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Kähler Spaces and Proper Open Morphisms

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Introduction

Several years ago, Hironaka [17] raised the following two problems:

Problem A. Let X be a Kähler space. Is the Douady space of X Kähler?

and its weaker version

Problem B. Let $\pi: X \to X'$ be a proper flat surjective morphism of complex spaces. If X is Kähler, is X' Kähler?

Actually problems A and B were raised for compact X but we will consider the non-compact case as well.

Problem A seems inaccessible for the moment.

Problem B was solved affirmatively in [23] for smooth X, X'. The aim of the present paper is to generalize the result to singular spaces. It appears that the flatness hypothesis on π is too strong, so it will be replaced by a less restrictive property which we call *geometric flatness.*

A closely related problem, raised by Lieberman [18] is

Problem C. Let X be a Kähler space and $\mathbf{B}_m(X)$ the Barlet space of compact complex m-cycles of X. Is $\mathbf{B}_m(X)$ Kähler?

A solution to problem C would imply one to problem B for geometrically flat π and reduced X'.

Finally a problem which is of fundamental importance in the theory of complex cycles is

Problem D. Let X be a complex space and $\xi \in H^m(X, \Omega_X^m)$. Is the function $F_\xi : c \mapsto (c \cdot \xi)$ holomorphic on $\mathbf{B}_m(X)$?

Our results can be summarized as follows: Problems B and C are reduced to problem D; problem D has a solution (for fixed X, m) if every compact m-dimensional complex-analytic subset of X has a smoothly embeddable neighborhood (Chap. I, Proposition 3.5.4).

In order to formulate our results completely, and as long as problem D remains unsolved in its full generality, we are led to introduce the notion of *weakly Kähler spaces.* The most useful properties of geometrically flat morphisms and weakly Kähler spaces are

(i) A geometrically flat morphism is proper open surjective with pure dimensional fibers and reduced base. The converse is true if the morphism is flat or the base is normal.

(ii) If G is a finite group of automorphisms of a reduced space X then the canonical projection $X \to X/G$ is geometrically flat.

(iii) Kähler spaces are weakly Kähler. Subspaces of weakly Kähler spaces are weakly Kähler. X is weakly Kähler iff X_{red} is weakly Kähler. A weakly normal space is weakly Kähler iff it is Kähler.

(iv) A compact space is weakly Kähler iff its weak normalization is Kähler.

Now we may enumerate our main results:

(i) If $\pi : X \to X'$ is geometrically flat with m-dimensional fibers, then problem B has a solution for π if problem D has a solution for X, m. Otherwise all we can say is that X' is weakly Kähler. But this is enough to ensure that X' is Kähler if it is normal (Chap. IV, Theorem 3–Corollary 1.2).

(ii) Problem C has a solution for X, m if problem D has a solution for X, m. Otherwise all we can say is that $\mathbf{B}_m(X)$ is weakly Kähler. But this is enough to ensure that, if X is compact, the weak normalization of $\mathbf{B}_m(X)$ is Kähler (Chap. IV, Theorem 4–Corollary 2.2).

(iii) The solution of problem B for normal X' implies that any reduced compact complex space in Fujiki's class $\mathscr{C}$ (holomorphic image of a compact Kähler space) is bimeromorphically equivalent to a compact Kähler space

(Chap. IV, Theorem 5). An alternative proof of this was given in [24] using the solution of problem C for smooth X.

Our paper is organized as follows:

In Chap. I we give a rapid discussion of the sheaf $\mathscr{C}_X^\infty$ in the sense we choose for a complex space X. $\mathscr{C}_X^\infty$ is *not* a subsheaf of the sheaf $\mathscr{C}_X$ of continuous complex-valued functions on X; there is only a canonical morphism $\varphi \mapsto [\varphi]$ from $\mathscr{C}_X^\infty$ to $\mathscr{C}_X$. This is important for the formulation of a smoothing lemma (2.5) for continuous strongly plurisubharmonic (p.s.h.) functions which is essentially due to Richberg [21]. We also remind some of the main properties of the Barlet space $\mathbf{B}_m(X)$ which we will use. Geometric flatness is defined in 3.3.

In Chap. II we define the notions of Kähler metrics, classes, spaces, and morphisms and prove *Theorem 1* (valid on any complex space) according to which, a space is Kähler if it admits an open covering $\mathscr{U}$ with 0-cochain $\varphi=(\varphi_\alpha)$ of *continuous* strongly p.s.h. functions and a 1-cocycle $h=(h_{\alpha\beta})$ of pluriharmonic functions such that $\delta\varphi=[h]$ in $C^1(\mathscr{U}, \mathscr{C}_X)$. (The cocycle condition on h is redundant only for X reduced). As a consequence we solve problem B for *finite* $\pi: X \to X'$ such that either π is flat and X' arbitrary (not necessarily reduced) or π is geometrically flat and X' reduced. If X is a Kähler space and G a finite group of automorphisms of $X, X/G$ is Kähler. In particular, $\operatorname{Sym}^k(X)$ is Kähler for any $k \geqq 1$ (Corollary 3.2.1). Finally we define weakly Kähler spaces in 4.1.

Chapter III is entirely devoted to the proof of *Theorem 2*: if X is a complex space and $m \geqq 0$ an integer, then there are open sets $U_\alpha \subset X$ and $U_{\alpha\beta}^j \subset U_\alpha \cap U_\beta$ such that any compact m-dimensional complex-analytic subset of X (resp. $U_\alpha \cap U_\beta$) is contained in some U_α (resp. $U_{\alpha\beta}^j$). Moreover, if ω is a Kähler form on X, then there are (m,m)-forms $\chi_\alpha = \bar{\chi}_\alpha$ on U_α, $\tau_{\alpha\beta}^j$ on $U_{\alpha\beta}^j$ such that $\omega^{m+1}|U_\alpha = i\partial\bar{\partial}\chi_\alpha$, $\bar{\partial}\tau_{\alpha\beta}^j = 0$, $(\chi_\alpha - \chi_\beta)|U_{\alpha\beta}^j = \tau_{\alpha\beta}^j + \bar{\tau}_{\alpha\beta}^j$ and the $\bar{\partial}$-cohomology class of $\tau_{\alpha\beta}^j$ lies in the image of the canonical morphism $H^m(U_{\alpha\beta}^j, \Omega^m) \to H_{\bar{\partial}}^{m,m}(U_{\alpha\beta}^j)$.

Theorem 2 is the main original element of this paper. It relies on Barlet's result [6] according to which m-dimensional compact complex-analytic subsets admit m-complete neighborhoods. For smooth X, Theorem 2 can be easily deduced from this [23, Lemma 3.6] and [24, 2.8] using the Dolbeault isomorphism. For singular X, this is considerably more difficult. Our method can be described as follows: When a complex of sheaves $(\mathscr{L}^\cdot, D)$ fails to be exact, we replace it by the single complex associated to the double complex (δ, D) where δ is the Čech differential with respect to some open covering. We call this new complex the *Čech transform* of $(\mathscr{L}^\cdot, D)$ and apply it to the $\partial\bar{\partial}$-complex $\mathscr{L}_m^\cdot$ (defined in 3.1). The key step is the existence of a cocycle Φ_{m+1} of degree $2m+2$ (defined in 4.3) of the Čech transform of the complex $\mathscr{L}_{m+1}^\cdot$ whose final component is ω^{m+1}. Using an elementary lemma of algebra (Lemma 2.2) we prove that Φ_{m+1} bounds near every m-dimensional compact complex-analytic subset of X, so ω^{m+1} is $\partial\bar{\partial}$-exact there. The last part of Theorem 2 relies on two morphisms β and γ (defined in 3.5) connecting the $\partial\bar{\partial}$-complex to the direct sum of the Dolbeault complex and its conjugate. Chapter III is self-contained.

Finally Chap. IV proves the main results we obtain as consequences of Theorems 1 and 2, namely Theorems 3–5 and corollaries.

List of Symbols

I. Ω_X^m
$\mathscr{C}_X$
$\mathscr{F}_{X,\mathbb{R}}$
$\mathscr{F}(U,\mathbb{R})$

1.1. $\mathscr{C}_X^\infty$
A_X^m
$A_X^{k,l}$
PH_X

1.2. $[\varphi]$
$[\mathscr{C}_X^\infty]$

2.1. P_X^0
SP_X^0
P_X^∞
SP_X^∞
$[P_X^\infty]$
$[SP_X^\infty]$

2.4. $SP^{0,\infty}(U,V)$

2.6. SP_π^∞

3.1. $\mathrm{Sym}^k(X)$
$\sum\{x_j\}$

3.2. $\mathbf{B}_m(X)$

3.3. $\mathbf{D}_m(X)$
$c(Y)$

3.4. $F_\varphi(c)$
$(c\cdot\xi)$

3.5. $\pi_*\varphi$
$\mathbf{B}_m(X)^{(0)}$

II.

1.1. $\mathscr{K}_X^1$
$\mathscr{K}_{X,\mathbb{R}}^1$
$\partial\bar{\partial}\kappa$

1.2. $\hat{c}_1$
c_1
$\tilde{c}_1$

3.1. $\mathrm{Tr}^{(c)}_{X/X'}$
$\mathrm{Tr}^{(h)}_{X/X'}$

4.1. $\mathscr{W}_X$
WPH_X
$\hat{X}$

III.

1.1. $\underline{X}$
$F:\underline{X}\to\underline{Y}$
$\underline{U}\ll\underline{X}$
$\underline{U}_1\cap\underline{U}_2$
ε
δ
$\varphi|\underline{U}$
T

1.2. $\varphi\cdot\psi$

2.1. $\check{C}^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot)$
Δ
$\check{Z}^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot)$
$\check{H}^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot)$

3.1. $\mathscr{L}_m^r$
D

3.3 φ^*
$\mathscr{L}_{m,\mathbb{R}}^r$

3.4. μ

3.5. $\mathscr{G}_m^q$
$\hat{d}$
β
γ

3.6. $\mathscr{E}_m^q(\underline{X})$
$\mathscr{E}_m^q(\underline{X},[\mathbb{R}])$
$\mathscr{E}_m^q(\underline{X},\mathbb{R})$

4.2. $\Phi_1(f,\varphi)$

4.3. $\mathscr{K}^m(\underline{X})$, $\mathscr{K}(\underline{X})$
$\mathscr{K}^m(\underline{X},[\mathbb{R}])$, $\mathscr{K}(\underline{X},[\mathbb{R}])$
$\mathscr{K}^m(\underline{X},\mathbb{R})$, $\mathscr{K}(\underline{X},\mathbb{R})$

4.4. $\Phi\times\Psi$

5.4. $\mathscr{D}_m^q(\underline{X})$
$\hat{\Delta}$

IV.

3.1. $\mathscr{C}$
$\mathscr{C}^*$

I. Preliminaries

X will always denote a complex space, not necessarily reduced unless explicitly stated. $X_{\mathrm{red}}\to X$ denotes the reduction of X. $\mathcal{O}_X=\Omega_X^0$ is the structure sheaf of X and Ω_X^m the sheaf of holomorphic m-forms on X. $\mathscr{C}_X$ is the sheaf of continuous functions on the topological space underlying to X. If $\mathscr{F}=\mathscr{F}_X$ is any sheaf on X, $\mathscr{F}(U)$ will denote $\Gamma(U,\mathscr{F}_X)$. If $\mathscr{F}_X$ is a sheaf of $\mathbb{C}$-vector spaces with a natural $\mathbb{C}$-antilinear involution, $\mathscr{F}_{X,\mathbb{R}}$ will denote the subsheaf of elements left fixed by the involution and $\mathscr{F}(U,\mathbb{R}):=\Gamma(U,\mathscr{F}_{X,\mathbb{R}})$. We always assume X countable at infinity.

1. $\mathscr{C}^\infty$ Forms and Functions on Complex Spaces

There are two inequivalent definitions of $\mathscr{C}_X^\infty$ in the literature. The first, which we call the "old" one [5, 10, 21] defines $\mathscr{C}_X^\infty$ as the subsheaf of $\mathscr{C}_X$ consisting of local restrictions of $\mathscr{C}^\infty$ functions under smooth embeddings. So $\mathscr{C}_X^\infty = \mathscr{C}_{X_{\mathrm{red}}}^\infty$ in this sense. The second which we will call thee "modern" one [8, 12] is the one we give below.

1.1. Definitions. We define on X the sheaves $\mathscr{C}_X^\omega$ of real-analytic functions, PH_X of pluriharmonic functions, $\mathscr{C}_X^\infty = A_X^0$ of $\mathscr{C}^\infty$ functions, A_X^m (resp. $A_X^{k,l}$) of $\mathscr{C}^\infty$ m-forms [resp. (k,l)-forms] as follows: For smooth X, they are well defined. Now suppose $X \to D$ is an embedding of X in a domain D of $\mathbb{C}^n$ and $\mathscr{I}_X \subset \mathcal{O}_D$ is the corresponding coherent ideal sheaf. Set

$$\mathscr{I}_X^\omega := (\mathscr{I}_X + \bar{\mathscr{I}}_X)\mathscr{C}_D^\omega, \qquad \mathscr{I}_X^\infty := (\mathscr{I}_X + \bar{\mathscr{I}}_X)\mathscr{C}_D^\infty$$

and

$$\mathscr{C}_X^\omega := \mathscr{C}_D^\omega / \mathscr{I}_X^\omega, \qquad \mathscr{C}_X^\infty := \mathscr{C}_D^\infty / \mathscr{I}_X^\infty, \qquad A_X^m := A_D^m / (\mathscr{I}_X^\infty A_D^m + d\mathscr{I}_X^\infty A_D^{m-1})$$

$$A_X^{k,l} := \text{the image of } A_D^{k,l} \text{ under the canonical morphism } \quad A_D^{k+l} \to A_X^{k+l}.$$

It is clear that these sheaves are independent of the choice of the embedding $X \to D$ so they extend to arbitrary X. There are canonical morphisms

$$\mathcal{O}_X \to \mathscr{C}_X^\omega \to \mathscr{C}_X^\infty \to \mathscr{C}_X.$$

1.2. Elementary Properties and Conventions. (i) The canonical morphisms $\mathcal{O}_X \to \mathscr{C}_X^\omega$ and $\mathscr{C}_X^\omega \to \mathscr{C}_X^\infty$ are injective. (The first is elementary and the second is a consequence of the fact that $\mathscr{C}_D^\infty$ is a faithfully flat $\mathscr{C}_D^\omega$-module by Malgrange [19, Chap. VI, Corollary 1.12].) They will be considered as inclusions

$$\mathcal{O}_X \subset \mathscr{C}_X^\omega \subset \mathscr{C}_X^\infty$$

and so we may define $PH_X := \mathcal{O}_X + \bar{\mathcal{O}}_X \subset \mathscr{C}_X^\omega$.

(ii) In $\mathscr{C}_X^\omega$ we have $\mathcal{O}_X \cap \bar{\mathcal{O}}_X = \mathbb{C}$ and there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{O}_X & \xrightarrow{-2\,\mathrm{Im}} & PH_{X,\mathbb{R}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\lambda} & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{O}_X \oplus \bar{\mathcal{O}}_X & \xrightarrow{(i\ \ -i)} & PH_X & \longrightarrow & 0, \end{array}$$

where $\lambda(f) = (f, \bar{f})$ and the unspecified morphisms are the canonical inclusions.

(iii) The canonical morphism $\varrho : \mathscr{C}_X^\infty \to \mathscr{C}_X$ is not injective in general even for X reduced; for $fg = 0$ in $\mathcal{O}_X$ does not imply $f\bar{g} = 0$ in $\mathscr{C}_X^\omega$. However, for X *reduced and locally irreducible*, ϱ is injective. (It is elementary that the restriction of ϱ to $\mathscr{C}_X^\omega$ is injective; we deduce that ϱ is injective by Malgrange [19, Chap. VI, Theorem 3.10].)

We write $[\varphi] := \varrho(\varphi)$, $[\mathscr{C}_X^\infty] := \varrho(\mathscr{C}_X^\infty)$. So $[\mathscr{C}_X^\infty]$ is the $\mathscr{C}^\infty$ sheaf of the "old" theory. For normal X the two theories coincide, by the above remark.

(iv) The kernel of the canonical morphism $PH_X \to \mathscr{C}_X$ is $\mathscr{N}_X + \bar{\mathscr{N}}_X$ where $\mathscr{N}_X$ is the sheaf of nilpotent sections of $\mathcal{O}_X$. In particular, for reduced X, PH_X may be considered as a subsheaf of $\mathscr{C}_X$.

(v) If $f : X \to Y$ is a morphism of complex spaces, $\varphi \in \mathscr{C}(Y)$ and $\psi \in \mathscr{C}^\infty(Y)$, write $\varphi \circ f \in \mathscr{C}(X)$ and $f^*\psi \in \mathscr{C}^\infty(X)$ for the corresponding induced elements. Write $\psi \circ f$ instead of $[\psi] \circ f$, so that $[f^*\psi] = \psi \circ f$ in $\mathscr{C}(X)$.

(vi) The canonical morphisms $\Omega_X^m \to A_X^{m,o}$ are injective and will be considered as inclusions.

(vii) The inclusions $A_X^{k,l} \subset A_X^{k+l}$ give a direct sum decomposition $A_X^m = \bigoplus_{k+l=m} A_X^{k,l}$ and $A_X^{\cdot}$ is a bigraded algebra with respect to the wedge product. The natural involution $\varphi \mapsto \bar{\varphi}$ applies $A_X^{k,l}$ on to $A_X^{l,k}$.

(viii) There is a canonical morphism $d = \partial + \bar{\partial}: A_X^m \to A_X^{m+1}$ satisfying the usual identities $d^2 = \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. However, none of the resulting complexes (Dolbeault, De Rham, etc. ...) is an exact sequence of sheaves in general.

(ix) Any morphism $f: X \to Y$ of complex spaces gives rise to a linear $f^*: A^m(Y) \to A^m(X)$ which is compatible with the wedge product, bigraduation and the operators $d, \partial, \bar{\partial}$. We have $(fg)^* = g^* f^*$.

2. Strongly Plurisubharmonic Functions

We write p.s.h. for plurisubharmonic.

2.1. Definitions. We define on X the sheaves of real convex cones P_X^0 (resp. SP_X^0) of continuous p.s.h. (resp. strongly p.s.h.) functions, P_X^∞ (resp. SP_X^∞) of $\mathscr{C}^\infty$ p.s.h. (resp. strongly p.s.h.) functions as the subsheaves of $\mathscr{C}_{X,\mathbb{R}}$ (resp. $\mathscr{C}_{X,\mathbb{R}}^\infty$) consisting of elements induced by corresponding functions on open sets of $\mathbb{C}^n$ under local embeddings. Also define $[P_X^\infty] := \varrho(P_X^\infty)$, $[SP_X^\infty] := \varrho(SP_X^\infty)$ where $\varrho: \mathscr{C}_X^\infty \to \mathscr{C}_X$ is the canonical morphism.

2.2. Examples. (i) On the subspace X of $\mathbb{C}^2$ defined by $z_1 z_2 = z_2^2 = 0$, set $\varphi_t(z_1, z_2) := z_1 \bar{z}_1 + t z_2 \bar{z}_2$ for real t. Then $[\varphi_t] \in SP^0(X)$ is independent of t, $\varphi_t \in \mathscr{C}^\infty(X, \mathbb{R})$ for all t but $\varphi_t \in P^\infty(X)$ only for $t \geqq 0$ and $\varphi_t \in SP^\infty(X)$ only for $t > 0$.

(ii) On the subspace X of $\mathbb{C}^2$ defined by $z_1 z_2 = 0$, set (for real t) $\varphi_t(z_1, z_2) := z_1 \bar{z}_1 + t(z_1 \bar{z}_2 + z_2 \bar{z}_1) + z_2 \bar{z}_2$. Then $[\varphi_t] \in SP^0(X)$ is independent of t, $\varphi_t \in \mathscr{C}^\infty(X, \mathbb{R})$ but $\varphi_t \in P^\infty(X)$ only for $|t| \leqq 1$ and $\varphi_t \in SP^\infty(X)$ only for $|t| < 1$.

(iii) On $\mathbb{C}^n$ set $\varphi(z_1, \dots, z_n) := \sum_{j=1}^n |t_j|^2$ where $t_1, \dots, t_n$ are the roots of $X^n - z_1 X^{n-1} + \dots + (-1)^n z_n$. Then $\varphi \in SP^0(\mathbb{C}^n)$.

2.3. The Cone $SP^{0,\infty}(U, V)$. This is an auxiliary notion introduced to give a meaning to smoothing lemmas of strongly p.s.h. functions. For U, V open in X, $SP^{0,\infty}(U, V)$ is defined as the set of pairs $\varphi = (\varphi^0, \varphi^\infty) \in SP^0(U) \times SP^\infty(U \cap V)$ such that $[\varphi^\infty] = \varphi^0|_{U \cap V}$. We set $[\varphi] := \varphi^0$. The following are obvious

(i) $SP^{0,\infty}(U, V) = SP^{0,\infty}(U, U \cap V)$.

(ii) $SP^{0,\infty}(U, \emptyset) \cong SP^0(U)$ canonically.

(iii) $SP^{0,\infty}(U, X) \cong SP^\infty(U)$ canonically.

(iv) For fixed V, $U \mapsto SP^{0,\infty}(U, V)$ is a sheaf on X.

(v) For $\varphi = (\varphi^0, \varphi^\infty) \in SP^{0,\infty}(U, V)$ and $h \in PH(U, \mathbb{R})$, the element $\varphi + h := (\varphi^0 + [h], \varphi^\infty + h|_{U \cap V})$ is in $SP^{0,\infty}(U, V)$.

The following is a slight improvement of a result of Richberg [21, Satz 4.1]. For $X = \mathbb{C}^n$, a complete proof is in [23].

2.4. Richberg's Lemma. *Let U, V, W be open in X with $U \subset\subset W$. Let $\varphi \in SP^{0,\infty}(W, V)$. Then there is a compact S such that $U \subset S \subset W$ and an ele-*

ment $\psi \in SP^{0,\infty}(W, U \cup V)$ *such that* $\varphi|_{W\setminus S} = \psi|_{W\setminus S}$ *in* $SP^{0,\infty}(W\setminus S, V) = SP^{0,\infty}(W\setminus S, U \cup V)$.

Sketch of Proof. Take a finite number of open sets $U_k \subset\subset V_k \subset\subset W_k$ $(1 \leqq k \leqq m)$ such that $U = \bigcup_{k=1}^{m} U_k$ and each W_k is embedded in an open subset D_k^0 of $\mathbb{C}^{n_k}$ such that $[\varphi]|_{W_k}$ is induced by an element of $SP^0(D_k)$. Using the method of [23] one can construct inductively elements $\varphi_k \in SP^{0,\infty}(W, U_1 \cup \ldots \cup U_k \cup V)$ such that $\varphi_k|_{W\setminus \bar{V}_k} = \varphi_{k-1}|_{W\setminus \bar{V}_k}$. Then set $S = \bar{V}_1 \cup \ldots \cup \bar{V}_m$ and $\psi = \varphi_m$.

2.5. The Fornaess-Narasimhan Theorem [10, Theorem 5.3.1]. *Let* $\varphi \in \mathscr{C}(X, \mathbb{R})$. *Suppose that for any holomorphic* $f: \Delta \to X$, *where* Δ *is the unit disc of* $\mathbb{C}$, $\varphi \circ f$ *is subharmonic on* Δ. *Then* $\varphi \in P^0(X)$.

2.6. The Cone $SP_\pi^\infty(X)$. Let $\pi: X \to Y$ be a morphism of complex spaces. Let $\varphi \in \mathscr{C}^\infty(X, \mathbb{R})$. We say that φ is *strongly p.s.h. relatively to* π and write $\varphi \in SP_\pi^\infty(X)$ if for any $x \in X$ there are open subsets $U \subset X$, $V \subset Y$ and $\psi \in SP^\infty(V)$ such that $x \in U \subset \pi^{-1}(V)$ and $(\varphi + \pi^*\psi)|_U \in SP^\infty(U)$.

3. Barlet's Space of Analytic Cycles

3.1. Symmetric Powers of Complex Spaces. If $k \geqq 1$ is an integer, let $\mathrm{Sym}^k(X) := X^k/\mathscr{S}_k$ be the quotient of X^k under the action of the symmetric group permuting components. Denote by $\sum_{j=1}^{k} \{x_j\}$ the image of $(x_1, \ldots, x_k)$ in $\mathrm{Sym}^k(X)$ under the canonical projection.

3.2. Analytic Families of Complex Cycles. $\mathbf{B}_m(X)$. Let X be reduced and $m \geqq 0$ an integer. A *compact complex-analytic m-cycle* (or briefly *m-cycle*) of X is a formal finite sum

$$c = \sum_{i \in I} n_i Y_i,$$

where $n_1 \geqq 1$ are integers and Y_i are compact irreducible m-dimensional complex-analytic subsets of X. $|c| := \bigcup_{i \in I} Y_i$ is called the *support* of c.

Let c be as above and $\sigma: V \to U \times B$ an embedding of an open set $V \subset X$ into a connected open set $U \times B$ of $\mathbb{C}^N = \mathbb{C}^m \times \mathbb{C}^{N-m}$. We say that $\mathscr{V} = (\sigma, V, U \times B)$ is a *well-adapted chart with respect to c* if σ extends to an embedding $\sigma_1: V_1 \to U_1 \times B_1$ such that $V \subset\subset V_1 \subset X$, $U \subset\subset U_1 \subset \mathbb{C}^m$, $B \subset\subset B_1 \subset \mathbb{C}^{N-m}$ and $\sigma_1(|c|) \cap (\bar{U} \times \partial B) = \emptyset$.

If we set $Z_i := \sigma(V \subset Y_i) \subset U \times B$, then the projection $U \times B \to U$ restricted to each Z_i is a branched covering $\pi_i: Z_i \to U$ of finite degree k_i and defines as such a morphism $\psi_i: U \to \mathrm{Sym}^{k_i}(B)$. Set $k := \sum n_i k_i$, $\psi := \sum n_i \psi_i: U \to \mathrm{Sym}^k(B)$, $\deg(c, \mathscr{V}) := k$.

Now let S be a reduced complex space and $(c_s)_{s \in S}$ a family of m-cycles of X parametrized by S. We say that (c_s) is an *analytic family of cycles* if for any $s_0 \in S$ and for *any* well-adapted chart $\mathscr{V}$ with respect to c_{s_0}, there is a neighborhood T of s_0 in S such that

(i) $\mathscr{V}$ is well-adapted with respect to c_s for all $s \in T$.
(ii) $\deg(c_s, \mathscr{V}) = k$ is independent of $s \in T$.
(iii) The resulting map $\psi: U \times T \to \mathrm{Sym}^k(B)$ is holomorphic.

The Barlet space $\mathbf{B}_m(X)$ of m-cycles of X is a reduced complex space, constructed in [3], whose points are the m-cycles of X forming a tautological analytic family and such that for any analytic family $(c_s)_{s\in S}$ of m-cycles of X, there is a unique morphism of complex spaces $H: S \to \mathbf{B}_m(X)$ such that

$$H(s) = c_s \quad \text{for all} \quad s \in S .$$

For X not necessarily reduced, we set

$$\mathbf{B}_m(X) := \mathbf{B}_m(X_{\mathrm{red}}) .$$

3.3. Proper Open Morphism. Geometric Flatness. Let $\mathbf{D}_m(X)$ be the Douady space [9] of compact subspaces of pure dimension m of X. In [3, Chap. 5], Barlet constructed a canonical morphism

$$c: (\mathbf{D}_m(X))_{\mathrm{red}} \to \mathbf{B}_m(X) .$$

If Y is a point of $\mathbf{D}_m(X)$ (a subspace of X) then $c(Y) = \sum n_i Y_i$ where Y_i are the irreducible components of Y_{red} and $n_i \geqq 1$ integers called multiplicities. If Y is generically reduced, all n_i are equal to 1.

Now suppose that $\pi: X \to X'$ is a morphism of complex spaces such that, for some fixed $m \geqq 0$

(3.3)
(i) π is proper open and surjective,
(ii) all fibers of π are of pure dimension m,
(iii) X' is reduced.

[If X, X' are pure dimensional, then (i) implies (ii).]

We will say that π is *geometrically flat* if there is a morphism of complex spaces

$$H: X' \to \mathbf{B}_m(X)$$

such that $H(x') = c(\pi^{-1}(x'))$ generically on X'. We call H the *classifying morphism of* π. The domain of validity of the equality $H(x') = c(\pi^{-1}(x'))$ is the dense Zariski open set U' of points of flatness of π (Frisch [11]).

3.3.1. Proposition. *Suppose* $\pi: X \to X'$ *satisfies* (3.3). *Then:*
(i) *If π is flat, then it is geometrically flat.*
(ii) *If X' is normal, then π is geometrically flat.*
(iii) *If π is geometrically flat, then H defines an isomorphism of X' onto a subspace of* $\mathbf{B}_m(X)$.

Proof. (i) If π is flat, then there is a morphism $X' \to \mathbf{D}_m(X)$, factoring through $(\mathbf{D}_m(X))_{\mathrm{red}}$ since X' is reduced, taking the value $\pi^{-1}(x')$ at x'. Composing with $c: (\mathbf{D}_m(X))_{\mathrm{red}} \to \mathbf{B}_m(X)$, we obtain the required H.

(ii) This is part of Theorem 1 of [3].

(iii) This is shown in [24, Appendix, p. 259].

3.3.2. Examples. (i) Let X be the union of two planes defined by $z_1 z_2 = z_1 z_4 = z_2 z_3 = z_3 z_4 = 0$ in $\mathbb{C}^4$, Y' the union of two lines defined by $x_1 x_2 = 0$ in $\mathbb{C}^2$, $\pi: X \to X' = \mathbb{C}^2$ and $\varrho: Y' \to X'$ defined by $\pi(z_1, z_2, z_3, z_4) = (z_1 + z_2, z_3 + z_4)$ and $\varrho(x_1, x_2) = (x_1, 0)$

$$\begin{array}{ccc} X & \longleftarrow & Y = X \times_{X'} Y' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ X' & \xleftarrow{\varrho} & Y' \end{array}$$

Then π is geometrically flat be 3.3.1 (ii) but π_1 is not since Y consists of one triple line over one branch of Y' and two single lines over the other. π is not flat.

(ii) Let X be the union of two single lines and one double line defined by $z_1 z_2 = z_2^2 - z_3^2 = 0$ in $\mathbb{C}^2$ and X' the union of two lines $z_1 z_2 = 0$ (as Y' above).

If $\pi(z_1, z_2, z_3) = (z_1, z_2)$, then $\pi: X \to X'$ is flat, X' is reduced but if $r: X_{\text{red}} \to X$ is the reduction of X then $\pi r: X_{\text{red}} \to X'$ is not geometrically flat.

3.4. Integration of Differential Forms. If $\varphi \in A^{m,m}(X)$ and $c = \sum n_i Y_i \in \mathbf{B}_m(X)$, define

$$F_\varphi(c) := \int_c \varphi = \sum n_i \int_{Y_i} \varphi .$$

If $\pi: X \to X'$ is geometrically flat with m-dimensional fibers and φ is a above, define

$$\pi_* \varphi := F_\varphi \circ H_\pi .$$

We have the following:

3.4.1. Proposition [4, 5, 23]. *With the above notations.*

(i) F_φ *(resp.* $\pi_* \varphi$*) is continuous on* $\mathbf{B}_m(X)$ *(resp.* X'*).*

(ii) *If* $d\varphi = 0$, *then* F_φ *and* $\pi_* \varphi$ *are locally constant.*

(iii) *If* $\varphi = \bar{\varphi}$ *and* $i\partial\bar{\partial}\varphi \geqq 0$ *then* F_φ *and* $\pi_* \varphi$ *are p.s.h.*

(iv) *If* $\varphi = \bar{\varphi}$ *and* $i\partial\bar{\partial}\varphi \gg 0$ *then* F_φ *and* $\pi_* \varphi$ *are strongly p.s.h.*

(v) *If* $\bar{\partial}\varphi = 0$ *then* F_φ *and* $\pi_* \varphi$ *are weakly holomorphic; if moreover* X *is smooth, they are holomorphic.*

3.4.2. Remark. Case (iii) above needs the Fornaess-Narasimhan theorem if we look at the proof of Proposition 1 of [5].

3.4.3. Definition. A $\bar{\partial}$-closed $\tau \in A^{m,m}(X)$ is said to *represent* an element $\xi \in H^m(X, \Omega_X^m)$ (or to be a *$\bar{\partial}$-closed representative of* ξ) if the class of τ in $H_{\bar{\partial}}^{m,m}(X)$ is the image of ξ under the canonical morphism $H^m(X, \Omega_X^m) \to H_{\bar{\partial}}^{m,m}(X)$. In that case we define $F_\xi(c) := F_\tau(c)$ for $c \in \mathbf{B}_m(X)$ and also write $(c \cdot \xi)$ for $F_\xi(c)$ (since it depends on ξ alone).

3.5. m-Complete and m-Admissible Neighborhoods. By the Andreotti-Grauert theorem [1], if X is a m-complete complex space, then for any coherent analytic sheaf $\mathcal{F}$ on X and any $q > m$ we have $H^q(X, \mathcal{F}) = 0$. We will use

3.5.1. Proposition. *Let Y be a compact m-dimensional complex-analytic subset of X. Then*

(i) *Y admits in X a fundamental system of m-complete neighborhoods (Barlet* [6]*).*

(ii) *Y admits in X a fundamental system of neighborhoods V such that $H^k(V, \mathbb{R})=0$ for $k>2m$ [23, Lemma 3.5].*

3.5.2. Definition. An open $U \subset X$ is said to be *m-admissible* if
(i) U is m-complete.
(ii) There is an open V such that $U \subset V \subset X$ and $H^k(V, \mathbb{R})=0$ for all $k>2m$.

3.5.3. Remark. If X is a Kähler manifold with a Kähler form ω and $U \subset X$ is 0-admissible, then one easily sees that $\omega|_U = i\partial\bar{\partial}\varphi$ for some $\varphi \in SP^\infty(U)$. This is the most trivial particular case of our Theorem 2.

3.5.3. Proposition. (i) *If $U \subset X$ is m-admissible and $k>2m$, then the canonical morphism $H^k(X, \mathbb{R}) \to H^k(U, \mathbb{R})$ is zero.*
(ii) *Any compact m-dimensional complex-analytic subset of X admits a fundamental system of m-admissible neighborhoods.*

Proof. (i) Is obvious by the definitions and (ii) is a restatement of 3.5.1.

3.5.4. Proposition. *Let $\mathbf{B}_m(X)^{(o)}$ be the open set of $\mathbf{B}_m(X)$ consisting of cycles whose support admits in X a smoothly embeddable neighborhood. Let $\xi \in H^m(X, \Omega_X^m)$. Then F_ξ is holomorphic on $\mathbf{B}_m(X)^{(o)}$.*

Sketch of Proof. For $c \in \mathbf{B}_m(X)^{(o)}$, $|c|$ admits a smoothly embeddable neighborhood V therefore by 3.5.1 a neighborhood U with an embedding $\sigma: U \to U_1$ in a smooth m-complete U_1.

If $\mathcal{N}$ is the coherent sheaf on U_1 defined by the exact sequence

$$0 \to \mathcal{N} \to \Omega_{U_1}^m \to \sigma_* \Omega_U^m \to 0,$$

then $H^{m+1}(U_1, \mathcal{N})=0$ and hence $\xi|_U$ is induced by some $\xi_1 \in H^m(U_1, \Omega_{U_1}^m)$. By 3.4.1(v), F_{ξ_1} is holomorphic on $\mathbf{B}_m(U_1)$ so F_ξ is holomorphic near c.

3.5.5. Corollary. *If $\pi: X \to X'$ is geometrically flat with m-dimensional fibers and U' is the set of $x' \in X'$ such that $\pi^{-1}(x')$ admits in X smoothly embeddable neighborhoods then for any $\xi \in H^m(X, \Omega_X^m)$, $\pi_* \xi|_{U'}$ is holomorphic.*

3.6. Note Added in Proof. After having submitted the manuscript, the author together with D. Barlet solved problem D of the Introduction. Proposition 3.5.4 and Corollary 3.5.5 above are now true with $\mathbf{B}_m(X)$ instead of $\mathbf{B}_m(X)^{(0)}$. The notion of a weakly Kähler space loses its importance and Theorems 3 and 4 below (Ch. IV) become

Theorem 3'. *If $\pi: X \to X'$ is geometrically flat with X Kähler and X' reduced, then X' is Kähler.*

Theorem 4'. *If X is Kähler then $\mathbf{B}_m(X)$ is Kähler.*

II. Theorem 1 and its First Consequences

1. Kähler Spaces and Kähler Metrics

Let X be a complex space.

1.1. The Sheaf $\mathscr{K}_X^1$. Define

$$\mathscr{K}_X^1 := \mathscr{C}_X^\infty / PH_X\,, \qquad \mathscr{K}_{X,\mathbb{R}}^1 := \mathscr{C}_{X,\mathbb{R}}^\infty / PH_{X,\mathbb{R}}\,,$$

$$\mathscr{K}^1(X) := H^0(X, \mathscr{K}_X^1)\,, \qquad \mathscr{K}^1(X,\mathbb{R}) := H^0(X, \mathscr{K}_{X,\mathbb{R}}^1)\,.$$

A section $\kappa \in \mathscr{K}^1(X)$ corresponds by definition to an open covering (U_α) of X together with elements $\varphi_\alpha \in \mathscr{C}^\infty(U_\alpha)$ such that $\varphi_\alpha - \varphi_\beta \in PH(U_\alpha \cap U_\beta)$. We write $\kappa = \{(U_\alpha, \varphi_\alpha)\}$. We have

$$\{(U_\alpha, \varphi_\alpha)\} = \{(V_j, \psi_j)\} \quad \text{iff} \quad (\varphi_\alpha - \psi_j)|_{U_\alpha \cap V_j} \in PH(U_\alpha \cap V_j)\,.$$

For such κ, we set $\partial\bar\partial\kappa := \omega \in A^{1,1}(X)$ where

$$\omega|_{U_\alpha} = \partial\bar\partial\varphi_\alpha\,.$$

Of course, ω is well-defined and $d\omega = 0$. We say that κ is *represented* by the φ_α.

1.2. Kähler Metrics, Kähler Classes. A *Kähler metric* on X is by definition an element $\kappa \in \mathscr{K}^1(X,\mathbb{R})$ represented by a system of sections of SP_X^∞. The *Kähler form* of (X,κ) is $\omega := i\partial\bar\partial\kappa$ $(i = \sqrt{-1})$. We will often write (X,ω) instead of (X,κ), although ω does not determine κ unless X is smooth.

Similarly, if $\pi : X \to Y$ is a morphism of complex spaces, a *relative Kähler metric for* π is an element κ_π of $\mathscr{K}^1(X,\mathbb{R})$ represented by sections of SP_π^∞.

To any element $\kappa \in \mathscr{K}^1(X)$ we associate three cohomology classes as follows:

From the exact sequence $0 \to PH_X \to \mathscr{C}_X^\infty \to \mathscr{K}_X^1 \to 0$, we deduce a canonical morphism

$$\hat c_1 : \mathscr{K}^1(X) \to H^1(X, PH_X) \tag{1.2.1}$$

which obviously sends $\mathscr{K}^1(X,\mathbb{R})$ into $H^1(X, PH_{X,\mathbb{R}})$. From the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r, "-2\mathrm{Im}"] \arrow[d] & PH_{X,\mathbb{R}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{O}_X \oplus \bar{\mathcal{O}}_X \arrow[r, "(i\ \ -i)"] \arrow[dr, "(d\ \ o)"'] & PH_X \arrow[r] \arrow[d, "-i\partial"] & 0 \\
 & & & d\mathcal{O}_X &
\end{tikzcd} \tag{1.2.2}$$

we deduce canonical morphisms $H^1(X, PH_X) \to H^2(X, \mathbb{C})$ and $H^1(X, PH_X) \to H^1(X, d\mathcal{O}_X)$ and, composing with $\hat c_1$, we obtain

$$\begin{aligned} c_1 &: \mathscr{K}^1(X) \to H^2(X, \mathbb{C})\,, \\ \tilde c_1 &: \mathscr{K}^1(X) \to H^1(X, d\mathcal{O}_X)\,. \end{aligned} \tag{1.2.3}$$

Of course c_1 sends $\mathscr{K}^1(X,\mathbb{R})$ into $H^2(X,\mathbb{R})$. $d\mathcal{O}_X$ is the subsheaf of Ω_X^1 consisting of locally exact holomorphic 1-forms. Sometimes we will replace $\tilde c_1(\kappa)$ by its image in $H^1(X, \Omega_X^1)$.

So we have a diagram

$$
(1.2.4)\qquad
\begin{tikzcd}
H^{1,1}_{\bar\partial}(X) & & H^1(X,\Omega^1_X) \arrow[ll] & & H^1(X,d\mathcal{O}_X) \arrow[ll] \\
Z^{1,1}_d(X) \arrow[u] \arrow[dr] & & \mathcal{K}^1(X) \arrow[ll, "i\partial\bar\partial"'] \arrow[urr, "\tilde{c}_1"] \arrow[rr, "\hat{c}_1"] \arrow[dr, "c_1"] & & H_1(X,PH_X) \arrow[u] \arrow[dl] \\
& H^2_d(X) & & H^2(X,\mathbb{C}) \arrow[ll] &
\end{tikzcd}
$$

which is commutative (see 4.2 of Chap. III). This means that if κ is a Kähler metric on X and $\omega=i\partial\bar\partial\kappa$ the corresponding Kähler form, then ω is a d-closed represenattive of $c_1(\kappa)$ in $H^2(X,\mathbb{R})$ and also a $\bar\partial$-closed representative of $\tilde{c}_1(\kappa)$ in $H^1(X,\Omega^1_X)$.

In [15] Grauert proved that if κ is a Kähler metric on a normal compact space X such that $c_1(\kappa)$ lies in the canonical image of $H^2(X,\mathbb{Q})$ in $H^2(X,\mathbb{R})$, then X is a projective variety.

1.3. Kähler Spaces, Kähler Morphisms. X is said to be a *Kähler space* if there exists a Kähler metric on X.

A morphism $\pi: X\to Y$ is a *Kähler morphism* if there exists a relative Kähler metric κ_π for π.

We have the following elementary properties:

1.3.1. Proposition. (i) *Subspaces of Kähler spaces are Kähler.*

(ii) *Smooth Kähler spaces are Kähler manifolds in the usual sense.*

(iii) $X\to\{y\}$ *is a Kähler morphism iff* X *is a Kähler space.*

(iv) *Kähler morphisms are preserved by composition and base-change* [8].

(v) *Projective morphisms (for example: finite morphisms and blow-ups) are Kähler* [8, 12].

(vi) *If* $\pi: X\to Y$ *is a Kähler morphism, and* Y *a Kähler space then any open* $U\subset\subset X$ *is Kähler. More precisely: If* κ_Y *is a Kähler metric on* Y *and* κ_π *a relative Kähler metric for* π, *then for any* $U\subset\subset X$ *there is a constant* $c_0>0$ *such that for any* $c>c_0$, $(\kappa_\pi+c\pi^*\kappa_Y)_{|U}$ *is a Kähler metric on* U [8, 12].

On the other hand,

1.3.2. Proposition. (i) *It is not always true that a reduced compact space is Kähler if its normalization is Kähler.*

(ii) *It is not always true that a compact space* X *is Kähler if* X_{red} *is Kähler. A counterexample* [8, *II*] *is given by an infinitesimal neighborhood of a* K3 *surface in its space of moduli.*

(iii) *It is not always true that a normal compact space is Kähler if the complement of a point is Kähler* [15, 20].

(iv) *It is not always true that small deformations of compact Kähler spaces are Kähler* [20].

(v) *It is not always true that a normal compact space that is both Moïezon and Kähler is projective* [20].

2. *Theorem 1*

2.1. Statement. Let X be a complex space. Suppose it admits an open covering $(U_\alpha)_{\alpha\in A}$ and a system of *continuous* strongly p.s.h. functions $\varphi_\alpha \in SP^0(U_\alpha)$ together with pluriharmonic functions $h_{\alpha\beta}\in PH(U_\alpha\cap U_\beta, \mathbb{R})$ such that

$$(2.1.1)\qquad \begin{aligned} &\text{(i) } \varphi_\alpha-\varphi_\beta=[h_{\alpha\beta}] \quad \text{in} \quad \mathscr{C}(U_\alpha\cap U_\beta,\mathbb{R}),\\ &\text{(ii) } h_{\alpha\beta}-h_{\alpha\gamma}+h_{\beta\gamma}=0 \quad \text{on} \quad U_\alpha\cap U_\beta\cap U_\gamma.\end{aligned}$$

Then there are elements $\psi_\alpha\in SP^\infty(U_\alpha)$ such that

$$(2.1.2)\qquad \psi_\alpha-\psi_\beta=h_{\alpha\beta} \quad \text{in}\,\mathscr{C}^\infty(U_\alpha\cap U_\beta,\mathbb{R}).$$

In particular, X is a Kähler space.

2.2. Remark. By Lemma 1.2(iv) of Chap. I, the cocycle condition (ii) is redundant for X reduced. For smooth X, Theorem 1 is proven in [23] and the proof we give there is valid for X reduced and locally irreducible. We will use the conventions stated in 2.4 of Chap. I.

2.3. Proof. Since X is paracompact, it admits two locally finite open coverings (V_k), (W_k) $(k\in\mathbb{N})$ such that $V_0=\emptyset$ and $V_k\subset\subset W_k\subset U_{\alpha_k}$ for each k. Set $T^k_{\alpha\beta}:=U_\alpha\cap U_\beta \cap(V_1\cup\ldots\cup V_k)$.

We will define inductively elements

$$\varphi^k_\alpha\in SP^{0,\infty}(U_\alpha, V_1\cup\ldots\cup V_k)$$

such that

(i) For some compact S_k, $V_k\subset S_k\subset W_k$,

$$\varphi^k_\alpha|_{U_\alpha\setminus S_k}=\varphi^{k-1}_\alpha|_{U_\alpha\setminus S_k}$$

in $SP^{0,\infty}(U_\alpha\setminus S_k, V_1\cup\ldots\cup V_k)=SP^{0,\infty}(U_\alpha\setminus S_k, V_1\cup\ldots\cup V_{k-1})$

$$(2.3.1)\qquad \begin{aligned} &\text{(ii) } [\varphi^k_\alpha]-[\varphi^k_\beta]=[h_{\alpha\beta}] \quad \text{in} \quad \mathscr{C}(U_\alpha\cap U_\beta,\mathbb{R}),\\ &\text{(iii) } (\varphi^k_\alpha-\varphi^k_\beta)|_{T^k_{\alpha\beta}}=h_{\alpha\beta}|_{T^k_{\alpha\beta}} \quad \text{in} \quad \mathscr{C}^\infty(T^k_{\alpha\beta},\mathbb{R}).\end{aligned}$$

We start by taking $\varphi^0_\alpha:=\varphi_\alpha$ the initial data.
Suppose φ^{k-1}_α is defined for all α.
Apply Richberg's lemma to $X=W_k$,

$$U=V_k\,,\qquad V=V_1\cup\ldots\cup V_{k-1}\,,\qquad \varphi=\varphi^{k-1}_{\alpha_k}|_{W_k}\,.$$

We obtain an element

$\psi\in SP^{0,\infty}(W_k, V_1\cup\ldots\cup V_k)$

and a compact S_k, $V_k\subset S_k\subset W_k$ such that

$$\psi|_{W_k\setminus S_k}=\varphi^{k-1}_{\alpha_k}|_{W_k\setminus S_k}\,.$$

Now we set

$$(2.3.2)\qquad \varphi^k_\alpha:=\begin{cases}\varphi^{k-1}_\alpha & \text{on} \quad U_\alpha\setminus S_k\\ \psi+h_{\alpha\alpha_k} & \text{on} \quad U_\alpha\cap W_k\,,\end{cases}$$

where the last expression is defined in 2.4(v) of Chap. I.

By the induction hypothesis, (2.3.1) is valid for the rank $k-1$, hence definition (2.3.2) is consistent. But this implies (2.3.1) for the rank k as well. Indeed, (i) is obvious. (ii) and (iii) can be easily checked on W_k by the cocycle condition (2.1.1)(ii) and outside S_k by the induction hypothesis. So (2.3.1) is valid.

Now since $S_k \subset W_k$, (S_k) is locally finite and, for fixed α, $(\varphi_\alpha^k)_{k\in\mathbb{N}}$ is locally stationary. We may set

$$\psi_\alpha := \lim_{k\to\infty} \varphi_\alpha^k \in SP^\infty(U_\alpha)$$

and the conclusion of Theorem 1 is satisfied.

2.4. Corollary. *The "old" and "modern" definition of a reduced Kähler space coincide.*

Proof. By 1.2(iv) of Chap. I, if X is reduced, PH_X can be identified to a subsheaf of $\mathscr{C}_X$. A Kähler metric in the "old" sense is a section of $\mathscr{C}_{X,\mathbb{R}}/PH_{X,\mathbb{R}}$ represented locally by sections of $[SP_X^\infty]$. Since $[SP_X^\infty]\subset SP_X^0$, Theorem 1 applies.

3. *Application to Finite Morphisms*

Theorem 1 implies that images of Kähler spaces under certain finite morphisms are Kähler. This solves a problem raised by Lieberman at the end of [18].

3.1. *Traces of Continuous and Holomorphic Functions*

3.1.1. Definitions. If X is reduced, $k\geqq 1$ an integer and $\varphi\in\mathscr{C}(X)$, then

$$\tilde{\varphi}: \sum_{j=1}^{k}\{x_j\}\mapsto \sum_{j=1}^{k}\varphi(x_j)$$

defines a continuous function on $\mathrm{Sym}^k(X)$. On the other hand, for arbitrary X we have

$$\mathbf{B}_0(X)=\coprod_{k\geqq 1}\mathrm{Sym}^k(X_{\mathrm{red}}).$$

Now suppose $\pi: X\to X'$ is a finite open surjective morphism with connected base X'. We examine the following two situations:

(1) X' is reduced and π is geometrically flat;

(2) X' is arbitrary and π is flat.

In the first case, there is an integer $k=k_\pi\geqq 1$ called the (geometric) *degree of* π such that the classifying morphism $H: X'\to\mathbf{B}_0(X)$ factors through $\mathrm{Sym}^k(X_{\mathrm{red}})$. We have for generic $x'\in X'$ (on the points of flatness of π)

$$H(x')=\sum_{x\in\pi^{-1}(x')}\{x\},$$

where the sum takes account of multiplicities.

Define a *continuous trace* morphism

$$\mathrm{Tr}_{X/X'}^{(c)}: \pi_*\mathscr{C}_X\to\mathscr{C}_{X'}$$

by $\varphi\mapsto\tilde{\varphi}\circ H$.

In the second case, there is an integer $r=r_\pi \geqq 1$ called the (algebraic) *degree of* π such that $\pi_* \mathcal{O}_X$ is a locally free $\mathcal{O}_{X'}$-module of rank r. Define the *holomorphic trace* morphism

$$\mathrm{Tr}^{(h)}_{X/X'}: \pi_* \mathcal{O}_X \to \mathcal{O}_{X'}$$

by $f \mapsto$ trace of the linear map $\{g \mapsto fg\}$.

r_π is preserved by base change and, if X' is reduced, coincides with k_π. For general X', define $\bar\pi$, Y, $\bar Y$ by the cartesian diagram

$$\begin{array}{ccc} X & \longleftarrow & Y := X \times_{X'} Y' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar\pi} \\ X' & \longleftarrow & Y' := X'_{\mathrm{red}}\,. \end{array}$$

Then we have $r_\pi = r_{\bar\pi} = k_{\bar\pi}$. We define

$$\mathrm{Tr}^{(c)}_{X/X'} := \mathrm{Tr}^{(c)}_{Y/Y'}$$

since $\mathscr{C}_X = \mathscr{C}_Y$ and $\mathscr{C}_{X'} = \mathscr{C}_{Y'}$. The two trace morphisms so defined are compatible, i.e. the diagram

$$\begin{array}{ccc} \pi_* \mathcal{O}_X & \xrightarrow{\varrho} & \pi_* \mathscr{C}_X \\ {\scriptstyle \mathrm{Tr}^{(h)}_{X/X'}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}^{(c)}_{X/X'}} \\ \mathcal{O}_{X'} & \xrightarrow{\varrho} & \mathscr{C}_{X'} \end{array}$$

is commutative, where $\varrho: f \mapsto [f]$ is the canonical morphism. The holomorphic trace morphism is obviously extended to $\pi_* PH_X \to PH_{X'}$.

We write $\pi_* \varphi$ for $\mathrm{Tr}^{(c)}_{X/X'} \varphi$ or $\mathrm{Tr}^{(h)}_{X/X'} \varphi$ indifferently.

3.1.2. Lemma [5, 23]. *If φ is p.s.h., strongly p.s.h., holomorphic or pluriharmonic on X, then $\tilde\varphi$ (resp. $\pi_* \varphi$) has the corresponding properties on* $\mathrm{Sym}^k(X_{\mathrm{red}})$ *(resp. X').*

3.1.3. Remark. (i) For the "p.s.h." part of the above lemma, the Fornaess-Narasimhan theorem is needed.

(ii) It is not true in general that $\pi_* \varphi$ is $\mathscr{C}^\infty$ if φ is $\mathscr{C}^\infty$ even if X and X' are smooth.

3.2. Theorem. *Let X be a Kähler space and $\pi: X \to X'$ a finite open surjective morphism such that either*

(i) *X' is reduced and π is geometrically flat or*

(ii) *π is flat.*

Then X' is Kähler.

Proof. It results from 3.1.2 and Theorem 1 (exactly as Proposition 2.1 of [23]).

3.2.1. Corollary. *If X is a reduced Kähler space and G a finite group of automorphisms of X, then X/G is Kähler. In particular* $\mathrm{Sym}^k(X)$ *is Kähler.*

Proof. It is clear that the canonical projection $X \to X/G$ is geometrically flat. For X smooth and G having isolated fixed points, this is shown by Fujiki [13, Proposition 1].

3.2.2. Corollary. *If $\pi: X \to X'$ is finite surjective with X Kähler and X' normal then X' is Kähler.*

4. *Weakly Kähler Metrics*

Because of the impossibility to solve (for the moment) problem 3.6 (Chap. I) we are forced to introduce the notion of weakly Kähler spaces.

4.1. Definitions. If X, Y are reduced spaces, a function $f: X \to Y$ is *weakly holomorphic* if it is *continuous* and generically holomorphic. Let $\mathscr{W}_X$ be the sheaf of weakly holomorphic complex-valued functions on X. Define the sheaf WPH_X of *weakly pluriharmonic* functions by $WPH_X := \mathscr{W}_X + \overline{\mathscr{W}_X}$. X is *weakly normal* iff $\mathscr{W}_X = \mathcal{O}_X$. The *weak normalization* of X is a weakly normal space $\hat{X}$ [2] together with a holomorphic homeomorphism $n: \hat{X} \to X$ such that $n_* \mathcal{O}_{\hat{X}} = \mathscr{W}_X$. If X is not reduced, define the weak normalization $\hat{X} \to X$ as that of X_{red} followed by the reduction $X_{\text{red}} \to X$.

A *weakly Kähler metric* on X is a section of the quotient sheaf $\mathscr{C}_{X,\mathbb{R}}/WPH_{X,\mathbb{R}}$ represented by a system of sections of SP^0_X. X is *weakly Kähler* if X_{red} admits a weakly Kähler metric. We have (for X, Y, Z reduced spaces):

4.1.1. Lemma. (i) *If $f: X \to Y$ and $g: Y \to Z$ are weakly holomorphic, then $g \circ f: X \to Z$ is weakly holomorphic.*

(ii) *If $f: X \to Y$ is weakly holomorphic and $h \in WPH(Y)$, then $h \circ f \in WPH(X)$.*

(iii) *X is weakly normal iff every local irreducible component of X is normal.*

The Fornaess-Narasimhan theorem implies:

4.1.2. Lemma. (i) *If $f: X \to Y$ is weakly holomorphic and $\varphi \in P^0(Y)$, then $\varphi \circ f \in P^0(X)$.*

(ii) *$WPH_X \subset P^0_X$ (weakly pluriharmonic functions are p.s.h.).*

(iii) *$WPH_X SP^0_X \subset SP^0_X$ [a consequence of (ii)].*

4.1.3. Lemma. *Let $n: X \to \hat{X}$ be the weak normalization of X. For $\varphi \in \mathscr{C}(\hat{X})$, set $n_*\varphi := \varphi \circ n^{-1} \in \mathscr{C}(X)$. Then*

(i) *If $\varphi \in P^0(\hat{X})$, then $n_*\varphi \in P^0(X)$.*

(ii) *If $\varphi \in SP^0(\hat{X})$, then $n_*\varphi \in SP^0(X)$.*

(iii) *If $\varphi \in \mathcal{O}(\hat{X})$, then $n_*\varphi \in \mathscr{W}(X)$.*

(iv) *If $\varphi \in PH(\hat{X})$, then $n_*\varphi \in WPH(X)$.*

4.2. *Relation with Kähler Metrics*

4.2.1. Lemma. *If X is weakly Kähler and weakly normal, then X is Kähler.*

Proof. Since $WPH_X = PH_X$, Theorem 1 applies.

4.2.2. Lemma. *If $\pi: X \to Y$ is a Kähler morphism and Y a weakly Kähler space, then any open $U \subset\subset X$ is weakly Kähler.*

Proof. By an elementary argument similar to 1.3.1 (vi).

4.2.3. Proposition. *Let X be a complex space and $n: \hat{X} \to X$ its weakly normalization. Then*

(i) *If $\hat{X}$ is Kähler, then X is weakly Kähler.*

(ii) *If X is weakly Kähler, then every open $U \subset\subset \hat{X}$ is Kähler.*

Proof. (i) Is a consequence of Lemma 4.1.3 above.
(ii) Since n is finite, it is Kähler morphism by 1.3.1 (v). We apply 4.2.2 and 4.2.1 to conclude.

4.2.4. Corollary. *If X is compact, then $\hat{X}$ is Kähler iff X is weakly Kähler.*

III. Theorem 2

1. Čech Spaces and Čech Open Sets

1.1. Definitions. A (topological or complex-analytic) *Čech space* will be by definition a pair

$$\underline{X}=(X,\mathscr{X}),$$

where X is a (topological or complex) space and $\mathscr{X}$ an open covering of X. We call X the *space underlying to* $\underline{X}$ and always denote both by the same letter. We will deal only with complex-analytic Čech spaces. If $\mathscr{X}=(X_\lambda)_{\lambda\in\Lambda}$, the X_λ will be called the *elementary* open sets of X.

Suppose $\underline{X}=(X,(X_\lambda)_{\lambda\in\Lambda})$ and $\underline{Y}=(Y,(Y_\mu)_{\mu\in M})$ are two Čech spaces. A morphism

$$F:\underline{X}\to\underline{Y}$$

will be a pair $F=(f,\mu)$ where $f:X\to Y$ is a morphism in the ordinary sense and $\mu:\Lambda\to M$ a map such that

$$X_\lambda\subset f^{-1}(Y_{\mu(\lambda)}) \tag{1.1.1}$$

for all $\lambda\in\Lambda$. We call f the *morphism underlying to* F. We will say that F is an *open inclusion* if f is one.

A *Čech open set* $U\ll X$ will be a Čech space chose underlying space is an open subset of X together with an open inclusion

$$j:\underline{U}\to\underline{X}.$$

Or course, j is not uniquely determined by $\underline{U}$.

If $\underline{U}_1=(U_1,(U_{1,\alpha})_{\alpha\in A_1})$ and $\underline{U}_2=(U_2,(U_{2,\beta})_{\beta\in A_2})$ are two Čech open sets of $\underline{X}$, define

$$\underline{U}_1\cap\underline{U}_2:=(U_1\cap U_2,(U_{1,\alpha}\cap U_{2,\beta})_{(\alpha,\beta)\in A_1\times A_2}). \tag{1.1.2}$$

Notice that there are two open inclusions

$$j_1,j_2:\underline{U}_1\cap\underline{U}_2\to\underline{X}$$

each factoring through $\underline{U}_1$ and $\underline{U}_2$, respectively.

If $\underline{X}=(X,\mathscr{X})$ is a Čech space and $\mathscr{F}$ a sheaf of abelian groups on X, write

$$C^q(\underline{X},\mathscr{F}),\ Z^q(\underline{X},\mathscr{F}),\ H^q(\underline{X},\mathscr{F})$$

for the groups of Čech contains, cocycles and cohomology classes of degree q of the covering $\mathscr{X}$ with coefficients in $\mathscr{F}$. Denote by

$$\varepsilon:H^0(X,\mathscr{F})\to C^0(\underline{X},\mathscr{F}) \tag{1.1.3}$$

the canonical inclusion and by

$$\delta : C^{q-1}(\underline{X}, \mathscr{F}) \to C^q(\underline{X}, \mathscr{F})$$

the Čech differential given by the usual formula

$$(\delta\varphi)_{\lambda_0 \ldots \lambda_q} := \sum_{r=0}^{q} (-1)^r \varphi_{\lambda_0 \ldots \hat{\lambda}_r \ldots \lambda_q}|_{X_{\lambda_0} \cap \ldots \cap X_{\lambda_q}}. \tag{1.1.4}$$

If $\underline{U} \ll \underline{X}$ is a Čech open set with an open inclusion $j : \underline{U} \to \underline{X}$, denote by

$$j^* : C^q(\underline{X}, \mathscr{F}) \to C^q(\underline{U}, \mathscr{F})$$

the obvious morphism. We will write

$$\varphi|_{\underline{U}} := j^*(\varphi) \tag{1.1.5}$$

if there is no ambiguity about j.

Now suppose there are two open inclusions

$$j_1, j_2 : \underline{U} \to \underline{X}$$

with $\underline{U} = (U, (U_\alpha)_{\alpha \in A})$, $\underline{X} = (X, (X_\lambda)_{\lambda \in \Lambda})$.

There is a homotopy operator

$$T : C^{q+1}(\underline{X}, \mathscr{F}) \to C^q(\underline{U}, \mathscr{F})$$

defined by

$$(T\varphi)_{\alpha_0 \ldots \alpha_q} := \sum_{r=0}^{q} (-1)^r \varphi_{\lambda_0 \ldots \lambda_r \mu_r \ldots \mu_q}|_{U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}}, \tag{1.1.6}$$

where

$$U_{\alpha_r} \subset X_{\lambda_r} \quad \text{by } j_1$$

and

$$U_{\alpha_r} \subset X_{\mu_r} \quad \text{by } j_2 .$$

T is extended by 0 on $C^0(\underline{X}, \mathscr{F})$ and $H^0(X, \mathscr{F})$. The following is obvious

$$\delta T + T\delta = j_2^* - j_1^* . \tag{1.1.7}$$

1.2. Cup-Products of Čech Cochains. Now suppose that $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ are sheaves of differential forms ($\mathscr{C}^\infty$, holomorphic or antiholomorphic) such that

$$\mathscr{F} \wedge \mathscr{G} \subset \mathscr{H} .$$

We define the cup-product

$$C^q(\underline{X}, \mathscr{F}) \times C^r(\underline{X}, \mathscr{G}) \to C^{q+r}(\underline{X}, \mathscr{H})$$

by the identity

$$(\varphi \cdot \psi)_{\alpha_0 \ldots \alpha_{q+r}} := (\varphi_{\alpha_0 \ldots \alpha_q} \wedge \psi_{\alpha_q \ldots \alpha_{q+r}})|_{X_{\lambda_0} \cap \ldots \cap X_{\lambda_{q+r}}}. \tag{1.2.1}$$

As an immediate consequence we have

$$\delta(\varphi \cdot \psi) = (\delta\varphi) \cdot \psi + (-1)^q \varphi \cdot \delta\psi , \tag{1.2.2}$$

$$T(\varphi \cdot \psi) = (T\varphi) \cdot j_2^* \psi + (-1)^q (j_1^* \varphi) \cdot T\psi . \tag{1.2.3}$$

1.3. m-Complete and m-Admissible Čech Open Sets. We extend the notion of m-admissible open sets (3.5.3 of Chap. I) to Čech open sets.

1.3.1. Definitions. A Čech space $\underline{X}$ is said to be *m-complete* if for any coherent analytic sheaf $\mathscr{F}$ on X and any $q>m$, we have $H^q(\underline{X},\mathscr{F})=0$.

A sufficient condition for this is that the underlying space X be m-complete and the elementary open sets of $\underline{X}$ be Stein.

If $\underline{U} \ll \underline{X}$ is a Čech open set of $\underline{X}$, we will say that $\underline{U}$ is *m-admissible in X* if

(i) $\underline{U}$ is m-complete.

(ii) There is a Čech open set $\underline{V}$ such that $\underline{U} \ll \underline{V} \ll \underline{X}$ and $H^k(\underline{V},\mathbb{R})=0$ for all $k>2m$.

Of course, if the above are satisfied, then the canonical morphism $H^k(\underline{X},\mathbb{R}) \to H^k(\underline{U},\mathbb{R})$ vanishes for $k>2m$, since it factors through $H^k(\underline{V},\mathbb{R})=0$.

This may be expressed as follows:

1.3.2. Lemma. *If $\underline{U} \ll \underline{X}$ is m-admissible, $k>2m$ and $a \in Z^k(\underline{X},\mathbb{R})$, then there is an element $b \in C^{k-1}(\underline{U},\mathbb{R})$ such that $a|_{\underline{U}}=\delta b$.*

1.3.3. Proposition. *Let $\underline{X}$ be a Čech space and $U \subset X$ an open set (in the ordinary sense) that is m-admissible. Then U is underlying to some m-admissible Čech open set $\underline{U} \ll \underline{X}$.*

Proof. By definition U is m-complete and there is an open V such that $U \subset V \subset X$ and $H^k(V,\mathbb{R})=0$ for all $k>2m$. If we take a sufficiently fine Leray open covering of V with respect to the constant sheaf such that $\underline{V} \ll \underline{X}$ and then a sufficiently fine Stein open covering of U such that $\underline{U} \ll \underline{V}$, it is clear that $\underline{U} \ll \underline{X}$ is m-admissible.

2. *Čech Transform of a Complex of Sheaves*

2.1. Definitions. Let $\underline{X}$ be a Čech space and

$$0 \to \mathscr{F} \xrightarrow{j} \mathscr{L}^0 \xrightarrow{D} \mathscr{L}^1 \xrightarrow{D} \dots \xrightarrow{D} \mathscr{L}^m \xrightarrow{D} \dots \tag{2.1.1}$$

a complex of sheaves of abelian groups on the underlying space X. We do not suppose it to be an exact sequence of sheaves.

The *Čech transform* of the complex (2.1.1) over $\underline{X}$ will be the single complex associated to the double complex

$$\begin{array}{ccccccc}
H^0(X,\mathscr{F}) & \to & H^0(X,\mathscr{L}^0) & \to & H^0(X,\mathscr{L}^1) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
C^0(\underline{X},\mathscr{F}) & \to & C^0(\underline{X},\mathscr{L}^0) & \to & C^0(\underline{X},\mathscr{L}^1) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
C^1(\underline{X},\mathscr{F}) & \to & C^1(\underline{X},\mathscr{L}^0) & \to & C^1(\underline{X},\mathscr{L}^1) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

More precisely, we define for $q \geqq 0$

$$\check{C}^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot) := C^q(\underline{X},\mathscr{F}) \oplus \left\{ \bigoplus_{k=1}^{q} C^{q-k}(\underline{X},\mathscr{L}^{k-1}) \right\} \oplus H^0(X,\mathscr{L}^q). \tag{2.1.2}$$

An element of $\check{C}^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot)$ has the form

$$\Phi = (f; \varphi^0, \dots, \varphi^{q-1}; \eta^q),$$

where

$$f \in C^q(\underline{X}, \mathscr{F}),$$
$$\varphi^{k-1} \in C^{q-k}(\underline{X}, \mathscr{L}^{k-1}) \quad \text{for} \quad k = 1, \dots, q,$$
$$\eta^q \in H^0(X, \mathscr{L}^q).$$

We will call f the *head* of Φ, φ^{k-1} the *k-th component* of Φ and η^q the *tail* of Φ. Define the differential

$$\Delta : \check{C}^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot) \to \check{C}^{q+1}(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot)$$

by

(2.1.3) $$\Delta := \delta + (-1)^{q+1} D$$

where

$$\delta\Phi := (\delta f; \delta\varphi^0, \dots, \delta\varphi^{q-1}, \varepsilon\eta^q; 0),$$
$$D\Phi := (0; jf, D\varphi^0, \dots, D\varphi^{q-1}; D\eta^q).$$

Sometimes we will change the sign convention

$$\Delta = \delta + (-1)^{q+1} D \quad \text{to} \quad \Delta = \delta + (-1)^q D.$$

We then define

(2.1.4) $$\check{Z}^q(\underline{X}, \mathscr{F}, \mathscr{L}^\cdot) := \operatorname{Ker}\{\check{C}^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot) \xrightarrow{\Delta} \check{C}^{q+1}(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot)\}$$

and the Čech hypercohomology groups

(2.1.5) $$\check{H}^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot) := \check{Z}^q(X; \mathscr{F}, \mathscr{L}^\cdot)/\Delta\check{C}^{q-1}(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot).$$

We will use the following.

2.2. Lemma. *Let $r^q : H^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot) \to H^q(\underline{X}, \mathscr{F})$ be the canonical morphism.*
(i) *If $H^{q-k}(\underline{X}, \mathscr{L}^{k-1}) = 0$ for $k = 1, \dots, q-1$ then r^q is injective.*
(ii) *If $H^{q-k}(\underline{X}, \mathscr{L}^k) = 0$ for $k = 0, \dots, q-1$ then r^q is surjective.*

Proof. It is an immediate consequence of the following elementary property of double complexes: If $M^\cdot$ is the single complex associated to a double complex $K^{\cdot,\cdot} = (K^{i,j})_{i,j \geq 0}$, then the canonical morphism $H^q(M^\cdot) \to H^q(K^{\cdot,0})$ is injective if $H^{q-j}(K^{\cdot,j}) = 0$ for $j = 1, \dots, q-1$ and surjective if $H^{q-j}(K^{\cdot,j+1}) = 0$ for $j = 0, \dots, q-1$. This is to be applied for

$$K^{i,j} = \begin{cases} 0 & \text{if} \quad i = j = 0 \\ H^0(X, \mathscr{L}^{j-1}) & \text{if} \quad j > i = 0 \\ C^{i-1}(\underline{X}, \mathscr{F}) & \text{if} \quad i > j = 0 \\ C^{i-1}(\underline{X}, \mathscr{L}^{j-1}) & \text{if} \quad i, j > 0. \end{cases}$$

Part (i) of the above lemma is equivalent to

2.3. Corollary. *If a cocycle $\Phi \in \check{Z}^q(X; \mathcal{F}, \mathcal{L}^\cdot)$ has a head that is δ-exact and if $H^{q-k}(\underline{X}, \mathcal{L}^{k-1})=0$ for $k=1, \ldots, q-1$, then Φ is Δ-exact and, in particular, the tail of Φ is D-exact.*

2.4. Remark. Definition 3.4.3 of Chap. I can be restated as follows: A $\bar{\partial}$-closed form $\tau \in A^{k,l}(X)$ is said to represent an element of $H^l(X, \Omega^k)$ if there is a cocycle of degree l

$$c \in \check{Z}^l(\underline{X}; \Omega^k, A^{k,\cdot})$$

of the Čech transform of the Dolbeault complex whose tail is τ, for some open covering of X.

3. The $\partial\bar{\partial}$-Complex $\mathcal{L}^\cdot_m$

Let X be a complex space. For any pair (p, q) of natural integers, there is a complex of sheaves on X of the form

$$\begin{array}{ccccccccccc} 0 \longrightarrow \mathbb{C} \xrightarrow{\lambda} \mathcal{L}^0_{p,q} \xrightarrow{D} \cdots \xrightarrow{D} & \mathcal{L}^{p+q-1}_{p,q} & \xrightarrow{D} & \mathcal{L}^{p+q}_{p,q} & \xrightarrow{D} \cdots \\ & \| & & \| & \\ & A_X^{p-1,q-1} & \xrightarrow{\partial\bar{\partial}} & A_X^{p,q} & \end{array}$$

defined in [7].

We will deal exclusively with the case $p=q$, so we write $\mathcal{L}^r_m$ for $\mathcal{L}^r_{m,m}$. The complex $\mathcal{L}^\cdot_m$ defined as follows (the suffix X will be omitted).

3.1. Definitions.

$$\mathcal{L}^r_m := \begin{cases} \Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r & \text{if } r<m \\ A^{m-1,r-m} \oplus \cdots \oplus A^{r-m,m-1} & \text{if } m \leqq r < 2m \\ A^{r-m,m} \oplus \cdots \oplus A^{m,r-m} & \text{if } r \geqq 2m. \end{cases} \tag{3.1}$$

Define $j: \mathbb{C} \xrightarrow{\binom{1}{1}} \Omega^0 \oplus \bar{\Omega}^0 = \mathcal{L}^0_m$ and

(i) For $0 \leqq r < m-1$,

$$\begin{array}{ccc} \mathcal{L}^r_m & \xrightarrow{\quad D \quad} & \mathcal{L}^{r+1}_m \\ \| & & \| \\ \Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r & \xrightarrow{\begin{pmatrix} d & 0 & 0 \\ (-1)^{r+1} & d & (-1)^r \\ 0 & 0 & d \end{pmatrix}} & \Omega^{r+1} \oplus A^r \oplus \bar{\Omega}^{r+1}. \end{array}$$

(ii) For $r=m-1$,

$$\begin{array}{ccc} \mathcal{L}^{m-1}_m & \xrightarrow{\quad D \quad} & \mathcal{L}^m_m \\ \| & & \| \\ \Omega^{m-1} \oplus A^{m-2} \oplus \bar{\Omega}^{m-1} & \xrightarrow{((-1)^m \;\; d \;\; (-1)^{m-1})} & A^{m-1}. \end{array}$$

(iii) For $m \leqq r < 2m-1$,

$$\begin{array}{ccc}
\mathscr{L}_m^r & \xrightarrow{\quad D \quad} & \mathscr{L}_m^{r+1} \\
\Big\| & \begin{pmatrix} \partial & \bar\partial & & 0 \\ 0 & \partial & \ddots & \\ & & \ddots & \bar\partial & 0 \\ 0 & & & \partial & \bar\partial \end{pmatrix} & \Big\| \\
A^{m-1,r-m}\oplus\ldots\oplus A^{r-m,m-1} & \longrightarrow & A^{m-1,r-m+1}\oplus\ldots\oplus A^{r-m+1,m-1}\,.
\end{array}$$

(iv) For $r = 2m-1$,

$$\begin{array}{ccc}
\mathscr{L}_m^{2m-1} & \xrightarrow{D} & \mathscr{L}_m^{2m} \\
\Big\| & & \Big\| \\
A^{m-1,m-1} & \xrightarrow{\partial\bar\partial} & A^{m,m}\,.
\end{array}$$

(v) For $r \geqq 2m$,

$$\begin{array}{ccc}
\mathscr{L}_m^r & \xrightarrow{D} & \mathscr{L}_m^{r+1} \\
\| & & \| \\
A^{r-m,m}\oplus\ldots\oplus A^{m,r-m} & \xrightarrow{d} & A^{r-m+1,m}\oplus\ldots\oplus A^{m,r-m+1}\,.
\end{array}$$

Actually, the part $\mathbb{C}\to\mathscr{L}_m^0\to\ldots\to\mathscr{L}_m^{2m-1}$ is the single complex associated to the truncated double complex

$$\begin{array}{ccccccc}
\mathbb{C} & \xrightarrow{\quad 1 \quad} & \Omega^0 & \xrightarrow{\quad d \quad} & \ldots & \xrightarrow{\quad d \quad} & \Omega^{m-1} \\
{\scriptstyle 1}\downarrow & & {\scriptstyle -1}\downarrow & & & & {\scriptstyle (-1)^m}\downarrow \\
\bar\Omega^0 & \xrightarrow{\quad 1 \quad} & A^{0,0} & \xrightarrow{\quad \partial \quad} & \ldots & \xrightarrow{\quad \partial \quad} & A^{m-1,0} \\
{\scriptstyle d}\downarrow & & {\scriptstyle \bar\partial}\downarrow & & & & {\scriptstyle \bar\partial}\downarrow \\
\vdots & & \vdots & & & & \vdots \\
{\scriptstyle d}\downarrow & & {\scriptstyle \bar\partial}\downarrow & & & & {\scriptstyle \bar\partial}\downarrow \\
\bar\Omega^{m-1} & \xrightarrow{(-1)^{m-1}} & A^{0,m-1} & \xrightarrow{\quad \partial \quad} & \ldots & \xrightarrow{\quad \partial \quad} & A^{m-1,m-1}
\end{array}$$

with the indicated sign conventions, and a similar observation may serve to define

$$\mathbb{C}\to\mathscr{L}_{p,q}^0\to\ \ldots\ \mathscr{L}_{p,q}^{p+q-1}\,.$$

3.2. Proposition (Bigolin [7]). *For smooth X, $(\mathscr{L}_{p,q}^{\cdot}, D)$ is an exact sequence of sheaves.*

3.3. The Involution on $\mathscr{L}_m^{\cdot}$. A $\mathbb{C}$-antilinear involution $\varphi\mapsto\varphi^*$ is defined on $\mathscr{L}_m^{\cdot}$ as follows:

(i) For $(g^r, \psi^{r-1}, \bar h^r)$ in $\mathscr{L}_m^r = \Omega^r\oplus A^{r-1}\oplus\bar\Omega^r$ $(r<m)$

$$(g^r, \psi^{r-1}, \bar h^r)^* := (h^r, -\bar\psi^{r-1}, \bar g^r)\,.$$

(ii) For ψ^{r-1} in $\mathscr{L}_m^r\subset A^{r-1}$, $(\psi^{r-1})^* := -\bar\psi^{r-1}$ $(m\leqq r<2m)$.

(iii) For ψ^r in $\mathscr{L}_m^r\subset A^r$, $(\psi^r)^* := \bar\psi^r$ $(r\geqq 2m)$.

It is obvious that $(D\varphi)^* = D(\varphi^*)$.

We denote by $\mathscr{L}_{m,\mathbb{R}}^{\cdot}$ the sub-complex of $\mathscr{L}_m^{\cdot}$ of fixed points under $(\cdot)^*$. We set $\mathrm{Re}\,\varphi := \frac{1}{2}(\varphi+\varphi^*)$. Note that a self-conjugate element of $\mathscr{L}_m^r$, for $r<2m$ has pure imaginary $\mathscr{C}^\infty$ components.

3.4. The Morphism $\mu\colon \mathscr{L}^{\cdot}_{m+1}\to\mathscr{L}^{\cdot}_m$. A morphism $\mu=\mu^r_m\colon \mathscr{L}^r_{m+1}\to\mathscr{L}^r_m$ is defined by

(i) For $r<m$, $\mathscr{L}^r_{m+1}=\Omega^r\oplus A^{r-1}\oplus\bar{\Omega}^r=\mathscr{L}^r_m$ and $\mu^r_m:=\mathrm{id}$. We define $\mu=\mathrm{id}$ on $\mathbb{C}$ as well.

(ii) For $m\leqq r<2m$, $\mathscr{L}^r_m$ is a direct summand of $\mathscr{L}^r_{m+1}$ and μ^r_m is defined as the canonical projection.

(iii) For $r=2m$,

$$\begin{array}{ccc} \mathscr{L}^{2m}_{m+1} & \xrightarrow{\mu^{2m}_m} & \mathscr{L}^{2m}_m \\ \| & & \| \\ A^{m,m-1}\oplus A^{m-1,m} & \xrightarrow{\frac{1}{2}(-\partial\ \ \bar{\partial})} & A^{m,m}. \end{array}$$

(iv) For $r=2m+1$,

$$\begin{array}{ccc} \mathscr{L}^{2m+1}_{m+1} & \xrightarrow{\mu^{2m+1}_m} & \mathscr{L}^{2m+1}_m \\ \| & & \| \\ A^{m,m} & \xrightarrow{\frac{1}{2}\begin{pmatrix}-\bar{\partial}\\ \partial\end{pmatrix}} & A^{m+1,m}\oplus A^{m,m+1}. \end{array}$$

(v) For $r>2m+1$, $\mathscr{L}^r_{m+1}$ is a direct summand of $\mathscr{L}^r_m$ and μ^r_m is defined as the canonical inclusion.

3.4.1. Lemma. *The above morphism* μ *commutes with* D *and the involution* $(\cdot)^*$.

3.5. Relation with the $(\bar{\partial}\oplus\partial)$*-Complex.* The $(\bar{\partial}\oplus\partial)$-complex $(\mathscr{G}^{\cdot}_m,\hat{d})$ is the direct sum of the Dolbeault complex and its conjugate

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathscr{G}^{-1}_m & \xrightarrow{j} & \mathscr{G}^0_m & \xrightarrow{\hat{d}}\cdots\xrightarrow{\hat{d}} & \mathscr{G}^q_m & \xrightarrow{\hat{d}}\cdots \\ & \| & & \| & & \| & \\ & \Omega^m\oplus\bar{\Omega}^m & \xrightarrow{\begin{pmatrix}1&0\\0&1\end{pmatrix}} & A^{m,0}\oplus A^{0,m} & \xrightarrow{\begin{pmatrix}\bar{\partial}&0\\0&\partial\end{pmatrix}}\cdots\xrightarrow{\begin{pmatrix}\bar{\partial}&0\\0&\partial\end{pmatrix}} & A^{m,q}\oplus A^{q,m} & \xrightarrow{\begin{pmatrix}\bar{\partial}&0\\0&\partial\end{pmatrix}}\cdots \end{array}$$

We define on $\mathscr{G}^{\cdot}_m$ the involution $(\varphi,\psi)\mapsto(\varphi,\psi)^*:=(\bar{\psi},\bar{\varphi})$. It is related to the $\partial\bar{\partial}$-complex by a homotopy operator $\beta\colon\mathscr{L}^{m+q+1}_{m+1}\to\mathscr{G}^q_m$ and a morphism of complexes $\gamma\colon\mathscr{L}^{m+q}_m\to\mathscr{G}^q_m$.

3.5.1. The Homotopy Operator $\beta\colon\mathscr{L}^{m+1+\cdot}_{m+1}\to\mathscr{G}^{\cdot}_m$. It is defined by

(i) For $q=-1$,

$$\begin{array}{ccc} \mathscr{L}^m_{m+1} & \xrightarrow{\beta} & \mathscr{G}^{-1}_m \\ \| & & \| \\ \Omega^m\oplus A^{m-1}\oplus\bar{\Omega}^m & \xrightarrow{\begin{pmatrix}-1&0&0\\0&0&1\end{pmatrix}} & \Omega^m\oplus\bar{\Omega}^m. \end{array}$$

(ii) For $0\leqq q<m$,

$$\begin{array}{ccc} \mathscr{L}^{m+q+1}_{m+1} & \xrightarrow{\beta} & \mathscr{G}^q_m \\ \| & & \| \\ A^{m,q}\oplus\ldots\oplus A^{q,m} & \xrightarrow{(-1)^{m-q}\begin{pmatrix}1&0\ldots0&0\\0&0\ldots0&1\end{pmatrix}} & A^{m,q}\oplus A^{q,m}. \end{array}$$

(iii) For $q=m$,

$$\begin{tikzcd}
\mathscr{L}^{2m+1}_{m+1} \arrow[r, "\beta"] \arrow[d, equal] & \mathscr{G}^{m}_{m} \arrow[d, equal] \\
A^{m,m} \arrow[r, "\frac{1}{2}\binom{1}{1}"] & A^{m,m} \oplus A^{m,m}.
\end{tikzcd}$$

(iv) For $q>m$, $\beta: \mathscr{L}^{m+q+1}_{m+1} \to \mathscr{G}^{q}_{m}$ is defined by 0.

3.5.2. The Morphism of Complexes $\gamma: \mathscr{L}^{m+\cdot}_{m} \to \mathscr{G}^{\cdot}_{m}$. It is defined by

(i) For $q=-1$,

$$\begin{tikzcd}
\mathscr{L}^{m-1}_{m} \arrow[r, "\gamma"] \arrow[d, equal] & \mathscr{G}^{-1}_{m} \arrow[d, equal] \\
\Omega^{m-1} \oplus A^{m-2} \oplus \bar{\Omega}^{m-1} \arrow[r, "\begin{pmatrix} d & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}"] & \Omega^{m} \oplus \bar{\Omega}^{m}.
\end{tikzcd}$$

(ii) For $0 \leqq q < m$,

$$\begin{tikzcd}
\mathscr{L}^{m+q}_{m} \arrow[r, "\gamma"] \arrow[d, equal] & \mathscr{G}^{q}_{m} \arrow[d, equal] \\
A^{m-1,q} \oplus \ldots \oplus A^{q,m-1} \arrow[r, "(-1)^{m-q}\begin{pmatrix} \partial & 0 \ldots 0 & 0 \\ 0 & 0 \ldots 0 & \bar{\partial} \end{pmatrix}"] & A^{m,q} \oplus A^{q,m}.
\end{tikzcd}$$

(iii) For $q \geqq m$,

$$\begin{tikzcd}
\mathscr{L}^{m+q}_{m} \arrow[r, "\gamma"] \arrow[d, equal] & \mathscr{G}^{q}_{m} \arrow[d, equal] \\
A^{q,m} \oplus \ldots \oplus A^{m,q} \arrow[r, "\begin{pmatrix} 0 & 0 \ldots 0 & 1 \\ -1 & 0 \ldots 0 & 0 \end{pmatrix}"] & A^{m,q} \oplus A^{q,m}.
\end{tikzcd}$$

In particular, for $q=m$, $\gamma(\alpha^{m,m}) = (\alpha^{m,m}, -\alpha^{m,m})$.

The following can be easily checked.

3.5.3. Lemma. (i) $\hat{d}\beta + \beta D = \gamma\mu$.

(ii) $\hat{d}\gamma = \gamma D$.

(iii) *If* $\eta^{m,m}$ *and* $\zeta^{m,m}$ *are* (m,m)*-forms, then* $\beta(\eta^{m,m}) + \gamma(\zeta^{m,m}) = (\varrho^{m,m}, \sigma^{m,m})$ *where* $\varrho^{m,m} + \sigma^{m,m} = \eta^{m,m}$.

(iv) β *and* γ *anticommute with the involutions* $(\cdot)^*$

$$\begin{tikzcd}
 & \mathscr{L}^{m+q-1}_{m} \arrow[dr, "\gamma"] \arrow[dd, "D"] & \\
\mathscr{L}^{m+q}_{m+1} \arrow[rr, "\beta"] \arrow[dr, "\mu"] \arrow[dd, "D"] & & \mathscr{G}^{q-1}_{m} \arrow[dd, "\hat{d}"] \\
 & \mathscr{L}^{m+q}_{m} \arrow[dr, "\gamma"] & \\
\mathscr{L}^{m+q+1}_{m+1} \arrow[rr, "\beta"] & & \mathscr{G}^{q}_{m}.
\end{tikzcd}$$

3.6. The Čech Transform of the $\partial\bar\partial$-Complex. For any Čech space $\underline{X}$, we denote by $\mathscr{E}^{\cdot}_m(\underline{X})$, $\mathscr{E}^{\cdot}_m(\underline{X},[\mathbb{R}])$ and $\mathscr{E}^{\cdot}_m(\underline{X},\mathbb{R})$ the Čech transforms of the complexes

$$0\to\mathbb{C}\to\mathscr{L}^0_m\to\mathscr{L}^1_m\to\dots,$$
$$0\to\mathbb{R}\to\mathscr{L}^0_m\to\mathscr{L}^1_m\to\dots,$$
$$0\to\mathbb{R}\to\mathscr{L}^0_{m,\mathbb{R}}\to\mathscr{L}^1_{m,\mathbb{R}}\to\dots,$$

respectively. So we set

$$\text{(3.6.1)}\qquad \begin{aligned} &\text{(i)}\quad \mathscr{E}^q_m(\underline{X}):=\check{C}^q(\underline{X};\mathbb{C},\mathscr{L}^{\cdot}_m),\\ &\text{(ii)}\quad \mathscr{E}^q_m(\underline{X},[\mathbb{R}]):=\check{C}^q(\underline{X};\mathbb{R},\mathscr{L}^{\cdot}_m),\\ &\text{(iii)}\quad \mathscr{E}^q_m(\underline{X},\mathbb{R}):=\check{C}^q(\underline{X};\mathbb{R},\mathscr{L}^{\cdot}_{m,\mathbb{R}}).\end{aligned}$$

Of course, $\mathscr{E}^q_m(\underline{X},\mathbb{R})\subset\mathscr{E}^q_m(X,[\mathbb{R}])\subset\mathscr{E}^q_m(X)$. Elements of $\mathscr{E}^q_m(\underline{X})$ will be written in a matrix form. For example an element of $\mathscr{E}^{2m}_m(\underline{X})$ will be written as

$$\text{(3.6.2)}\qquad \Phi=\begin{array}{|c|ccc|c|}
\cline{1-4} a & g^0 & \cdots & g^{m-1} & \multicolumn{1}{c}{} \\
\cline{1-4} \bar h^0 & \varphi^{0,0} & \cdots & \varphi^{m-1,0} & \multicolumn{1}{c}{} \\
\vdots & \vdots & & \vdots & \multicolumn{1}{c}{} \\
\bar h^{m-1} & \varphi^{0,m-1} & \cdots & \varphi^{m-1,m-1} & \multicolumn{1}{c}{} \\
\hline \multicolumn{4}{c|}{} & \eta^{m,m} \\
\cline{5-5}
\end{array}$$

where

$$a\in C^{2m}(\underline{X},\mathbb{C})$$
$$g^k\in C^{2m-k-1}(\underline{X},\Omega^k)$$
$$\bar h^l\in C^{2m-l-1}(\underline{X},\bar\Omega^l)$$
$$\varphi^{k,l}\in C^{2m-k-l-2}(\underline{X},A^{k,l})$$
$$\eta^{m,m}\in H^0(X,A^{m,m})$$

a is the head and $\eta^{m,m}$ the tail of Φ.

$$\Phi\in\mathscr{E}^{2m}_m(\underline{X},[\mathbb{R}])\quad\text{iff}\quad a=\bar a\in C^{2m}(\underline{X},\mathbb{R})$$
$$\Phi\in\mathscr{E}^{2m}_m(\underline{X},\mathbb{R})\quad\text{iff}\quad a=\bar a,$$

$g^k=h^k$, $\varphi^{k,l}+\bar\varphi^{l,k}=0$ and $\eta^{m,m}=\bar\eta^{m,m}$.

If we apply $\Delta:\mathscr{E}^{2m}_m(\underline{X})\to\mathscr{E}^{2m+1}_m(\underline{X})$ we obtain

$$\Delta\Phi=\begin{array}{|c|ccc|c|c|}
\cline{1-4} b & u^0 & \cdots & u^{m-1} & \multicolumn{2}{c}{} \\
\cline{1-4} \bar v^0 & \psi^{0,0} & \cdots & \psi^{m-1,0} & \multicolumn{2}{c}{} \\
\vdots & \vdots & & \vdots & \multicolumn{2}{c}{} \\
\bar v^{m-1} & \psi^{0,m-1} & \cdots & \psi^{m-1,m-1} & \multicolumn{2}{c}{} \\
\hline \multicolumn{4}{c|}{} & \psi^{m,m} & \lambda^{m+1,m} \\
\cline{5-6} \multicolumn{4}{c|}{} & \lambda^{m,m+1} & \multicolumn{1}{c}{} \\
\cline{5-5}
\end{array}$$

where

$$(3.6.3)\quad \begin{aligned}
&\text{(i)} && b=\delta a\\
&\text{(ii)} && u^0=\delta g^0-a\\
&\text{(iii)} && u^k=\delta g^k-dg^{k-1} \quad \text{for} \quad 1\leqq k<m\\
&\text{(iv)} && \bar{v}^0=\delta \bar{h}^0-a\\
&\text{(v)} && \bar{v}^l=\delta \bar{h}^l-d\bar{h}^{l-1} \quad \text{for} \quad 1\leqq l<m\\
&\text{(vi)} && \psi^{0,0}=\delta\varphi^{0,0}+g^0-\bar{h}^0\\
&\text{(vii)} && \psi^{k,0}=\delta\varphi^{k,0}+(-1)^k g^k-\partial\varphi^{k-1,0} \quad \text{for} \quad 1\leqq k<m\\
&\text{(viii)} && \psi^{0,l}=\delta\varphi^{0,l}-\bar{\partial}\varphi^{0,l-1}+(-1)^{l-1}\bar{h}^l \quad \text{for} \quad 1\leqq l<m\\
&\text{(ix)} && \psi^{k,l}=\delta\varphi^{k,l}-\bar{\partial}\varphi^{k,l-1}-\partial\varphi^{k-1,l} \quad \text{for} \quad 1\leqq k,l<m\\
&\text{(x)} && \psi^{m,m}=\varepsilon(\eta^{m,m})-\partial\bar{\partial}\varphi^{m-1,m-1}\\
&\text{(xi)} && \lambda^{m+1,m}=\partial\eta^{m,m}\\
&\text{(xii)} && \lambda^{m,m+1}=\bar{\partial}\eta^{m,m}.
\end{aligned}$$

In the next section we construct, for any Kähler space (X,ω), an open covering $\mathscr{X}$ such that on the resulting Čech space $\underline{X}$ and for any integer $m>0$, there is a cocycle in $\mathscr{E}_m^{2m}(\underline{X},[\mathbb{R}])$ whose tail is ω^m.

4. The Čech Cochains Associated to a Kähler Metric

We first note that, if X is a Kähler space, it admits by definition an open covering (U_α) such that there are elements $\varphi_\alpha\in SP^\infty(U_\alpha)$ such that $\varphi_\alpha-\varphi_\beta$ is *locally* the real part of a holomorphic function on $U_\alpha\cap U_\beta$. We show that "locally" can be omitted.

4.1. Covering Lemma. *Let X be a paracompact topological space and $(U_\alpha)_{\alpha\in A}$ an open covering of X such that, for every $\alpha,\beta\in A$, $(U^j_{\alpha\beta})_{j\in J_{\alpha\beta}}$ is an open covering of $U_\alpha\cap U_\beta$. Let $J=\bigcup_{\alpha,\beta} J_{\alpha\beta}$.*

Then there exists a refinement

$$\mathscr{X}=(X_\lambda)_{\lambda\in\Lambda}$$

of (U_α) together with two maps

$$\alpha:\Lambda\to A$$

$$j:\Lambda\times\Lambda\to J$$

such that

$$(4.1.1)\quad \begin{aligned}
&\text{(i)} && X_\lambda\subset U_{\alpha(\lambda)}\\
&\text{(ii)} && X_\lambda\cap X_\mu\subset U^{j(\lambda,\mu)}_{\alpha(\lambda)\alpha(\mu)}.
\end{aligned}$$

Proof. Since X paracompact, (U_α) admits a refinement $(\bar{V}_\alpha)_{\alpha \in A}$ indexed by the same set A such that $\bar{V}_\alpha \subset U_\alpha$ and $(\bar{V}_\alpha)$ is locally finite. Let Λ be the set of all multi-indices

(4.1.2) $$\lambda = (\alpha_0, \dots, \alpha_s; j_0, \dots, j_s) \quad (s \in \mathbb{N})$$

such that the α_r are pairwise distinct elements of A and $j_r \in J_{\alpha_0 \alpha_r}$ for $0 \leqq r \leqq s$. Set

(4.1.3) $$X_\lambda := V_{\alpha_0} \cap \bigcap_{r=0}^{s} U_{\alpha_0\alpha_r}^{j_r} \Big\backslash \bigcup_{\beta \neq \alpha_0, \dots, \alpha_s} \bar{V}_\beta .$$

X_λ is open since $(\bar{V}_\beta)$ is locally finite.

Define $\alpha(\lambda) := \alpha_0$. Then obviously $X_\lambda \subset U_{\alpha(\lambda)}$.

Now suppose that
$$\mu = (\beta_0, \dots, \beta_t; k_0, \dots, k_t)$$
is a multi-index in Λ such that $X_\lambda \cap X_\mu \neq \emptyset$. Then β_0 must be equal to one (and only one) of the α_r for otherwise $\bar{V}_{\beta_0} \cap X_\lambda$ would be empty by construction of X_λ. If $\beta_0 = \alpha_r$, set
$$j(\lambda, \mu) := j_r .$$
It is clear that
$$X_\lambda \cap X_\mu \subset U_{\alpha_0\alpha_r}^{j_r} = U_{\alpha_0\beta_0}^{j_r} = U_{\alpha(\lambda)\alpha(\mu)}^{j(\lambda,\mu)}$$
as required. Finally it is true that the X_λ ($\lambda \in \Lambda$) cover X; for if $x \in X$ is arbitrary, take $\alpha \in A$ such that $x \in V_\alpha$. The set S of $\beta \in A$ such that $x \in \bar{V}_\beta$ is finite containing α [since $(\bar{V}_\beta)$ is locally finite]; let
$$S = \{\alpha_0, \dots, \alpha_s\} \quad \text{with} \quad \alpha_0 = \alpha .$$
For all
$$r \in \{0, \dots, s\}, \qquad x \in V_{\alpha_0} \cap \bar{V}_\alpha \subset U_{\alpha_0} \cap U_{\alpha_r}$$
hence $x \in U_{\alpha_0\alpha_r}^{j_r}$ for some $j_r \in J_{\alpha_0\alpha_r}$. So we obtain a multi-index $\lambda \in \Lambda$ with $x \in X_\lambda$. Since $x \in X$ was arbitrary, the proof is complete.

4.1.1. Corollary. *Let X be a Kähler space with a fixed Kähler metric κ. Then X admits an open covering $\mathscr{X} = (X_\lambda)$ in which is represented by elements*
$$\varphi_\lambda \in SP^\infty(X_\lambda)$$
such that
$$\varphi_\lambda - \varphi_\mu = f_{\lambda\mu} + \bar{f}_{\lambda\mu}, \qquad f_{\lambda\mu} \in \mathcal{O}(X_\lambda \cap X_\mu).$$

Proof. By definition, there is an open covering (U_α) together with $\psi_\alpha \in SP^\infty(U_\alpha)$ such that $\psi_\alpha - \psi_\beta \in PH(U_\alpha \cap U_\beta, \mathbb{R})$. This means that $U_\alpha \cap U_\beta$ admits an open covering $(U_{\alpha\beta}^j)_{j \in J_{\alpha\beta}}$ such that
$$(\psi_\alpha - \psi_\beta)|_{U_{\alpha\beta}^j} = g_{\alpha\beta}^j + \bar{g}_{\alpha\beta}^j, \qquad g_{\alpha\beta}^j \in \mathcal{O}(U_{\alpha\beta}^j).$$

Apply the Covering Lemma above to obtain an open covering (X_λ) of X with $X_\lambda \subset U_{\alpha(\lambda)}$ and $X_\lambda \cap X_\mu \subset U_{\alpha(\lambda)\alpha(\mu)}^{j(\lambda,\mu)}$. Then if we set
$$\varphi_\lambda := \psi_{\alpha(\lambda)}|_{X_\lambda}$$
$$f_{\lambda\mu} := g_{\alpha(\lambda)\alpha(\mu)}^{j(\lambda,\mu)}|_{X_\lambda \cap X_\mu}$$
these elements satisfy the required conditions.

4.2. Kähler-Čech Pairs. It will be convenient to multiply the above elements φ_λ and $f_{\lambda\mu}$ by $i=\sqrt{-1}$ to obtain

$$\text{(4.2.1)}\qquad \begin{aligned} &\text{(i)} && \varphi_\lambda-\varphi_\mu=f_{\lambda\mu}-\bar{f}_{\lambda\mu}\\ &\text{(ii)} && -i\varphi_\lambda\in SP^\infty(X_\lambda)\\ &\text{(iii)} && \partial\bar\partial\varphi_\lambda=\omega|_{X_\lambda}.\end{aligned}$$

So the Kähler metric of X is

$$\kappa=\{(X_\lambda,-i\varphi_\lambda)\}.$$

A pair (f,φ) with $f\in C^1(\underline{X},\Omega^0)$ and $\varphi\in C^0(\underline{X},A^0)$ satisfying (4.2.1) will be called a *Kähler-Čech pair* and $\underline{X}=(X,\mathscr{X})$ will be called a *Kähler-Čech space.* Since $(\delta\varphi)_{\lambda\mu}=\varphi_\mu-\varphi_\lambda$, we have the identities

$$\text{(A)}\qquad \begin{aligned} &(1) && \delta\varphi=\bar f-f\\ &(2) && \delta f=\delta\bar f\\ &(3) && d\delta f=0\\ &(4) && \partial\delta\varphi=-df\\ &(5) && \bar\partial\delta\varphi=d\bar f\\ &(6) && \partial\bar\partial\varphi=\varepsilon(\omega)\\ &(7) && d\omega=0.\end{aligned}$$

Identity (A2) shows that $\delta f\in Z^2(\underline{X},\mathbb{R})$. The diagram

$$\text{(4.2.2)}\qquad \begin{tikzcd} & & -i\delta\varphi \arrow[dd, "-i\partial"] & i\varphi \arrow[l, "\delta"'] \arrow[dd, "-i\partial"'] \arrow[dr, "i\partial\bar\partial"] & & \\ \delta f & f \arrow[l, "\delta"'] \arrow[ur, "-2\,\mathrm{Im}"] \arrow[dr, "d"'] & & & \partial\bar\partial\varphi & \omega \arrow[l, "\varepsilon"'] \\ & & df & \partial\varphi \arrow[l, "\delta"] \arrow[ur, shift left, "d"] \arrow[ur, shift right, "\bar\partial"'] & & \end{tikzcd}$$

shows that $-2\,\mathrm{Im}f=-i\delta\varphi$ represents the Kähler class $\hat{c}_1(\kappa)$ of (X,κ) in $H^1(X,PH_{X,\mathbb{R}})$, δf represents $c_1(\kappa)\in H^2(X,\mathbb{R})$ and df represents $\tilde{c}_1(\kappa)\in H^1(X,\Omega^1_X)$. Moreover (4.2.2) confirms that ω is a d-closed representative of $c_1(\kappa)$ and a $\bar\partial$-closed representative of $\tilde{c}_1(\kappa)$, i.e. that diagram (1.2.4) of Chap. II is indeed commutative.

In terms of the $\partial\bar\partial$-complex $\mathscr{L}^{\cdot}_{1,\mathbb{R}}$ given by

$$\text{(4.2.3)}\qquad \begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "\binom{1}{1}"] & \mathscr{L}^0_{1,\mathbb{R}} \arrow[r, "D"] \arrow[d, equal] & \mathscr{L}^1_{1,\mathbb{R}} \arrow[r, "D"] \arrow[d, equal] & \mathscr{L}^2_{1,\mathbb{R}} \arrow[r, "D"] \arrow[d, equal] & \mathscr{L}^3_{1,\mathbb{R}} \arrow[d, equal] \\ & & (\Omega^0\oplus\bar\Omega^0)_{\mathbb{R}} \arrow[r, "(-1\ \ 1)"] & (A^{0,0})_{i\mathbb{R}} \arrow[r, "\partial\bar\partial"] & A^{1,1}_{\mathbb{R}} \arrow[r, "d"] & (A^{1,2}\oplus A^{2,1})_{\mathbb{R}} \end{tikzcd}$$

[where $(\cdot)_{\mathbb{R}}$ denotes self-conjugate elements and $(\cdot)_{i\mathbb{R}}$ anti-self-conjugate elements] we constructed an element

$$\text{(4.2.4)} \qquad \Phi_1(f,\varphi):=\begin{array}{|c|c|c} \hline \delta f & f & \\ \hline \bar f & \varphi & \\ \hline & & \multicolumn{1}{|c|}{\omega} \\ \cline{3-3} \end{array} \in \mathscr{E}_1^2(\underline{X},\mathbb{R})$$

(with the notations of 3.6) and relations (A) mean precisely that $\Delta\Phi_1(f,\varphi)=0$.

4.3. Generalization to Higher Powers. We now construct the announced element

$$\Phi_m(f,\varphi)\in \check{Z}^{2m}(X;\mathbb{R},\mathscr{L}_m^{\cdot})$$

whose head is $(\delta f)^m$ and tail ω^m, and whose existence is the key step in the proof of Theorem 2. Actually, if we set

$$\text{(i)}\quad \tilde{\mathscr{K}}^m(\underline{X}):=\check{Z}^{2m}(\underline{X};\mathbb{C},\mathscr{L}_m^{\cdot}),\qquad \tilde{\mathscr{K}}(\underline{X}):=\bigoplus_{m\geqq 0}\tilde{\mathscr{K}}^m(\underline{X})$$

$$\text{(4.3.1)(ii)}\quad \tilde{\mathscr{K}}^m(\underline{X},[\mathbb{R}]):=\check{Z}^{2m}(\underline{X};\mathbb{R},\mathscr{L}_m^{\cdot}),\qquad \tilde{\mathscr{K}}(\underline{X},[\mathbb{R}]:=\bigoplus_{m\geqq 0}\tilde{\mathscr{K}}^m(\underline{X},[\mathbb{R}])$$

$$\text{(iii)}\quad \tilde{\mathscr{K}}^m(\underline{X},\mathbb{R}):=\check{Z}^{2m}(\underline{X};\mathbb{R},\mathscr{L}_{m,\mathbb{R}}^{\cdot}),\qquad \tilde{\mathscr{K}}(\underline{X},\mathbb{R}):=\bigoplus_{m\geqq 0}\tilde{\mathscr{K}}^m(\underline{X},\mathbb{R})$$

then there is an associative product law on $\tilde{\mathscr{K}}(\underline{X})$ with respect to which it is a graded $\mathbb{C}$-algebra admitting $\tilde{\mathscr{K}}(\underline{X},[\mathbb{R}])$ as a $\mathbb{R}$-subalgebra, but not $\tilde{\mathscr{K}}(\underline{X},\mathbb{R})$. Then $\Phi_m(f,\varphi)$ is simply the m-th power of $\Phi_1(f,\varphi)$ in $\tilde{\mathscr{K}}(\underline{X},\mathbb{R})$.

$\Phi_m(f,\varphi)$ is defined by

$$\text{(4.3.2)}\quad \Phi_m(f,\varphi):=\begin{array}{|c|ccc|c} \hline a_m & g_m^0 & \cdots & g_m^{m-1} & \\ \hline \bar h_m^0 & \varphi_m^{0,0} & \cdots & \varphi_m^{m-1,0} & \\ \vdots & \vdots & & \vdots & \\ \bar h_m^{m-1} & \varphi_m^{0,m-1} & \cdots & \varphi_m^{m-1,m-1} & \\ \hline & & & & \multicolumn{1}{|c|}{\eta_m^{m,m}} \\ \cline{5-5} \end{array} \in \tilde{\mathscr{K}}^m(\underline{X},[\mathbb{R}]),$$

where $a_m\in C^{2m}(\underline{X},\mathbb{R})$, $g_m^k\in C^{2m-k-1}(\underline{X},\Omega^k)$,

$$\bar h_m^l\in C^{2m-l-1}(\underline{X},\bar\Omega^l),\qquad \varphi_m^{k,l}\in C^{2m-k-l-2}(\underline{X},A^{k,l}),\qquad \eta_m^{m,m}\in H^0(X,A^{m,m})$$

are given by the relations (B) below. Recall that $\delta f=\delta\bar f$ by (A2). We use the cup-product of Čech cochains as defined in 1.2.

(B)

$$(1)\quad a_m=(\delta f)^m$$

$$(2)\quad g_m^k=(-1)^k(df)^k\cdot f\cdot(\delta f)^{m-k-1}$$

$$(3)\quad \bar h_m^l=(\delta f)^{m-l-1}\cdot\bar f\cdot(d\bar f)^l$$

$$(4)\quad \varphi_m^{k,l}=(-1)^{k+l}(df)^k\cdot f\cdot(\delta f)^{m-k-l-2}\cdot\bar f\cdot(d\bar f)^l\quad\text{for}\quad k+l<m-1$$

$$(5)\quad \varphi_m^{k,l}=(-1)^{m-l-1}(df)^{m-l-1}\cdot\delta\varphi\cdot(d\bar f)^{m-k-2}\cdot\bar\partial\varphi\wedge\omega^{k+l-m+1}$$

$$\text{for}\quad k<m-1\leqq k+l$$

(6) $\varphi_m^{m-1,l}=(-1)^{m-l-1}(df)^{m-l-1}\cdot\varphi\wedge\omega^l$

(7) $\eta_m^{m,m}=\omega^m$.

Domains of validity of formulae (B)

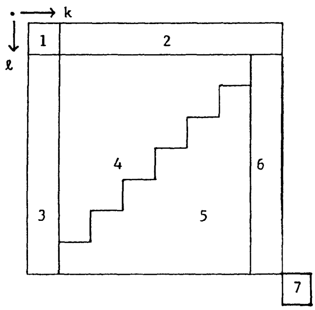

Fig. 1

Before proving that $\Delta\Phi_m(f,\varphi)=0$ we mention

4.4. Relation Between $\Phi_m(f,\varphi)$, $\Phi_n(f,\varphi)$, and $\Phi_{m+n}(f,\varphi)$. A formal consequence of identities (A) and (B) is the following:

(C)

(1) $a_{m+n}=a_m\cdot a_n$

(2) $g_{m+n}^k=g_m^k\cdot a_n$ for $0\leqq k<m$

(3) $\quad=(-1)^m dg_m^{m-1}\cdot g_n^{k-m}$ for $m\leqq k<m+n$

(4) $\bar h_{m+n}^l=a_m\cdot\bar h_n^l$ for $0\leqq l<n$

(5) $\quad=\bar h_m^{l-n}\cdot d\bar h_n^{n-1}$ for $n\leqq l<m+n$

(6) $\varphi_{m+n}^{k,l}=(-1)^l g_m^k\cdot\bar h_n^l$ for $0\leqq k<m,\quad 0\leqq l<n$

(7) $\quad=(-1)^m dg_m^{m-1}\cdot\varphi_n^{k-m,l}$ for $m\leqq k<m+n,\quad 0\leqq l<n$

(8) $\quad=(-1)^n\varphi_m^{k,l-n}\cdot d\bar h_n^{n-1}$ for $n\leqq l<m+n-k-1$

(9) $\quad=(-1)^{m-1}\delta\varphi_m^{m+n-l-1,l-n}\cdot\bar\partial\varphi_n^{k+l-m-n+1,n-1}$
for $l\geqq n,\quad m+n-1\leqq k+l<m+2n-1$

(10) $\quad=\varphi_m^{k-n,l-n}\wedge\eta_n^{n,n}$ for $k+l\geqq m+2n-1$

(11) $\eta_{m+n}^{m+n,m+n}=\eta_m^{m,m}\wedge\eta_n^{n,n}$.

Domains of validity of formulae (C)

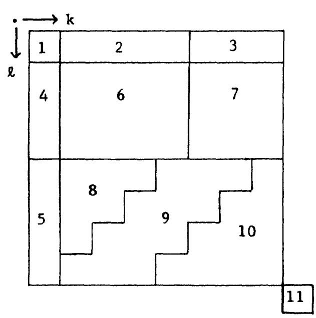

Fig. 2

Actually, the identities (C) define the announced product law

$$\tilde{\mathscr{K}}^m(\underline{X}) \times \tilde{\mathscr{K}}^n(\underline{X}) \to \tilde{\mathscr{K}}^{m+n}(\underline{X}).$$

It is true (the proof be omitted) that the law in question is associative (it will be denoted by the symbol $\times$) and, if Φ and Ψ are Δ-closed, $\Phi \times \Psi$ is also Δ-closed. However it is not compatible with the involution defined in 3.3 and this is the reason for which we work in $\tilde{\mathscr{K}}^m(\underline{X}, [\mathbb{R}])$ instead of $\tilde{\mathscr{K}}^m(\underline{X}, \mathbb{R})$.

Identities (C) will be used to prove

4.5. The Relation $\Delta\Phi_m(f, \varphi) = 0$. In order to prove that the element $\Phi_m(f, \varphi)$ defined in (4.3.3) is Δ-closed, we must prove according to (3.6.3) the relations

$$\text{(D)} \quad \begin{aligned}
&(1) \quad \delta a_m = 0\\
&(2) \quad \delta g_m^0 = a_m\\
&(3) \quad \delta g_m^k = d g_m^{k-1} \quad \text{for} \quad 1 \leqq k < m\\
&(4) \quad \delta \bar{h}_m^0 = a_m\\
&(5) \quad \delta \bar{h}_m^l = d \bar{h}_m^{l-1} \quad \text{for} \quad 1 \leqq l < m\\
&(6) \quad \delta \varphi_m^{0,0} = -g_m^0 + \bar{h}_m^0\\
&(7) \quad \delta \varphi_m^{k,0} = (-1)^{k-1} g_m^k + \partial \varphi_m^{k-1,l} \quad \text{for} \quad 1 \leqq k < m\\
&(8) \quad \delta \varphi_m^{0,l} = \bar{\partial} \varphi_m^{0,l-1} + (-1)^l \bar{h}_m^l \quad \text{for} \quad 1 \leqq l < m\\
&(9) \quad \delta \varphi_m^{k,l} = \bar{\partial} \varphi_m^{k,l-1} + \partial \varphi_m^{k-1,l} \quad \text{for} \quad 1 \leqq k, l < m\\
&(10) \quad \varepsilon(\eta_m^{m,m}) = \partial \bar{\partial} \varphi_m^{m-1,m-1}\\
&(11) \quad d\eta_m^{m,m} = 0 .
\end{aligned}$$

Proof of (D1). It is obvious.

Proof of (D2). $\delta g_m^0 = \delta(f \cdot (\delta f)^{m-1}) = (\delta f)^m = a_m$.

Proof of (D3).

$$\begin{aligned}
\delta g_m^k &= \delta((-1)^k (df)^k \cdot f \cdot (\delta f)^{m-k-1}) = (df)^k \cdot (\delta f)^{m-k}\\
&= d((-1)^{k-1} (df)^{k-1} \cdot f \cdot (\delta f)^{m-k}) = d g_m^{k-1} .
\end{aligned}$$

Proof of (D4).

$$\begin{aligned}
\delta \bar{h}_m^0 &= \delta((\delta f)^{m-1} \cdot \bar{f}) = (\delta f)^m \quad \text{by (A2)}\\
&= a_m .
\end{aligned}$$

Proof of (D5).

$$\begin{aligned}
\delta \bar{h}_m^l &= \delta((\delta f)^{m-l-1} \cdot \bar{f} \cdot (d\bar{f})^l) = (\delta f)^{m-l} \cdot (d\bar{f})^l \quad \text{by (A2)}\\
&= d((\delta f)^{m-l} \cdot \bar{f} \cdot (d\bar{f})^{l-1}) = d \bar{h}_m^{l-1} .
\end{aligned}$$

Proof of (D6).

$$\delta\varphi_m^{0,0}=\delta(f\cdot(\delta f)^{m-2}\cdot\bar{f})=(\delta f)^{m-1}\cdot\bar{f}-f\cdot(\delta f)^{m-1}=\bar{h}_m^0-g_m^0 .$$

Proof of (D7). *Case 1.* $k<m-1$

$$\begin{aligned}\delta\varphi_m^{k,0}&=\delta((-1)^k dg_k^{k-1}\cdot\varphi_{m-k}^{0,0}) \quad \text{by (C7)}\\&=dg_k^{k-1}\cdot\delta\varphi_{m-k}^{0,0}=dg_k^{k-1}\cdot(-g_{m-k}^0+\bar{h}_{m-k}^0) \quad \text{by (D6)}\\&=-dg_k^{k-1}\cdot g_{m-k}^0+\partial(g_k^{k-1}\cdot\bar{h}_{m-k}^0)\\&=(-1)^{k-1}g_m^k+\partial\varphi_m^{k-1,0} \quad \text{by (C3) and (C6)}.\end{aligned}$$

Case 2. $k=m-1$

$$\begin{aligned}\delta\varphi_m^{m-1,0}&=\delta((-1)^{m-1}(df)^{m-1}\cdot\varphi)=(df)^{m-1}\cdot\delta\varphi=(df)^{m-1}\cdot(-f+\bar{f}) \quad \text{by (A1)}\\&=-(df)^{m-1}\cdot f+\partial((-1)^{m-2}(df)^{m-2}\cdot f\cdot\bar{f})\\&=(-1)^m g_m^{m-1}+\partial\varphi_m^{m-2,0} \quad \text{by (B2) and (B5)}.\end{aligned}$$

Proof of (D8). *Case 1.* $l<m-1$

$$\begin{aligned}\delta\varphi_m^{m-1,0}&=\delta((-1)^{m-1}(df)^{m-1}\cdot\varphi)=(df)^{m-1}\cdot\delta\varphi=(df)^{m-1}\cdot(-f+\bar{f}) \quad \text{by (A1)}\\&=-(df)^{m-1}\cdot f+\partial((-1)^{m-2}(df)^{m-2}\cdot f\cdot\bar{f})\\&=(-1)^m g_m^{m-1}+\partial\varphi_m^{m-2,0} \quad \text{by (B2) and (B5)}.\end{aligned}$$

Proof of (D8). *Case 1.* $l<m-1$

$$\begin{aligned}\delta\varphi_m^{0,l}&=\delta((-1)^l\varphi_{m-l}^{0,0}\cdot d\bar{h}^{l-1}) \quad \text{by (C8)}\\&=(-1)^l\delta\varphi_{m-l}^{0,0}\cdot d\bar{h}_l^{l-1}=(-1)^l(-g_{m-l}^0+\bar{h}_{m-l}^0)\cdot d\bar{h}_l^{l-1} \quad \text{by (D6)}\\&=\bar{\partial}((-1)^{l-1}g_{m-l}^0\cdot\bar{h}^{l-1})+(-1)^l\bar{h}_{m-l}^0\cdot d\bar{h}_l^{l-1}\\&=\bar{\partial}\varphi_m^{0,l-1}+(-1)^l\bar{h}_m^l \quad \text{by (C6) and (C5)}.\end{aligned}$$

Case 2. $l=m-1$

$$\begin{aligned}\delta\varphi_m^{0,m-1}&=\delta(\delta\varphi\cdot(d\bar{f})^{m-2}\cdot\bar{\partial}\varphi)=(-1)^{m-1}\delta\varphi\cdot(d\bar{f})^{m-1} \quad \text{by (A5)}\\&=(-1)^m(f-\bar{f})\cdot(d\bar{f})^{m-1} \quad \text{by (A1)}\\&=\bar{\partial}((-1)^{m-2}f\cdot\bar{f}\cdot(d\bar{f})^{m-2})+(-1)^{m-1}\bar{f}\cdot(d\bar{f})^{m-2}\\&=\bar{\partial}\varphi_m^{0,m-2}+(-1)^{m-1}\bar{h}_m^{m-1} \quad \text{by (B4) and (B5)}.\end{aligned}$$

Proof of (D9). *Case1.* $k+l<m-1$.

We can write $m=r+s$ with $r>k$ and $s>l$. Then

$$\begin{aligned}\delta\varphi_m^{k,l}&=\delta\varphi_{r+s}^{k,l}=\delta((-1)^l g_r^k\cdot\bar{h}_s^l) \quad \text{by (C6)}\\&=(-1)^l\delta g_r^k\cdot\bar{h}_s^l+(-1)^{k+l-1}g_r^k\cdot\delta\bar{h}_s^l\\&=(-1)^l dg_r^{k-1}\cdot\bar{h}_s^l+(-1)^{k+l-1}g_r^k\cdot d\bar{h}_s^{l-1} \quad \text{by (D3) and (D5)}\\&=\partial((-1)^l g_r^{k-1}\cdot\bar{h}_s^l)+\bar{\partial}((-1)^{l-1}g_r^k\cdot\bar{h}_s^{l-1})\\&=\partial\varphi_m^{k-1,l}+\bar{\partial}\varphi_m^{k,l-1} \quad \text{by (C6)}.\end{aligned}$$

Case 2. $k+l=m-1$

$$\begin{aligned}\delta\varphi_m^{k,l} &= \delta((-1)^k (df)^k \cdot \delta\varphi \cdot (d\bar{f})^{l-1} \cdot \bar{\partial}\varphi) \quad \text{by (B5)}\\ &= (-1)^l (df)^k \cdot \delta\varphi \cdot (d\bar{f})^l \quad \text{by (A5)}\\ &= (-1)^{l-1}(df)^k \cdot f \cdot (d\bar{f})^l + (-1)^l (df)^k \cdot \bar{f} \cdot (d\bar{f})^l \quad \text{by (A1)}\\ &= (-1)^{k+l-1} g_{k+1}^k \cdot d\bar{h}_l^{l-1} + (-1)^l dg_k^{k-1} \cdot \bar{h}_{l+1}^l \quad \text{by (B2) and (B3)}\\ &= \bar{\partial}((-1)^{l-1} g_{k+1}^k \cdot \bar{h}_l^{l-1}) + \partial((-1)^l g_k^{k-1} \cdot \bar{h}_{l+1}^l)\\ &= \bar{\partial}\varphi_m^{k,l-1} + \partial\varphi_m^{k-1,l} \quad \text{by (C6)}.\end{aligned}$$

Case 3. $k<m-1<k+l$

$$\begin{aligned}\delta\varphi_m^{k,l} &= (-1)^{m-k-1}(df)^{m-l-1} \cdot \delta\varphi \cdot (d\bar{f})^{m-k-1} \wedge \omega^{k+l-m+1} \quad \text{by (B5) and (A5)}\\ &= (-1)^{m-k-1}(df)^{m-l-1} \cdot \delta\varphi \cdot (d\bar{f})^{m-k-1} \cdot \partial\bar{\partial}\varphi \wedge \omega^{k+l-m} \quad \text{by (A6)}\\ &= \bar{\partial}((-1)^{m-l}(df)^{m-l} \cdot \delta\varphi \cdot (d\bar{f})^{m-k-2} \cdot \bar{\partial}\varphi \wedge \omega^{k+l-m})\\ &\quad + \partial((-1)^{m-l-1}(df)^{m-l-1} \cdot \delta\varphi \cdot (d\bar{f})^{m-k-1} \cdot \bar{\partial}\varphi \wedge \omega^{k+l-m}) \quad \text{by (A4), (A5), (A6)}\\ &= \bar{\partial}\varphi_m^{k,l-1} + \partial\varphi_m^{k-1,l} \quad \text{by (B5)}.\end{aligned}$$

Case 4. $k=m-1$

$$\begin{aligned}\delta\varphi_m^{m-1,l} &= (df)^{m-l-1} \cdot \delta\varphi \wedge \omega^l \quad \text{by (B6)}\\ &= \bar{\partial}((-1)^{m-l}(df)^{m-l} \cdot \varphi \wedge \omega^{l-1}) + \partial((-1)^{m-l-1}(df)^{m-l-1} \cdot \delta\varphi \cdot \bar{\partial}\varphi \wedge \omega^{l-1})\\ &= \bar{\partial}\varphi_m^{m-1,l-1} + \partial\varphi_m^{m-2,l} \quad \text{by (B5) and (B6)}.\end{aligned}$$

Finally, (D10) and (D11) are obvious since $\varphi_m^{m-1,m-1} = \varphi\omega^{m-1}$ and $\eta_m^{m,m} = \omega^m$. Therefore the proof of the relation $\Delta\Phi_m(f,\varphi)=0$ is complete.

4.5.1. Remark. There are several alternative ways of proving $\Delta\Phi_m(f,\varphi)=0$. For example, identities (C) written only for $n=1$ give a relation between $\Phi_m(f,\varphi)$ and $\Phi_{m+1}(f,\varphi)$, and the relation $\Delta\Phi_m(f,\varphi)$ can be proven by induction on m. Otherwise, one can prove directly that $\Delta\Phi_m = \Delta\Phi_n = 0$ implies $\Delta(\Phi_m \times \Phi_n) = 0$ using (A), (B), and (C) but the calculations would be longer than the above (30 verifications are needed).

5. *Theorem 2*

5.1. Statement of Theorem 2. Let (X,ω) be a Kähler space and $m \geqq 0$ an integer. Then there exist open sets $U_\alpha \subset X$ $(\alpha \in A)$ and $U_{\alpha\beta}^j \subset U_\alpha \cap U_\beta$ $(j \in J_{\alpha\beta})$ depending on X and m alone such that

(i) Any compact m-dimensional complex-analytic subset of X is contained in some U_α.

(ii) Any compact m-dimensional complex-analytic subset of $U_\alpha \cap U_\beta$ is contained in some $U_{\alpha\beta}^j$.

(iii) There exist elements $\chi_\alpha \in A^{m,m}(U_\alpha, \mathbb{R})$ such that

$$\omega^{m+1}|_{U_\alpha} = i\partial\bar{\partial}\chi_\alpha.$$

(iv) There exist elements $\tau^j_{\alpha\beta} \in A^{m,m}(U^j_{\alpha\beta})$ such that

$$\bar{\partial}\tau^j_{\alpha\beta} = 0 \quad \text{and} \quad (\chi_\alpha - \chi_\beta)|_{U_{\alpha\beta}} = \tau^j_{\alpha\beta} + \bar{\tau}^j_{\alpha\beta}.$$

(v) The $\tau^j_{\alpha\beta}$ are $\bar{\partial}$-closed representatives of elements $\xi^j_{\alpha\beta} \in H^m(U^j_{\alpha\beta}, \Omega^m)$.

5.2. Proof of (i) and (ii). We take an open covering $\mathscr{X}$ of X such that $\underline{X} = (X, \mathscr{X})$ is a Kähler-Čech space with a Kähler-Čech pair (f, φ) as in 4.2.

The U_α are taken as the m-admissible open sets of X and the $U^j_{\alpha\beta}$ as the m-admissible open sets of $U_\alpha \cap U_\beta$. Parts (i) and (ii) of Theorem 2 are restatements of Lemma 3.5.4 of I. By Proposition 1.3.3, each U_α is underlying to some m-admissible $\underline{U}_\alpha \ll \underline{X}$ and each $U^j_{\alpha\beta}$ to some m-admissible $\underline{U}^j_{\alpha\beta} \ll \underline{U}_\alpha \cap \underline{U}_\beta$.

5.3. Proof of (iii). We use the element

$$\Phi_{m+1}(f, \varphi) \in \check{\mathscr{K}}^{m+1}(X, [\mathbb{R}]) = \check{Z}^{2m+2}(\underline{X}; \mathbb{R}, \mathscr{L}^{\cdot}_{m+1})$$

which is Δ-closed in the Čech transform of the complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{L}^0_{m+1} \longrightarrow \ldots \longrightarrow \mathscr{L}^{2m+1}_{m+1} \xrightarrow{\partial\bar{\partial}} \mathscr{L}^{2m+2}_{m+1} \longrightarrow \ldots.$$

Take the restriction [in the sense of (1.1.5)]

$$\Phi_{m+1,\alpha} := \Phi_{m+1}(f, \varphi)|_{\underline{U}_\alpha} \in \check{\mathscr{K}}^{m+1}(\underline{U}_\alpha, [\mathbb{R}]). \tag{5.3.1}$$

Since $\underline{U}_\alpha$ is m-complete, we have

$$H^{2m-k+1}(\underline{U}_\alpha, \mathscr{L}^k_{m+1}) = 0 \quad \text{for} \quad 0 \leqq k \leqq 2m.$$

Indeed, for $k \leqq m$ this is due to the m-completeness of $\underline{U}_\alpha$ and the fact that $\mathscr{L}^k_{m+1} = \Omega^k \oplus A^{k-1} \oplus \bar{\Omega}^k$; for $k > m$, it is due to the fact that $\mathscr{L}^k_{m+1}$ is a fine sheaf.

Corollary 2.3 applies and $\Phi_{m+1,\alpha}$ is Δ-exact if its head is δ-exact, since the canonical morphism

$$\check{H}^{2m+2}(\underline{U}_\alpha; \mathbb{R}, \mathscr{L}^{\cdot}_{m+1}) \to H^{2m+2}(\underline{U}_\alpha, \mathbb{R})$$

is injective. But the head of $\Phi_{m+1,\alpha}$ is $(\delta f)^{m+1}|_{\underline{U}_\alpha}$ whose class in $H^{2m+2}(\underline{U}_\alpha, \mathbb{R})$ is 0 by Lemma 1.3.2, since $\underline{U}_\alpha \ll \underline{X}$ is m-admissible. Therefore

$$\Phi_{m+1,\alpha} = \Delta\Theta_{m+1,\alpha} \tag{5.3.2}$$

for some $\Theta_{m+1,\alpha} \in \mathscr{E}^{2m+1}_{m+1}(\underline{U}_\alpha, [\mathbb{R}])$. In particular, if $\psi_\alpha \in A^{m,m}(U_\alpha)$ is the tail of $\Theta_{m+1,\alpha}$, we have

$$\omega^{m+1}|_{U_\alpha} = \partial\bar{\partial}\psi_\alpha. \tag{5.3.3}$$

It is then sufficient to set

$$\chi_\alpha := \frac{i}{2}(\bar{\psi}_\alpha - \psi_\alpha) \tag{5.3.4}$$

to satisfy condition (iii) of Theorem 2.

5.4. Proof of (iv) and (v). Take a fixed $\underline{U} = \underline{U}^j_{\alpha\beta} \ll \underline{U}_\alpha \cap \underline{U}_\beta$.

There are open inclusions of Čech open sets

$$\begin{tikzcd} \underline{U}_\alpha \arrow[d] & \underline{U} \arrow[l, "i_\alpha"'] \arrow[r, "i_\beta"] \arrow[dl, "j_\alpha"] \arrow[dr, "j_\beta"'] & \underline{U}_\beta \arrow[d] \\ \underline{X} & & \underline{X}. \end{tikzcd}$$

We may then apply the operator T of (1.1.6) relatively to $j_\alpha, j_\beta: \underline{U} \to \underline{X}$ and set

$$(5.4.1) \qquad \tilde{\Theta}_{m+1} := T\Phi_{m+1}(f, \varphi) \in \mathscr{E}^{2m+1}_{m+1}(\underline{U}, [\mathbb{R}]).$$

This element satisfies the conditions

$$(5.2.1) \qquad \begin{aligned} &\text{(i)} \quad \Delta\tilde{\Theta}_{m+1} = j_\beta^*\Phi_{m+1}(f, \varphi) - j_\alpha^*\Phi_{m+1}(f, \varphi) \\ &\text{(ii)} \quad \text{The tail of } \tilde{\Theta}_{m+1} \text{ is } 0. \end{aligned}$$

Indeed, (i) is a consequence of (1.1.7) and (ii) of the fact that T induces 0 on 0-cochains and global sections. Now set

$$(5.4.3) \qquad \Theta_{m+1} := i_\alpha^*(\Theta_{m+1,\alpha}) - i_\beta^*(\Theta_{m+1,\beta}) + \tilde{\Theta}_{m+1} \in \mathscr{E}^{2m+1}_{m+1}(\underline{U}, [\mathbb{R}]).$$

This element satisfies, by (5.3.2) and (5.4.2)

$$(5.4.4) \qquad \begin{aligned} &\text{(i)} \quad \Delta\Theta_{m+1} = 0 \\ &\text{(ii)} \quad \text{The tail of } \Theta_{m+1} \text{ is } \quad \psi := (\psi_\alpha - \psi_\beta)|_{\underline{U}}. \end{aligned}$$

We notice that Lemma 2.2(i) does not apply to the canonical morphism

$$\check{H}^{2m+1}(\underline{U}; \mathbb{R}, \mathscr{L}^{\cdot}_{m+1}) \to H^{2m+1}(\underline{U}, \mathbb{R})$$

for among the groups $H^{2m-k}(\underline{U}, \mathscr{L}^k_{m+1})$ there is $H^m(\underline{U}, \mathscr{L}^m_{m+1}) = H^m(\underline{U}, \Omega^m \oplus \bar{\Omega}^m)$ which is not 0 in general. So we apply the operator μ defined in 3.4 to obtain $\mu\Theta_{m+1} \in \mathscr{E}^{2m+1}_m(\underline{U}, [\mathbb{R}])$.

Since μ commutes with D (and δ), $\mu\Theta_{m+1}$ is Δ-closed. This time the canonical morphism

$$\check{H}^{2m+1}(\underline{U}; \mathbb{R}, \mathscr{L}^{\cdot}_m) \to H^{2m+1}(\underline{U}, \mathbb{R})$$

is injective since the groups $H^{2m-k}(\underline{U}, \mathscr{L}^k_m)$ are all 0 for $0 \leq k \leq 2m-1$. Indeed, for $k < m$ this is due to the m-completeness of $\underline{U}$ and, for $k \geq m$, to the fact that $\mathscr{L}^k_m$ is a fine sheaf. So, by Corollary 2.3, $\mu\Theta_{m+1}$ is Δ-exact if its head is δ-exact in $C^{\cdot}(\underline{U}, \mathbb{R})$. But the head of $\mu\Theta_{m+1}$ is equal to the head of Θ_{m+1} which is of the form $c_{m+1}|_{\underline{U}}$ with

$$c_{m+1} \in Z^{2m+1}(\underline{U}_\alpha \cap \underline{U}_\beta, \mathbb{R}).$$

Since $\underline{U} \ll \underline{U}_\alpha \cap \underline{U}_\beta$ is m-admissible, $c_{m+1}|_{\underline{U}}$ is δ-exact (Lemma 1.4.2) and therefore

$$(5.4.5) \qquad \mu\Theta_{m+1} = \Delta Z_m$$

for some $Z_m \in \mathscr{E}^{2m}_m(\underline{U}, \mathbb{R})$.

Now we use the operators β and γ defined in 3.5. Denote by $\mathscr{D}^{\cdot}_m(\underline{U})$ the Čech transform of the $(\bar{\partial} \oplus \partial)$-complex over $\underline{U}$, i.e.

$$(5.4.6) \qquad \mathscr{D}^q_m(\underline{U}) := \check{C}^q(\underline{U}; \Omega^m \oplus \bar{\Omega}^m, \mathscr{G}^{\cdot}_m)$$

with differential

$$\hat{\Delta} := \delta + (-1)^{m+q+1}\hat{d} : \mathscr{D}^q_m(\underline{U}) \to \mathscr{D}^{q+1}_m(\underline{U}). \tag{5.4.7}$$

Notice that this sign convention differs from (2.1.3).

Diagram (3.5.3) becomes

$$\begin{tikzcd}
 & \mathscr{E}^{2m}_m(\underline{U}) \arrow[d, "\Delta=\delta-D"] \arrow[dr, "\gamma"] & \\
\mathscr{E}^{2m+1}_{m+1}(\underline{U}) \arrow[rr, "\beta"] \arrow[dr, "\mu"] \arrow[dd, "\Delta=\delta+D"'] & & \mathscr{D}^m_m(\underline{U}) \arrow[dd, "\hat{\Delta}=\delta-\hat{d}"] \\
 & \mathscr{E}^{2m+1}_m(\underline{U}) \arrow[dr, "\gamma"] & \\
\mathscr{E}^{2m+2}_{m+1}(\underline{U}) \arrow[rr, "\beta"] & & \mathscr{D}^{m+1}_m(\underline{U}).
\end{tikzcd} \tag{5.4.8}$$

By Lemma 3.5.3 and the sign convention (5.4.7) on $\hat{\Delta}$ we have on $\mathscr{E}^{2m+1}_{m+1}(\underline{U})$

$$\begin{aligned}\beta\Delta - \hat{\Delta}\beta &= \beta(\delta + D) - (\delta - \hat{d})\beta = (\beta\delta - \delta\beta) + (\beta D + \hat{d}\beta) \\ &= \beta D + \hat{d}\beta = \gamma\mu.\end{aligned} \tag{5.4.9}$$

On the other hand, we have on $\mathscr{E}^{2m}_m(\underline{U})$

$$\gamma\Delta = \hat{\Delta}\gamma. \tag{5.4.10}$$

If we apply (5.4.9) to Θ_{m+1} and (5.4.10) to Z_m, we get

$$-\hat{\Delta}\beta\Theta_{m+1} = (\beta\Delta - \hat{\Delta}\beta)\Theta_{m+1} = \gamma\mu\Theta_{m+1} = \gamma\Delta Z_m = \hat{\Delta}\gamma Z_m$$

which means that the element

$$\Lambda_m := \beta\Theta_{m+1} + \gamma Z_m \in \mathscr{D}^m_m(\underline{U}) \tag{5.4.11}$$

satisfies

$$\hat{\Delta}\Lambda_m = 0.$$

The tail of Λ_m has the form

$$(\varrho^{m,m}, \sigma^{m,m}) \in A^{m,m}(U) \oplus A^{m,m}(U)$$

with $\bar{\partial}\varrho^{m,m} = \partial\sigma^{m,m} = 0$ (since $\hat{\Delta}\Lambda_m = 0$) and

$$\varrho^{m,m} + \sigma^{m,m} = \psi \tag{5.4.12}$$

by Lemma 3.5.3(iii).

The fact that Λ_m is a $\hat{\Delta}$-cocycle means precisely that $\varrho^{m,m}$ and $\bar{\sigma}^{m,m}$ represent elements of $H^m(\underline{U}, \Omega^m)$. So if we set

$$\tau^j_{\alpha\beta} := \frac{i}{2}(\bar{\sigma}^{m,m} - \varrho^{m,m}) \tag{5.4.13}$$

it is clear that conditions (iv) and (v) of Theorem 2 are satisfied.

5.5. *Remark.* (1) We did not use the positivity of ω in the proof of Theorem 2. The result we can actually prove by our method is the following: If U_α and $U^j_{\alpha\beta}$ are the open sets of Theorem 2, then conditions (i) and (ii) remain unchanged. If moreover $\kappa_0, \dots, \kappa_m$ are arbitrary elements of $\mathscr{K}^1(X)$ and $\omega_q := \partial\bar{\partial}\kappa_q$ for $0 \leqq q \leqq m$, then

(iii) There are elements $\psi_\alpha \in A^{m,m}(U_\alpha)$ such that $(\omega_0 \wedge \dots \wedge \omega_m)|_{U_\alpha} = \partial\bar{\partial}\psi_\alpha$.

(iv) There are elements $\varrho^j_{\alpha\beta}, \sigma^j_{\alpha\beta} \in A^{m,m}(U^j_{\alpha\beta})$ such that $\bar{\partial}\varrho^j_{\alpha\beta} = \partial\sigma^j_{\alpha\beta} = 0$ and $(\psi_\alpha - \psi_\beta)|_{U^j_{\alpha\beta}} = \varrho^j_{\alpha\beta} + \sigma^j_{\alpha\beta}$.

(v) $\varrho^j_{\alpha\beta}$ and $\bar{\sigma}^j_{\alpha\beta}$ represent cohomology classes of $H^m(U^j_{\alpha\beta}, \Omega^m)$.

(2) The proof we gave was a reasoning on $\mathscr{E}^{\cdot}_m(\underline{X}, [\mathbb{R}])$. We could have chosen $\mathscr{E}^{\cdot}_m(\underline{X}, \mathbb{R})$ as well, replacing $\Phi_{m+1}(f, \varphi)$ by

$$\mathrm{Re}(\Phi_{m+1}(f, \varphi)) = \tfrac{1}{2}(\Phi_{m+1}(f, \varphi) + \Phi_{m+1}(f, \varphi)^*)$$

and using Lemma 3.5.3(iv).

IV. The Main Results

1. Stability Theorems

We are now in position to prove that some proper images of Kähler spaces are Kähler.

1.1. Theorem 3. *Let $\pi: X \to X'$ be a geometrically flat morphism of complex spaces with m-dimensional fibers (π is proper surjective and X' reduced by definition). Suppose X is Kähler. Then X' is weakly Kähler.*

If moreover there is a discrete $D' \subset X'$ such that for any $x' \in X' \backslash D'$, either

(i) *X' is weakly normal at x' or*

(ii) *$\pi^{-1}(x')$ admits in X a smoothly embeddable neighborhood*

then X' is Kähler.

Proof. With the notations of Theorem 2, set

$$\begin{aligned}
V'_\alpha &:= \{x' \in X' \mid \pi^{-1}(x') \subset U_\alpha\}\\
V_\alpha &:= \pi^{-1}(V'_\alpha)\\
V'^j_{\alpha\beta} &:= \{x' \in X' \mid \pi^{-1}(x') \subset U^j_{\alpha\beta}\}\\
V^j_{\alpha\beta} &:= \pi^{-1}(V'^j_{\alpha\beta})\\
\psi_\alpha &:= \pi_*(\chi_\alpha|_{V_\alpha})\\
g^j_{\alpha\beta} &:= \pi_*(\tau^j_{\alpha\beta}|_{V^j_{\alpha\beta}}).
\end{aligned}$$

Since π is surjective, the sets V'_α cover X' and, for fixed α, β, the $V'^j_{\alpha\beta}$ cover $V'_\alpha \cap V'_\beta$. By Proposition 3.4.1 of Chap. I, $\psi_\alpha \in SP^0(V'_\alpha)$, $g^j_{\alpha\beta} \in \mathscr{W}(V'^j_{\alpha\beta})$ and, since $(\psi_\alpha - \psi_\beta)|_{V'^j_{\alpha\beta}} = g^j_{\alpha\beta} + \bar{g}^j_{\alpha\beta}$, $\psi_\alpha - \psi_\beta \in WPH(V'_\alpha \cap V'_\beta, \mathbb{R})$. So X' is weakly Kähler. Now if conditions (i) and (ii) are fulfilled, then $g^j_{\alpha\beta}$ is holomorphic on $V'^j_{\alpha\beta} \backslash D'$ and $\psi_\alpha - \psi_\beta$ pluriharmonic on $V'_\alpha \cap V'_\beta \backslash D'$. If we take a refinement (W'_λ) of (V'_α) such that each point of D' belongs at most to one W'_λ, then it is clear that Theorem 1 applies and X' is Kähler.

1.2. Corollary. *Let $\pi: X \to X'$ be a proper open surjective morphism. Suppose X is Kähler and X' normal. Then X' is Kähler.*

Many other consequences may be formulated. For example

1.3. Corollary. *Let $\pi: X \to X'$ be a flat projective morphism. Suppose X is Kähler and X' reduced. Then X' is Kähler.*

Proof. The fibers of a projective morphism have smoothly embeddable neighborhoods by construction of $\mathbb{P}(\mathscr{F})$ for a coherent sheaf $\mathscr{F}$.

1.4. Remark. Conditions (i) and (ii) of Theorem 3 are actually unnecessary. See note 3.6 of Chap. I.

2. *The Space of Cycles of a Kähler Space*

We use the notations of Chap. I, 3.

2.1. Theorem 4. *Let X be a Kähler space and $m \geqq 0$ an integer. Then the Barlet space $\mathbf{B}_m(X)$ of m-cycles of X is weakly Kähler. Moreover, the open subset $\mathbf{B}_m(X)^{(0)}$ of $\mathbf{B}_m(X)$ is Kähler.*

Proof. By an argument similar to the above, set

$$W_\alpha := \{c \in \mathbf{B}_m(X) \mid |c| \subset U_\alpha\}$$

$$W^j_{\alpha\beta} := \{c \in \mathbf{B}_m(X) \mid |c| \subset U^j_{\alpha\beta}\}$$

$$\Phi_\alpha := F_{\chi_\alpha}, \qquad G^j_{\alpha\beta} := F_{\tau^j_{\alpha\beta}},$$

Then $\Phi_\alpha \in SP^0(W_\alpha)$, $G^j_{\alpha\beta}$ is weakly holomorphic on $W^j_{\alpha\beta}$ and holomorphic on $W^j_{\alpha\beta} \cap \mathbf{B}_m(X)^{(0)}$, $(\Phi_\alpha - \Phi_\beta)|_{W^j_{\alpha\beta}} = G^j_{\alpha\beta} + \bar{G}^j_{\alpha\beta}$ and the result follows.

2.2. Corollary. *Let X be a Kähler space. Then the weak normalization of $\mathbf{B}_m(X)$ is Kähler.*

Proof. By a well-known result [5, 12, 18] every connected component of $\mathbf{B}_m(X)$ is compact and, by Theorem 4 above, weakly Kähler. The result follows from Proposition 4.2.4 of Chap. II.

3. *Fujiki's Class $\mathscr{C}$*

3.1. Definition (Fujiki [12]). A reduced compact complex space X is said to belong to class $\mathscr{C}$ if it is a holomorphic image of a compact Kähler space.

By Hironaka's resolution of singularities it is sufficient to take holomorphic images of compact Kähler manifolds.

Let us define for the moment the class $\mathscr{C}^*$ of reduced compact spaces bimeromorphically equivalent to compact Kähler manifolds, i.e. admitting compact Kähler modifications.

It is then true that $\mathscr{C}$ is stable under holomorphic images and subspaces; but it seems difficult to prove, for example, that a reduced subspace of a space in $\mathscr{C}^*$ is in $\mathscr{C}^*$. Of course, $\mathscr{C}^* \subset \mathscr{C}$.

On the other hand, several important results are valid for compact manifolds in $\mathscr{C}^*$. For example:

(i) If X is a manifold in $\mathscr{C}^*$ and $H^0(X, \Omega_X^2)=0$ then X is Moišezon [14].

(ii) If X is a manifold in $\mathscr{C}^*$, $n=\dim X$ and $\pi: X \to S$ a surjective morphism of X on a complex space S, then $R^q\pi_*(\Omega_X^n)=0$ for all $q>\dim X-\dim S$ (Takegoshi [22]). It seems difficult to prove such results with the hypothesis $X \in \mathscr{C}$. But we have

3.2. Theorem 5. $\mathscr{C}=\mathscr{C}^*$.

Proof. Let X be a compact complex space in $\mathscr{C}$. By definition there is a compact Kähler space X_1 and a surjective morphism $\varrho: X_1 \to X$. By Hironaka's flattening theorem [16], there is a commutative diagram

$$\begin{array}{ccc} X_1 & \xleftarrow{\sigma_1} & Y_1 \\ {\scriptstyle\varrho}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xleftarrow{\sigma} & Y, \end{array}$$

where σ, σ_1 are projective modifications and π is flat. Since σ_1 is a Kähler morphism and X_1 a compact Kähler space, Y_1 is Kähler. Moreover Y can be chosen to be normal, since flatness is preserved by base-change. If we apply Corollary 1.2 to $\pi: Y_1 \to Y$, then we deduce that Y is Kähler and $X \in \mathscr{C}^*$ as required.

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