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Group Actions on Strongly Monotone Dynamical Systems

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It is well known (see [8–10, 13, 14]) that parabolic partial differential equations and systems admitting the strong comparison principle define strongly monotone dynamical systems. If the domain and the coefficients in such an equation exhibit a symmetry then this reflects in the dynamical system being equivariant, i.e. the flow commutes with the action Γ of some group G (see Sect. 3 for examples and e.g. [6, 18, 20] for the general background).

As shown by Hirsch [9, 10] trajectories in strongly monotone dynamical systems have strong tendency to be nonchaotic: almost all of them are quasiconvergent, that is, their ω-limit sets (limit sets, in short) consist of equilibria. More precisely, points which are not quasiconvergent (see Subsect. 1.2 for the precise definition) but have compact trajectory closures form a meagre set.

Our purpose in this paper is to show that if the action Γ is monotone [the homeomorphism $\Gamma(g): X \to X$ is monotone for each $g \in G$] then, loosely speaking, "symmetry is included in nonchaotic behaviour". For instance, almost all trajectories (in the sense as above) eventually symmetrize (their limit sets consist of symmetric equilibria).

To be more specific, suppose that X is a strongly ordered metric space with order relation $\leqq$, Φ is a strongly monotone flow on X and $\Gamma: G \to \operatorname{Hom} X$ is a monotone representation of a compact connected metrizable group G (see Sect. 1 for definitions). Assume (for simplicity) that all trajectories are relatively compact. Then our results assert that:

1) every equilibrium stable from above (or from below) is symmetric: $\Gamma(g)(x) = x$ for all $g \in G$,

2) if X is a separable Banach space then the set Y of points whose limit sets consist of symmetric equilibria is residual in X,

3) if the flow is order-compact then Y is open and dense in X.

Note that we do not impose any smoothness conditions on the flow (so that, unlike [11], the principle of linearized stability is not assumed to hold).

The following fact appears to be crucial in our reasoning. If $x \in X$ and $g \in G$ then one cannot have $\Gamma(g)(x) \leqq x$ unless $\Gamma(g)(x) = x$ (see Proposition 1.3). Therefore in any neighbourhood of a nonsymmetric equilibrium x one can find another one

$\Gamma(g)(x)$ which is not in the relation $<$ to x. This prevents x from being stable. For such an equilibrium even more is true: the group orbit Gx is not stable.

One of fundamental technical tools made use of is Hirsch's Limit Set Dichotomy (see Theorem 1.1). We also use other results of Hirsch and Matano to get more precise information about the behaviour of a flow.

The paper is organized as follows. In Sect. 1 we collect definitions and basic facts concerning strongly monotone dynamical systems and monotone group actions. Section 2 contains main results. In Sect. 3 we give some examples of semilinear and quasilinear parabolic equations for which our abstract theorems from Sect. 2 apply.

1. Preliminaries

1.1. Strongly Ordered Metric Spaces

Let X be a metric space with metric d. By an *ordered space* we mean X endowed with a closed partial order relation $R \subset X \times X$. We write

$x \leqq y$ if $(x, y) \in R$,

$x < y$ if $x \leqq y$ and $x \neq y$,

$x \ll y$ if $(x, y) \in \operatorname{int} R$ and $x \neq y$, where int denotes the interior of a set.

For two subsets $A, B \subset X$ we write $A \leqq B$ ($A < B$, $A \ll B$ respectively) if $x \leqq y$ ($x < y$, $x \ll y$ respectively) for all $x \in A$, $y \in B$.

The reversed relation signs are used in the usual way.

Following [10, Sect. 1] we say that an ordered space X is *strongly ordered* if every open set $U \subset X$ satisfies the following:

(SO) If $x \in U$ then there exist a, $b \in U$ such that $a \ll x \ll b$.

We define the *closed order interval*

$$[a, b] := \{x \in X : a \leqq x \leqq b\},$$

and the *open order interval*

$$[[a, b]] := \{x \in X : a \ll x \ll b\}.$$

More generally, for two subsets $A, B \subset X$ we introduce the notation

$$[[A, B]] := \{x \in X : A \ll x \ll B\}.$$

A set $U \subset X$ is *order-bounded* if we have $U \subset [[A, B]]$ for some compact nonempty sets $A, B \subset X$. $U \subset X$ is *order-convex* if it contains $[x, y]$ whenever $x, y \in U$.

The space X can be topologized by taking the collection of all open order intervals as the neighbourhood base. This topology is called the *order topology* (for a set $U \subset X$, $U^\wedge$ will stand for U endowed with the relative order topology). The identity map $\mathrm{id}: X \to X^\wedge$ is continuous (the order topology is not finer than the original one). Clearly, if K is compact then $K = K^\wedge$.

We say that a metric $d^\wedge$ for $X^\wedge$ is *ordered* if

$$d^\wedge(a, b) \leqq d^\wedge(u, v)$$

provided that $a < b$, $u < v$ and $[a, b] \subset [u, v]$. For technical reasons, our standing assumption will be that there is an ordered metric $d^\wedge$ for $X^\wedge$ (it will be clear, however, that many of our results hold without this hypothesis).

When X is an open subset of a Banach space V with norm $\|\cdot\|$, it is always understood that V is strongly ordered by a cone with nonempty interior, d is the metric induced by $\|\cdot\|$ and $d^{\wedge}$ is the (ordered) metric induced by the order-unit norm $\|\cdot\|_u$ for some $u \gg 0$ (see [1] or [10]).

1.2. Strongly Monotone Dynamical Systems

A map $f: X \to Y$ between ordered spaces is called *monotone* if $x \leqq y$ implies $f(x) \leqq f(y)$, and *strongly monotone* if $x < y$ implies $f(x) \ll f(y)$.

By a *dynamical system* we understand a pair (X, Φ) consisting of a metric space X and a continuous map $\Phi: D(\Phi) \to X$ such that

i) The domain $D(\Phi)$ is an open set in $[0, \infty) \times X$ containing $\{0\} \times X$.
ii) For every $x \in X$, $\Phi(0, x) = x$.
iii) For every $x \in X$, $s \geqq 0$, $t \geqq 0$, we have

$$\Phi(s, \Phi(t, x)) = \Phi(t+s, x),$$

where the equality sign is to be understood in the sense that if one side is defined then so is the other and the equality holds.

We call Φ the *flow*. The map Φ_t is defined as

$$\Phi_t(x) := \Phi(t, x).$$

The set $J_x := \{t \geqq 0 : x \in D(\Phi_t)\}$ is a half-open interval $[0, \tau_x)$, $0 < \tau_x \leqq \infty$, where τ_x is called the *escape time* of x.

We often write $x \cdot t$ instead of $\Phi(t, x)$. By the *trajectory* of $x \in X$ we mean the image of the map

$$\Phi(\cdot, x): J_x \to X.$$

A set $K \subset X$ is said to be *invariant* if it contains the trajectories of all its members, and *totally invariant* if all its members have infinite escape times and $\Phi_t(K) = K$ for all $t \geqq 0$.

The *limit set* of $x \in X$ is

$$\omega(x) := \{y \in X: \text{there is a sequence } t_k \to \tau_x \text{ such that } x \cdot t_k \to y\}.$$

Its members are called *limit points* of x. If the trajectory of x is *relatively compact* (i.e. its closure is compact) then $\omega(x)$ is nonempty, compact, connected and totally invariant.

An *equilibrium* is a point $p \in X$ such that $p \cdot t = p$ for all $t \geqq 0$. The set of equilibria is denoted by E. If $x \cdot t \to p$ as $t \to \tau_x$ then $p \in E$ and $\tau_x = \infty$. In this case x (and its trajectory) are called *convergent* (we say also that the trajectory of x *converges* to p). A point $x \in X$ is called *quasiconvergent* if its trajectory is relatively compact and $\omega(x) \subset E$.

Let $x, y \in E$. A *trajectory connection* from x to y is given by a continuous map $c: \mathbb{R} \to X$ such that

a) $\lim_{t \to -\infty} c(t) = x$,
b) $\lim_{t \to \infty} c(t) = y$,
c) $c(s+t) = c(s) \cdot t$ for all $s \in \mathbb{R}$, $t \geqq 0$.

A dynamical system (X, Φ) (or its flow Φ) is *strongly monotone* if for each $t > 0$, Φ_t is a strongly monotone map. From now on, it is assumed that every flow is strongly monotone.

We say that Φ is *compact* (resp. *order-compact*) if $\Phi_t(B)$ has compact closure whenever $t>0$ and $B \subset D(\Phi_t)$ is bounded (resp. order-bounded). If Φ is compact (resp. order-compact) then every bounded (resp. order-bounded) trajectory has compact closure and is therefore *global* (i.e. its escape time is infinity).

An equilibrium x is said to be *stable from above* if for every $\varepsilon>0$ there is a $\delta>0$ such that if $y \geqq x$, $d(y,x)<\delta$ and $0 \leqq t < \tau_y$ then $d(y \cdot t, x)<\varepsilon$. An invariant set K is said to be *stable* if for every $\varepsilon>0$ there is a $\delta>0$ such that if

$$d(y,K) := \inf\{d(y,x) : x \in K\} < \delta$$

and $0 \leqq t < \tau_y$ then $d(y \cdot t, K) < \varepsilon$.

If one replaces in the above definitions the metric d by the metric $d^\wedge$, then one obtains the definitions of an equilibrium *order-stable from above* and of an *order-stable* invariant set, respectively. Stability (and order-stability) from below are defined in an analogous way. (Note that our order-stability from above is called upper stability in [10].)

If there exists an ordered metric for $X^\wedge$ (as is the case for X an open subset of a strongly ordered Banach space, see the preceding subsection), then it is straightforward that stability implies order-stability. The notions of stability and order-stability coincide if the flow Φ is order-compact (see [9, p. 47]).

Finally we state without proof a result which will be extensively used in the sequel.

Theorem 1.1 (Limit Set Dichotomy, [10, Theorem 6.8]). *Assume that $x<y$ and that their trajectories are relatively compact. Then either*

$$\omega(x) \ll \omega(y),$$

or else

$$\omega(x) = \omega(y) \subset E.$$

In the latter case for any sequence $t_k \to \infty$ and any $p \in E$ we have $x \cdot t_k \to p \Leftrightarrow y \cdot t_k \to p$.

1.3. Equivariant Group Actions on Strongly Monotone Dynamical Systems

From now on G is a metrizable group with unit element e.

We begin by stating a useful fact.

Lemma 1.2. *Let G be compact. Then for every $g \in G$ there exists a sequence $\{n_k\}$ of positive integers, $n_k \to \infty$ as $k \to \infty$, such that $g^{n_k} \to e$ as $k \to \infty$.*

Proof. By [15, Sect. 1.22] there is a right-invariant metric ϱ on G. Suppose to the contrary that for some $\varepsilon>0$ one has $\varrho(e, g^n) > \varepsilon$ for all n. This implies that $\varrho(g^n, g^m) > \varepsilon$ for all positive integers $n \neq m$. We have a discrete infinite subset $\{g^n : n \in \mathbb{Z}_+\}$ of the compact space G, a contradiction. □

We say that G *acts* on a topological space Y if there is a group homomorphism $\Gamma : G \to \operatorname{Hom} Y$ into the group $\operatorname{Hom} Y$ of homeomorphisms of Y such that the map

$$\gamma : G \times Y \to Y, \qquad \gamma(g,y) := \Gamma(g)(y),$$

is jointly continuous. We will call Γ (or γ) a *group action*. It is well known (see [5, Theorem 1]) that if G is a Baire space (in particular, if G is compact) and Y is metrizable then it suffices to verify only separate continuity in the definition above.

For $y \in Y$ the *orbit* of y is the set

$$Gy := \{\Gamma(g)(y) : g \in G\}.$$

(Note that we use the word "orbit" in connection with the action of the group G, whereas "trajectory" refers to the flow.) By the continuity of γ, if G is compact and/or connected then so is Gy, for each $y \in Y$.

The set $I_y := \{g \in G : \Gamma(g)(y) = y\}$ is called the *isotropy subgroup* of $y \in Y$.

Proposition 1.3. *Let a compact group G act monotonely on an ordered space Y (i.e. for every $g \in G$ the map $\Gamma(g)$ is monotone). Then for every $g \in G$, $y \in Y$, the relation $\Gamma(g)(y) \leqq y$ implies $\Gamma(g)(y) = y$.*

Proof. Assume that $\Gamma(g)(y) \leqq y$ for some $g \in G$, $y \in Y$. Since $\Gamma(g)$ is monotone, we have

$$\ldots \leqq \Gamma(g^{n+1})(y) \leqq \Gamma(g^n)(y) \leqq \ldots \leqq \Gamma(g)(y) \leqq y.$$

By Lemma 1.2 and the continuity of γ,

$$y \geqq \gamma(g, y) \geqq \gamma(g^{n_1}, y) \geqq \ldots \geqq \gamma(g^{n_k}, y) \geqq \gamma(g^{n_{k+1}}, y) \to y.$$

Because $\leqq$ is closed, $y \geqq \gamma(g, y) \geqq y$, so $y = \Gamma(g)(y)$. $\square$

We say that the triple (X, Φ, G) satisfies *Axiom* (GO) if (X, Φ) is a strongly monotone dynamical system, G is a compact connected group, and the following holds.

$(GO1)$ G acts monotonely on X.

$(GO2)$ $\Gamma(g)(x \cdot t) = \Gamma(g)(x) \cdot t$ for all $x \in X$, $g \in G$ and $0 \leqq t < \min(\tau_x, \tau_{\Gamma(g)(x)})$.

The condition $(GO2)$, referred to as *equivariance*, implies that $\tau_x = \tau_{\Gamma(g)(x)}$ for each $x \in X$, $g \in G$. Indeed, if this were not true, then we would have for some $x \in X$ and some $g \in G$

$$\tau_x > \tau_{\Gamma(g)(x)} =: T.$$

But for $s \in [0, T)$, $\Gamma(g)(x) \cdot s = \Gamma(g)(x \cdot s)$, hence, by continuity, $\Gamma(g)(x) \cdot s$ has limit as $s \to T$. From the definition of the flow we deduce that $T < \tau_{\Gamma(g)(x)}$, a contradiction.

By (GO) and the continuity of $\Gamma(g)$, the trajectory of $\Gamma(g)(x)$ is relatively compact (resp. quasiconvergent, convergent) if and only if so is the trajectory of x.

For (X, Φ, G) satisfying Axiom (GO), a point $x \in X$ is said to be *symmetric* if $\Gamma(g)(x) = x$ for any $g \in G$. Otherwise it is called *nonsymmetric*. The set of quasiconvergent points whose limit sets contain only symmetric equilibria is denoted by H.

2. Main Results

In the present section our standing assumption is that (X, Φ, G) satisfies Axiom (GO).

Since in this section we consider a fixed group action, we suppress Γ notationally: we write simply gx instead of $\Gamma(g)(x)$.

Let X_C denote the set of all points having compact trajectory closures, and let Q denote the set of all quasiconvergent points.

Lemma 2.1. *Let $x \in E$. Then for any compact totally invariant set K such that $K > x$ one has $K \gg Gx$.*

Proof. By strong monotonicity and total invariance, $K \gg x$. Consider the set $U := \{z \in Gx : z \ll K\}$. U is nonempty (since $x \in U$) and open in the relative topology of Gx. For each $z \in \operatorname{cl} U \subset E$ (where cl denotes the closure) we have $z \leqq K$, and, by Proposition 1.3, $z < K$, and, again owing to strong monotonicity and total invariance, $z \ll K$. Therefore U is open and closed in the connected space Gx, so $U = Gx$. □

Proposition 2.2. *Let $x \in E$ be nonsymmetric. Then x is isolated in $M^\wedge$ (hence in M), where*

$$M := \bigcup \{\omega(z) : z \in X_C, z \geqq x\}.$$

Proof. Assume to the contrary that x is not isolated in $M^\wedge$. Then there is a sequence $y_n \in M \setminus \{x\}$ such that x is its limit in the order topology. By the Limit Set Dichotomy (Theorem 1.1) and Lemma 2.1, for any $z \in X_C$, $z > x$, one has either $\omega(z) = \{x\}$ or $\omega(z) \gg Gx$. Therefore $Gx \ll y_n$ for all n, and, because of the closedness of the relation $\leqq$ in $X^\wedge \times X^\wedge$ (see [10]), $Gx \leqq x$. Proposition 1.3 yields $Gx = \{x\}$, a contradiction. □

Proposition 2.3. *Let $x \in E$ be nonsymmetric. Then there do not exist three points $z_1, z_2, z_3 \in X_C$, $z_1 < z_2 < z_3$, such that $x \in \omega(z_1) \cap \omega(z_3)$.*

Proof. Suppose that such three points exist. By the Limit Set Dichotomy $\omega(z_1) = \omega(z_2) = \omega(z_3)$. Without loss of generality assume $z_1 \ll z_2 \ll z_3$ (if not, replace them by $z_1 \cdot 1$, $z_2 \cdot 1$, $z_3 \cdot 1$, respectively).

Let $t_k \to \infty$ be a sequence such that $z_2 \cdot t_k \to x$. Set $P := \{g \in G : gz_2 \in [[z_1, z_3]]\}$. P is a neighbourhood of e. It follows from the Limit Set Dichotomy that $gz_2 \cdot t_k \to x$ for all $g \in P$ (recall that $gz_2 \in X_C$). On the other hand, from equivariance and continuity of the group action we deduce that $gz_2 \cdot t_k \to gx$. So $P \subset I_x$. Thus I_x has nonempty interior, this implies that I_x equals the connected component of e in G. By connectedness, $I_x = G$, so x is symmetric, a contradiction. □

Theorem 2.4. *Let $x \in E$ be nonsymmetric. Assume that x is not isolated in*

$$\{z \in X_C : z \geqq x\}.$$

Then x is not stable from above. Furthermore, Gx is not stable.

If x is not isolated in

$$\{z \in X_C : z \geqq x\}^\wedge,$$

then the statement holds for order-stability.

Proof. Suppose that a nonsymmetric equilibrium x is stable from above. Due to strong monotonicity, the hypothesis enables us to construct a sequence $z_n \in X_C$, $x \ll z_{n+1} \ll z_n$, $z_n \to x$. From Proposition 2.3 and the stability of x it follows that arbitrarily close to x there is a $y \gg x$, y being a limit point for some $z_n \in X_C$, contrary to Proposition 2.2.

Now suppose that Gx is stable. A reasoning similar to the above one assures us of the existence of a sequence $y_n \gg x$, $y_n \in \omega(z_n)$ for some $z_n \in X_C$, satisfying $d(y_n, Gx) \to 0$ as $n \to \infty$. By Lemma 2.1, $y_n \gg Gx$ for all n. Since Gx is compact, by passing to a subsequence if necessary we may assume that $y_n \to y$ for some $y \in Gx$. This contradicts Proposition 2.2 (with x replaced with y).

The proof for order-stability is quite similar. □

Corollary 2.5. *If the flow Φ is compact (resp. order-compact), then no nonsymmetric equilibrium is stable (resp. order-stable) from above.*

Proof. If $x \in E$ is stable from above then for any $z \gg x$, z near x, the trajectory of z is bounded, hence relatively compact. Similarly for order-stability. □

Remark. Needless to say, all the above results have their analogues for the reversed inequality sign.

Corollary 2.6. *Assume that X is an open order interval in a strongly ordered Banach space V, and that the flow Φ is order-compact. Let $x \in E$ be nonsymmetric. If*

$$S^+(x) := \{z \in E : z > x\}$$

is nonempty, then there exists a symmetric $v \in E$ such that

(i) $v \gg Gx$,
(ii) *v is least in $S^+(y)$ for each $y \in Gx$,*
(iii) *for every $y \in Gx$ there is a trajectory connection from y to v.*

Proof. First observe that the assumptions imply that for all $B \subset X$ and all $t > 0$, $\Phi_t(B)$ has compact closure (so, in particular, $X = X_C$).

A theorem of Matano [13, Theorem 5] asserts that there exists a continuous map $c : (-\infty, 0] \to X$ such that

a) $c(t) \gg x$ for each $t \in (-\infty, 0]$,
b) $c(t) \cdot s = c(t+s)$ for $t \leqq 0$, $s \geqq 0$ with $t + s \leqq 0$,
c) $\lim_{t \to -\infty} c(t) = x$.

From a) and b) we deduce that

$$c(T) \ll c(0) \quad \text{for some} \quad T < 0. \tag{2.1}$$

According to the convergence criterion for strongly monotone flows [10, Theorem 6.4], (2.1) together with relative compactness of trajectories implies that $\omega(c(0)) = \{v\}$ for some $v \in E$, $v \gg c(0) \gg x$. We have obtained a trajectory connection from x to v. Because $c(T) \ll c(0) \ll v$ and these points have $\{v\}$ as their common limit set, from Proposition 2.3 it follows that v is symmetric.

Again by a) and c), for any $z > x$ we have $\omega(z) \geqq v$. Therefore v is least in $S^+(x)$.

Let for some $y \in Gx$ (say $y = gx$) w be a point obtained as above. We have

$$w = g^{-1}w > g^{-1}y = g^{-1}gx = x,$$

so $w > x$, hence $w \geqq v$. Interchanging x and y we get $w \leqq v$, so $w = v$. This concludes the proof of parts (ii) and (iii). Part (i) follows from Lemma 2.1. □

Recall that H denotes the set of all quasiconvergent points whose limit sets consist of symmetric equilibria.

Theorem 2.7. *Let $L \subset X_C$ be simply ordered. Then for each $r \in (L \cap Q) \setminus H$ there is a neighbourhood U_r of r in L such that for every $u \in U_r \setminus \{r\}$ its trajectory converges to a symmetric equilibrium. In particular, the set $(L \cap Q) \setminus H$ is discrete.*

Proof. For any $r \in (L \cap Q) \setminus H$ take a nonsymmetric equilibrium $x \in \omega(r)$. By Proposition 2.3 and the Limit Set Dichotomy, for every $u \in L$, $u \neq r$, we have either $\omega(u) \ll \omega(r)$, or $\omega(u) \gg \omega(r)$. Proposition 2.2 assures us that there exist $y_1, y_2, y_1 \ll x \ll y_2$ such that

$$\omega(u) \notin [[y_1, y_2]] \quad \text{for all} \quad u \in L, u \neq r.$$

Since $x \in \omega(r)$, there is a $T > 0$ such that $r \cdot T \in [[y_1, y_2]]$. By continuity, we can find a neighbourhood $\tilde{U}_r$ of r such that $u \cdot T \in [[y_1, y_2]]$ for all $u \in \tilde{U}_r$.

Define $U_r := \tilde{U}_r \cap L$. If $u \in U_r$ and $u < r$, then for some $s > 0$ we have

$$u \cdot (T+s) \ll y_1 \ll u \cdot T.$$

Since u has compact trajectory closure, the convergence criterion for strongly monotone flows [10, Theorem 6.5] implies that there exists a $v \in E$ such that $\omega(u) = \{v\}$ and $v \ll u \cdot t$ for all $t > T$. Then there are three points $v \ll u \cdot (T+s) \ll u \cdot T$ having $\{v\}$ as their common limit set, hence by virtue of Proposition 2.3 v is symmetric. The argument for $u \in U_r$, $u > r$ is quite similar. □

Recall that a set is said to be *meagre* if it is contained in a countable union of closed nowhere dense sets.

Theorem 2.8. *Let X be an open subset of a strongly ordered separable Banach space V. Then the set $X_C \backslash H$ is meagre.*

Proof. For any simply ordered $L \subset X_C$, $L \backslash Q$ is countable by [10, Theorem 7.3]. The preceding theorem asserts that $(L \cap Q) \backslash H$ is discrete, hence countable. But a subset of a strongly ordered separable Banach space is meagre provided that all its simply ordered subsets are countable [10, Lemma 7.4]. □

Remarks. (a) Note that by [10, Lemma 7.7] one can prove that under the assumptions of Theorem 2.8, $\mu(X_C \backslash H) = 0$ for any Gaussian measure μ on V.

(b) The conclusion of Theorem 2.8 holds if, instead of separability, the ambient Banach space V satisfies any of the hypotheses (a), (b) in [10, Theorem 7.3].

Theorem 2.9. *If, in addition to the hypotheses of Theorem 2.8, $X = X_C$ and the flow is order-compact, then H contains an open dense subset of X.*

Proof. By [10, Theorem 8.12 and Proposition 9.5], the set $\operatorname{int} Q$ is dense in X. We shall prove that $\operatorname{int} H$ is dense in X. Take any open nonempty set $U \subset X$. We have

$$B := U \cap \operatorname{int} Q \neq \emptyset.$$

If $B \subset H$ we are through. Otherwise, let $r \in B \backslash H \subset Q \backslash H$. By an argument similar to that used in the proof of Theorem 2.7 we find an open neighbourhood $V \subset B$ of r such that the trajectory of any member of the nonempty open set

$$D := V \cap \{y \in X : y > r\} \subset U$$

converges to a symmetric equilibrium. Hence $D \subset H$. □

Remark. If for some strongly monotone dynamical system it is known that there exists an open dense subset $X_1 \subset X$ such that for any $x \in X_1$ its trajectory converges to a stable equilibrium (as is the case for some smooth dynamical systems [11, 17]), then Theorem 2.9 follows immediately from Theorem 2.4.

3. Examples from Parabolic Equations

In this section we give examples of second order parabolic partial differential equations which generate strongly monotone dynamical systems satisfying Axiom (*GO*).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the initial-boundary value problem:

$$u_t(t,x) = \mathcal{F}(u(t,x)) \quad \text{for} \quad t>0, x \in \bar{\Omega} := \Omega \cup \partial\Omega, \tag{3.1}$$

$$\mathcal{B}u(t,x) = 0 \quad \text{for} \quad t>0, x \in \partial\Omega, \tag{3.2}$$

$$u(0,x) = u_0(x) \quad \text{for} \quad x \in \bar{\Omega}, \tag{3.3}$$

where u takes on values in $\mathbb{R}^N$, $N \geqq 1$, $\mathcal{F} : (C^2(\bar{\Omega}))^N \to (C^0(\bar{\Omega}))^N$ is an autonomous strongly elliptic partial differential operator of second order (semilinear or quasilinear), and $\mathcal{B}$ is a boundary operator which is for each component u_i of u either of Dirichlet type:

$$u_i(t,x) = 0 \quad \text{for} \quad t>0, x \in \partial\Omega,$$

or of Neumann type:

$$\partial u_i(t,x)/\partial \nu = 0 \quad \text{for} \quad t>0, x \in \partial\Omega$$

(here ν is the unit normal vector field on $\partial\Omega$ pointing out of Ω).

Under appropriate smoothness conditions imposed on functions included in the operator $\mathcal{F}$ the problem (3.1)–(3.3) defines a flow on a closed subspace X (corresponding to the boundary conditions) of a Sobolev-Slobodeckiĭ space (see [19]) $W_p^\sigma := W_p^\sigma(\Omega, \mathbb{R}^N) \cong (W_p^\sigma(\Omega))^N$, $n<p<\infty$, $1+n/p<\sigma<2$ (see [2, 3, 4] or, in the case of semilinear equations, [7]). For semilinear equations one can also get a flow on a subspace of $(C^1(\bar{\Omega}))^N$ (see [16]). Some results of Amann and Mora are collected in [10]. In both cases X is continuously and densely embedded into the product of N spaces, each of them being either $C^0(\bar{\Omega})$ or

$$C_0^1(\bar{\Omega}) := \{u \in C^1(\bar{\Omega}) : u(x) = 0 \ \text{for} \ x \in \partial\Omega\}.$$

The latter two spaces are strongly ordered [10, Sect. 1], so X is strongly ordered by the ordering:

$$u \leqq v \quad \text{if} \quad u_i(x) \leqq v_i(x) \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad i \in \{1, \ldots, N\}$$

(see [10, Corollary 1.12]).

Assume that G is a compact connected subgroup of the group $SO(n)$ of all orthogonal orientation-preserving linear transformations of $\mathbb{R}^n$ whose action leaves the domain $\bar{\Omega}$ invariant:

$$g\bar{\Omega} = \bar{\Omega} \quad \text{for all} \quad g \in G.$$

For a function $v : \bar{\Omega} \to \mathbb{R}^N$ let $\Gamma(g)(v)$ be defined by the formula:

$$\Gamma(g)(v)(x) := v(gx) \quad \text{for} \quad g \in G, x \in \bar{\Omega}. \tag{3.4}$$

It is clear that the map $v \mapsto \Gamma(g)(v)$ does not influence the boundary conditions (3.2). Furthermore, it is a linear isometry on $(C^1(\bar{\Omega}))^N$ as well as on W_p^σ. Appealing to a theorem due to Chernoff and Marsden [5, Theorem 1], we will show that (3.4) defines an action of G on X if we prove that the map $g \mapsto \Gamma(g)(v)$ is continuous for each fixed $v \in X$. It is straightforward that for each $v \in (C^2(\bar{\Omega}))^N$ this is continuous as a map into $(C^2(\bar{\Omega}))^N$, hence as a map into X. Now we use the density of $(C^2(\bar{\Omega}))^N \cap X$ in X (see [19]).

For each $g \in G$, $\Gamma(g)$ is positive, hence monotone. So the triple (X, Φ, G), where Φ is the flow induced on X by (3.1)–(3.3) and G acts on X according to (3.4), satisfies Axiom (GO), provided that Φ is strongly monotone and equivariance $(GO2)$ holds. The latter is the case if $\mathscr{F}$ commutes with the action of G:

$$\mathscr{F}(\Gamma(g)(u)(x)) = \Gamma(g)(\mathscr{F}(u(x))) \quad \text{for} \quad g \in G, u \in (C^2(\bar{\Omega}))^N .$$

Now we give examples of such $\mathscr{F}$'s.

Example 1 (Scalar equation). Let $N=1$ and

$$\mathscr{F}(u(x)) := a(x, u, \nabla u)\Delta u + f(x, u, \nabla u),$$

where $a, f: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are sufficiently smooth (in general C^2 is needed) and $a(\cdot) \geqq \alpha$ for some positive constant α. By [2, 3, 4] there is a solution flow for (3.1)–(3.3) in a subspace of W_p^σ with σ, p as above. For a independent of u and ∇u one can make use of the results contained in [7] or [16] as well. In this case the resulting flow is compact, and order-compact when in addition f does not depend on ∇u.

The strong comparison principle guarantees that the flow is strongly monotone [10]. As to equivariance it is enough for $h := (a, f)$ to satisfy:

$$h(gx, u, g^{-1}z) = h(x, u, z) \quad \text{for} \quad g \in SO(n), x \in \bar{\Omega}, u \in \mathbb{R}, z \in \mathbb{R}^n,$$

for instance $h = h(r, u, u_r)$ in the polar coordinates.

Example 2 (Strongly cooperative system). Let $N \geqq 1$ and

$$\mathscr{F}(u(x)) := a(u)\Delta u + f(u),$$

where $a(\cdot)$ is an $N \times N$ diagonal matrix function with all entries greater than some positive constant, $f: \mathbb{R}^N \to \mathbb{R}^N$, a and f are sufficiently smooth and $f = (f_1, \ldots, f_N)$ satisfies the strong cooperativity condition: f_i is strictly increasing in u_j for all $i, j \in \{1, \ldots, N\}$, $i \neq j$. Using the theory presented in [2, 3, 4] (or in [7] if a is a constant matrix) one obtains a solution flow for (3.1)–(3.3). If a is constant the flow is order-compact. Strong cooperativity in conjunction with the strong comparison principle implies the strong monotonicity of the flow (cf. [12]). Equivariance is obvious.

Remark. Observe that applying results contained in [3, 4] one can extend our theory to the case of quasilinear parabolic equations under nonlinear boundary conditions.

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