ respectively. The proof is concluded as in (i) by showing that $B_1=\{\varepsilon_j b_j\}_{j=1}^{\infty}$. □

Theorem 2.13. *Suppose E has a* (Δ) *basis* $B=\{b_i\}_{i\geqq 1}$. *Then every onto isometry of E acts as a permutation and changes of signs on B.*

Proof. If g is an isometry of E, them $\{g(b_i)\}_{i\geqq 1}$ is also a (Δ) basis for E, so there exist $\varepsilon_i=\pm 1$ such that $\{g(b_i)\}_{i\geqq 1}=\{\varepsilon_i b_i\}_{i\geqq 1}$. □

Remarks. (1) Examples of finite-dimensional spaces having (Δ) bases are very easy to construct with the aid of Theorem 3.1 which claims that if G is a finite group of operators on R^n which contains $-I$, then there exists a Banach space E such that $G(E)=G$. Let therefore E_n $(n=1,2,\ldots)$ be finite-dimensional spaces having (Δ) bases, and let $E=\left(\sum_{n\geqq 1}\oplus E_n\right)_{l_p}$ $(1\leqq p<\infty)$, then E has a natural unique (Δ) basis in the sense of Theorem 2.12. Thus, it is easy to construct spaces having a (Δ) basis of a particular kind, and not having a 1-symmetric basis for example.

(2) The uniqueness of a (Δ) basis cannot be guaranteed if one $G(E_i)$ were infinite, for example when E_i is a Hilbert space for some i.

Each of conditions (a), (b) in defining the (Δ) property cannot be generally weakened if one aims at retaining the uniqueness property. For example, if $4|n$ one can construct a Banach space $E=(R^n,\|\cdot\|)$ by using Theorem 3.1 with two 1-uc permutation transitive normalized bases $\{b_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^n$ where each x_i has the form $\pm 2^{-1}(b_{i_1}\pm b_{i_2})\pm 2^{-1/2}b_{i_3}$. For simplicity, if $n=4$, let $x_1=2^{-1/2}b_1-2^{-1}(b_3-b_4)$, $x_2=2^{-1/2}b_2-2^{-1}(b_3+b_4)$, $x_3=2^{-1/2}b_3+2^{-1}(b_1+b_2)$, $x_4=2^{-1/2}b_4-2^{-1}(b_1-b_2)$, where $\{b_i\}_1^4$ is the standard basis of l_2^4, so $\{x_i\}_2^4$ is also orthonormal.

Let G be the group of operators on R^4 generated by $-I$ and the operators $I-2b_i\otimes b_i$, $I-2x_i\otimes x_i$ and the permutations g_1, g_2 defined by $g_1: b_1\to b_2, b_2\to b_1, b_3\to b_4, b_4\to b_3$, and $g_2: b_1\to b_3, b_3\to b_1, b_2\to b_4, b_4\to b_2$. It is easy to check that G is a finite group, transitive on $\{b_i\}$ and also on $\{x_i\}$. By Theorem 3.1 G can be realized as $G(E_4)$ for some $E_4=(R^4,\|\cdot\|)$. Both bases $\{b_i\}$ and $\{x_i\}$ are then 1-uc normalized and permutation transitive. This can be done for any $n=4k$, by taking $E_n=(E_4\oplus E_4\oplus\ldots\oplus E_4)_{l_1}$. Similarly, examples with two "different" 1-uc normalized permutation transitive bases can be constructed in the cases when n is odd and divisible by 7, or n is even, and in other situations as well where the conditions are weakened. We omit the details which are straightforward but lengthy.

3. Realization of Groups of Linear operators as Isometries

We saw in § 1 that if $E=(R^n,\|\cdot\|)$ is a normed space then there exists an invertible operator T an R^n such that $TG(E)T^{-1}$ is a subgroup of O_n. The following Theorem implies that any finite group of linear operators on R^n which contains $-I$ can be realized as the group of isometries of some Banach space.

Theorem 3.1. *Let $G \subset O_n$ be a finite subgroup which contains $-I$, and let $\varepsilon > 0$. Then there exists a Banach space E such that $d(E, l_2^n) \leqq 1 + \varepsilon$ and $G(E) = G$.*

Proof. Select $x_1 \in S_n$ such that $x_1 \neq g(e_i)$ for all $g \in G$ and all $i = 1, 2, \ldots, n$, and $x_1 \neq g(x_1)$ for all $g \in G \sim \{I\}$. Let 4δ be the least positive distance between the distinct points in the set $\{g(e_i), g(x_1);\ g \in G, 1 \leqq i \leqq n\}$.

Select now the points $x_2, x_3, \ldots, x_n \in S_n$ to satisfy the following requirements:

(1) $\operatorname{span}\{x_i\}_{i=1}^n = R^n$.
(2) $\|x_i - x_j\|_2 < \delta$, for all $1 \leqq i, j \leqq n$.
(3) If $T \in O_n$ satisfies $\{T(x_i)\}_{i=1}^n = \{x_i\}_{i=1}^n$, then $T = I$.

Condition (3) is satisfied if the points x_i are chosen so that $(x_i, x_j) = (x_k, x_l)$ implies $\{i, j\} = \{k, l\}$, or, $i = j$ and $k = l$.

Let $B = \operatorname{co}\{g(x_i), g(e_i);\ g \in G, 1 \leqq i \leqq n\}$, and let E_0 be the normed space with B as its unit ball. Since Σ_n is the ellipsoid of least volume which contains $\{\pm e_i\}_{i=1}^n$ $(\subset B)$, Σ_n is therefore also the ellipsoid of least volume which contains B, hence $G(E_0) \subset O_n$, and clearly $G \subseteqq G(E_0)$.

We claim that $G(E_0) = G$. To see this let $h \in G(E_0)$. Since

$$h(\operatorname{Ext}(B)) = \operatorname{Ext}(B), \quad \{h(x_i)\}_{i=1}^n \subset \{g(x_i), g(e_i);\ g \in G, 1 \leqq i \leqq n\}.$$

If $h(x_j) = g(e_i)$, then taking $k \neq j$ $\|h(x_j) - h(x_k)\|_2 = \|x_j - x_k\|_2 < \delta$. On the other hand, if $h(x_k) = g'(e_p)$ for some p and $g' \in G$, then $\|g(e_i) - g'(e_p)\|_2 \geqq 4\delta$, which is a contradiction, and if $h(x_k) = g'(x_p)$ then

$$\|g(e_i) - g'(x_p)\|_2 \geqq \|g(e_i) - g'(x_1)\|_2 - \|g'(x_1) - g'(x_p)\|_2 \geqq 4\delta - \delta = 3\delta,$$

which is again impossible. Hence

$$\{h(x_i)\}_{i=1}^n \subseteqq \{g(x_i);\ 1 \leqq i \leqq n, g \in G\}.$$

Assume $h(x_j) = g(x_i)$ and $h(x_k) = g'(x_l)$ where $j \neq k$ and $g \neq g'$. Then since $g \neq g'$

$$\begin{aligned} \delta > \|x_j - x_k\|_2 &= \|g(x_i) - g'(x_l)\|_2 \\ &\geqq \|g(x_1) - g'(x_1)\|_2 - \|g(x_1) - g(x_i)\|_2 - \|g'(x_1) - g'(x_l)\|_2 \\ &\geqq 4\delta - \delta - \delta = 2\delta. \end{aligned}$$

which is a contradiction. Hence, there exists $g_0 \in G$ such that $\{h(x_i)\}_{i=1}^n = \{g_0(x_i)\}_{i=1}^n$, but by (3) it follows that $h = g_0$.

To finish the proof let $C = \operatorname{co}\{B \cup (1+\varepsilon)^{-1}\Sigma_n\}$, and let $E = (R^n, \|\cdot\|)$ be the space with C as its unit ball. Clearly $G \subseteqq G(E)$, and $G(E) \subset O_n$ because Σ_n is the ellipsoid of least volume which contains C. Let $h \in G(E)$, and $x \in \operatorname{Ext}(B)$, then $h(x) \in \operatorname{Ext}(C)$, and $\|h(x)\|_2 = \|x\|_2 = 1$, so $h(x) \in \operatorname{Ext}(B)$, that is $h(B) \subseteqq B$ so $h \in G(E_0) = G$, therefore $G = G(E)$. Finally, it is clear that $(1+\varepsilon)C \supset \Sigma_n \supset C$, so $d(E, l_2^n) \leqq 1 + \varepsilon$. $\square$

Let G be any subgroup of the group of operators mapping R^ω to R^ω generated by the finite permutations and sign changes on the basis $\{e_n\}_{n=1}^\infty$ of R^ω. That is, for each $g \in G$, there exists a sequence $\varepsilon_n = \pm 1$, with $\varepsilon_n = -1$ for finitely many n, and a finite permutation π of the positive integers such that $g(e_n) = \varepsilon_n e_{\pi(n)}$, $n = 1, 2, \ldots$. Let $n_1 < n_2 < \ldots$ be a sequence of integers, and let

$$G_{n_i} = \{g|_{R^{n_i}};\, g \in G, g(R^{n_i}) = R^{n_i}\}.$$

$\{G_{n_i}\}_{i=1}^{\infty}$ will be called a *consistent* sequence of groups derived from G if $-I_{n_i} \in G_{n_i}$ for all i, where I_n is the identity on R^n.

Theorem 3.2. *Let $\{G_{n_i}\}_{i=1}^{\infty}$ be a consistent sequence of groups derived from G, and let $\varepsilon > 0$. There exists a sequence $E_{n_i} = (R^{n_i}, |\cdot|_{n_i})$ of normed spaces, such that for all i:*

(1) $G(E_{n_i}) = G_{n_i}$
(2) $B(E_{n_i}) = R^{n_i} \cap B(E_{n_{i+1}})$
(3) $(1-\varepsilon)\Sigma_{n_i} \subset B(E_{n_i}) \subset \Sigma_{n_i}$.

Proof. Let $C_1 = \left\{m^{-1/2} \sum_{k=1}^{m} \varepsilon_k e_{i_k};\, 1 \leqq m \leqq n_1, \varepsilon_k = \pm 1,\ 1 \leqq i_1 < i_2 < \ldots < i_m \leqq n_1\right\}$ and select $a_1 = \sum_{j=1}^{n_1} a_{1,j} e_j \in S_{n_1}$ so that $a_1 \notin C_1$, $0 < a_{1,j} < (n_1 - 1)^{-1/2}$ for all $1 \leqq j \leqq n_1$ and $g(a_1) \neq a_1$ for all $g \in G_{n_1} \sim \{I_{n_1}\}$.

Let $4\delta_1$ be the least positive distance between the distinct points contained in the set $C_1 \cup \{g(a_1);\, g \in G_{n_1}\}$. Select now the points $a_i = \sum_{j=1}^{n_1} a_{i,j} e_j$ $(i = 1, 2, \ldots, n_1)$ in S_{n_1} so that

(1) $0 < a_{i,j} < (n_1 - 1)^{-1/2}$ for all $1 \leqq i, j \leqq n_1$.
(2) $\|a_i - a_k\|_2 < \delta_1$ for all $1 \leqq i,\ k \leqq n_1$.
(3) $\operatorname{span}\{a_i\}_{i=1}^{n_1} = R^{n_1}$.
(4) If $T \in O_{n_1}$ satisfies $\{T(a_i)\}_{i=1}^{n_1} = \{a_i\}_{i=1}^{n_1}$, then $T = I_{n_1}$.

Let $B_{n_1} = \operatorname{co}\{C_1 \cup \{g(a_i);\, g \in G_{n_1},\ 1 \leqq i \leqq n_1\}\}$, and let $F_{n_1} = (R^{n_1}, \|\cdot\|_{n_1})$ be the normed space in which B_{n_1} is its unit ball. Since $\{\pm e_i\}_{i=1}^{n_1} \subset B_{n_1} \subset \Sigma_{n_1}$, Σ_{n_1} is the ellipsoid of least volume which contains B_{n_1}, hence $G(F_{n_1}) \subset O_{n_1}$. Obviously, $G_{n_1} \subseteqq G(F_{n_1})$. As in Theorem 3.1, our choice of $\{a_i\}_{i=1}^{n_1}$ insures that $G_{n_1} = G(F_{n_1})$.

We shall now construct $F_{n_2} = (R^{n_2}, \|\cdot\|_{n_2})$. Let

$$C_2 = \left\{m^{-1/2} \sum_{k=1}^{m} \varepsilon_k e_{i_k};\, \varepsilon_k = \pm 1, 1 \leqq m \leqq n_2, 1 \leqq i_1 < i_2 < \ldots < i_m \leqq n_2\right\}.$$

As for a_1, let $a_{n_1+1} = \sum_{1}^{n_2} a_{n_1+1,j} e_j \in S_{n_2} \sim C_2$, $0 < a_{n_1+1,j} < (n_2 - 1)^{-1/2}$ for all $1 \leqq j \leqq n_2$, and $g(a_{n_1+1}) \neq a_{n_1+1}$ for all $g \in G_{n_2} \sim \{I_{n_2}\}$.

Let $D_2 = \{g(a_i);\, 1 \leqq i \leqq n_1, g \in G$ and $g(a_i)$ is in $R^{n_2}\}$ (we consider $R^m \subsetneqq R^n \subsetneqq R^\omega$ for $1 \leqq m < n < \omega$). Let $4\delta_2$ be the least positive distance between distinct points in the sets D_2, C_2, and $\{g(a_{n_1+1});\ g \in G_{n_2}\}$, and select the points $a_i = \sum_{j=1}^{n_2} a_{i,j} \in S_{n_2} (n_1 < i \leqq n_1 + n_2)$ so that:

(5) $0 < a_{i,j} < (n_2 - 1)^{-1/2}$ for all $n_1 < i \leqq n_1 + n_2$, $1 \leqq j \leqq n_2$.
(6) $\|a_i - a_k\|_2 < \delta_2$ for all $n_1 < i, k \leqq n_1 + n_2$.
(7) $R^{n_2} = \operatorname{span}\{a_{n_1+1}, \ldots, a_{n_1+n_2}\}$.
(8) If $T \in O_{n_2}$ satisfies $\{T(a_i)\}_{i=n_1+1}^{n_1+n_2} = \{a_i\}_{i=n_1+1}^{n_1+n_2}$, then $T = I_{n_2}$.

Let B_{n_2} be the convex hull of $C_2 \cup D_2 \cup \{g(a_i); n_1 < i \leqq n_1 + n_2, g \in G_{n_2}\}$, and let F_{n_2} be the normed space with B_{n_2} its unit ball. It follows easily that $G_{n_2} \subseteqq G(F_{n_2}) \subset O_{n_2}$, and as above by noting that for all $h \in G(F_{n_2})$, $\{h(a_i)\}_{n_1+1}^{n_1+n_2} \subset \mathrm{Ext}(B_{n_2})$, we obtain $G_{n_2} = G(F_{n_2})$.

To prove $B_{n_1} = B_{n_2} \cap R^{n_1}$ it suffices to observe that $B_{n_1} \subset B_{n_2}$ and to prove that if P is the orthogonal projection of R^{n_2} on R^{n_1}, then $P(x) \in B_{n_1}$ for every $x \in \mathrm{Ext}(B_{n_2})$.

If $x = m^{-1/2} \sum_1^m \varepsilon_k e_{i_k} \in C_2$, where $1 \leqq i_1 < \ldots < i_p \leqq n_1 < i_{p+1} < \ldots < i_m \leqq n_2$, then $P(x) = m^{-1/2} \sum_1^p \varepsilon_k e_{i_k}$. If $m \leqq n_1$, since $p \leqq m$, it is clear that $P(x)$ is in the convex hull of points in C_1, so $P(x) \in B_{n_1}$. If $m > n_1$, $P(x)$ is then in the convex hull of the points $n_1^{-1/2} \sum_1^{n_1} \pm e_k$ of C_1, hence $P(x) \in B_{n_1}$.

If $x = g(a_i)$, $n_1 < i \leqq n_1 + n_2$, $g \in G_{n_2}$, then $P(x)$ has n_1 non-zero coordinates whose absolute values $< (n_2 - 1)^{-1/2} \leqq n_1^{-1/2}$ [by (5)], hence $P(x)$ is in the convex hull of the points $n_1^{-1/2} \sum_1^{n_1} \pm e_j$, so is in B_{n_1}.

If $x = g(a_i)$, $1 \leqq i \leqq n_1$, $g \in G$ and $x \in R^{n_2}$, then either $x \in R^{n_1}$ or $x \notin R^{n_1}$. If $x \in R^{n_1}$, since $a_{i,j} > 0$ for all $1 \leqq j \leqq n_1$, and since g acts as a permutation and sign changes on $\{e_j\}_{j=1}^\infty$, it follows that $g(R^{n_1}) = R^{n_1}$, so $g|_{R^{n_1}} \in G_{n_1}$, hence $x \in \mathrm{Ext}(B_{n_1})$. If $x \notin R^{n_1}$, then $P(x) \in R^{n_1}$ and has at most $(n_1 - 1)$ non-zero coordinates whose absolute values $< (n_1 - 1)^{-1/2}$, hence $P(x)$ is in the convex hull of the points

$$(n_1 - 1)^{-1/2} \sum_1^{n_1 - 1} \varepsilon_k e_{i_k}, \qquad \varepsilon_k = \pm 1, 1 \leqq i_1 < \ldots < i_{n_1 - 1} \leqq n_1,$$

therefore $P(x) \in \mathrm{co}(C_1) \subseteqq B_{n_1}$. We have proved $B_{n_1} = B_{n_2} \cap R^{n_1}$.

We shall now demonstrate the construction of B_{n_3} out of B_{n_2}, this will show us how to obtain B_{n_k} by induction. Choose $a_{n_1+n_2+1} \in S_{n_3}$,

$$a_{n_1+n_2+1} \notin C_3 = \left\{ m^{-1/2} \sum_{k=1}^m \varepsilon_k e_{i_k}; \varepsilon_k = \pm 1, 1 \leqq m \leqq n_3, 1 \leqq i_1 < \ldots < i_m \leqq n_3 \right\},$$

with $0 < a_{n_1+n_2+1,j} < (n_3 - 1)^{-1/2}$ for all $1 \leqq j \leqq n_3$, and $g(a_{n_1+n_2+1}) \neq a_{n_1+n_2+1}$ for all $g \in G_{n_3} \sim \{I_{n_3}\}$.

Let $D_3 = \{g(a_i); 1 \leqq i \leqq n_1 + n_2, g \in G$ and $g(a_i) \in R^{n_3}\}$, and select $\{a_i\}_{i=n_1+n_2+1}^{n_1+n_2+n_3}$ in S_{n_3} in an analogous manner as before for the suitable $\delta_3 > 0$. B_{n_3} will be the convex hull of $C_3 \cup D_3 \cup \{g(a_i); n_1 + n_2 < i \leqq n_1 + n_2 + n_3, g \in G_{n_3}\}$ and will be the unit ball of F_{n_3}. As above we obtain that $G_{n_3} = G(F_{n_3})$, and utilizing the orthogonal projection of R^{n_3} onto R^{n_2} it follows similarly that $B_{n_2} = B_{n_3} \cap R^{n_2}$.

To terminate the proof, let $M_{n_k} = \mathrm{co}\{B_{n_k} \cup (1-\varepsilon)\Sigma_{n_k}\}$. Then, $(1-\varepsilon)\Sigma_{n_k} \subset M_{n_k} \subset \Sigma_{n_k}$, so if E_{n_k} is the space with M_{n_k} its unit ball, since $\{\pm e_i\}_{i=1}^{n_k} \subset M_{n_k} \subset \Sigma_{n_k}$, it follows that $G_{n_k} \subseteqq G(E_{n_k}) \subset O_{n_k}$. As concluded in the proof of Theorem 3.1 we have in fact $G(E_{n_k}) = G_{n_k}$. It is trivial to check that $M_{n_{k+1}} \cap R^{n_k} = M_{n_k}$ for all k. □

If $B = \{e_n\}_{n=1}^\infty$ is a basis for a Banach space E, we say that an isometry $g \in G(E)$ is a *finitely-represented* isometry with respect to B, if there exists n such that $g(e_i) = e_i$ for all $i > n$, and $g([e_i]_1^n) = [e_i]_1^n$. Denote by $FG(E; B)$ the set of all finitely-represented isometries with respect to B.

Theorem 3.3. *Let $\{G_{n_i}\}_{i=1}^{\infty}$ be a consistent sequence of groups derived from G and $\varepsilon>0$. Then there exists a norm $\|\cdot\|$ on R^{ω}, such that if E is the closure of the normed space $(R^{\omega}, \|\cdot\|)$, then $B=\{e_n\}_{n=1}^{\infty}$ is a normalized basis for E, $d(E, l_2)<1+\varepsilon$, $G \subsetneqq FG(E; B)$, and for every $g \in FG(E; B)$ if n_k is any integer such that $g(R^{n_k})=R^{n_k}$ and $g(e_n)=e_n$ for all $n>n_k$, then $g|_{R^{n_i}} \in G_{n_i}$ for all $i \geqq k$.*

Proof. Construct E_{n_i} as in Theorem 3.2. If $x=\sum_1^m x_i e_i \in R^{\omega}$, then Theorem 3.2 (2) implies $|x|_{n_i}=|x|_{n_k}$ whenever $n_i>n_k \geqq m$. Let $\|x\|$ denote this common norm. (3) implies $(\Sigma x_i^2)^{1/2} \leqq \|x\| \leqq (1-\varepsilon)^{-1}(\Sigma x_i^2)^{1/2}$, so $\{e_i\}$ will be a normalized basis for E, defined to be the completion of the space $(R^{\omega}, \|\cdot\|)$ and $d(E, l_2) \leqq (1-\varepsilon)^{-1}$.

Let $g \in G$ and $x=\sum_{j=1}^m x_j e_j \in R^{\omega}$. There exists $n_k(\geqq m)$ so that $g(R^{n_k})=R^{n_k}$ and $g(e_j) = e_j$ for all j $(>n_k)$. Then $g|_{R^{n_k}} \in G_{n_k}=G(E_{n_k})$ and $x \in E_{n_k}$, so $|x|_{n_k}=|g(x)|_{n_k}$, implying $\|x\|=\|g(x)\|$. Hence, $G \subsetneqq FG(E; B)$.

Let now $g \in FG(E; B)$ and let n_k be as in the statement of the theorem. Then $g|_{R^{n_i}}$ is an isometry of E_{n_i} for all $i \geqq k$, so $g|_{R^{n_i}} \in G(E_{n_i})=G_{n_i}$. This establishes the proof. □

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