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Kähler Spaces and Proper Open Morphisms

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Introduction

Several years ago, Hironaka [17] raised the following two problems:

Problem A. Let X be a Kähler space. Is the Douady space of X Kähler?

and its weaker version

Problem B. Let $\pi: X \to X'$ be a proper flat surjective morphism of complex spaces. If X is Kähler, is X' Kähler?

Actually problems A and B were raised for compact X but we will consider the non-compact case as well.

Problem A seems inaccessible for the moment.

Problem B was solved affirmatively in [23] for smooth X, X'. The aim of the present paper is to generalize the result to singular spaces. It appears that the flatness hypothesis on π is too strong, so it will be replaced by a less restrictive property which we call *geometric flatness*.

A closely related problem, raised by Lieberman [18] is

Problem C. Let X be a Kähler space and $\mathbf{B}_m(X)$ the Barlet space of compact complex m-cycles of X. Is $\mathbf{B}_m(X)$ Kähler?

A solution to problem C would imply one to problem B for geometrically flat π and reduced X'.

Finally a problem which is of fundamental importance in the theory of complex cycles is

Problem D. Let X be a complex space and $\xi \in H^m(X, \Omega_X^m)$. Is the function $F_\xi : c \mapsto (c \cdot \xi)$ holomorphic on $\mathbf{B}_m(X)$?

Our results can be summarized as follows: Problems B and C are reduced to problem D; problem D has a solution (for fixed X, m) if every compact m-dimensional complex-analytic subset of X has a smoothly embeddable neighborhood (Chap. I, Proposition 3.5.4).

In order to formulate our results completely, and as long as problem D remains unsolved in its full generality, we are led to introduce the notion of *weakly Kähler spaces*. The most useful properties of geometrically flat morphisms and weakly Kähler spaces are

(i) A geometrically flat morphism is proper open surjective with pure dimensional fibers and reduced base. The converse is true if the morphism is flat or the base is normal.

(ii) If G is a finite group of automorphisms of a reduced space X then the canonical projection $X \to X/G$ is geometrically flat.

(iii) Kähler spaces are weakly Kähler. Subspaces of weakly Kähler spaces are weakly Kähler. X is weakly Kähler iff X_{red} is weakly Kähler. A weakly normal space is weakly Kähler iff it is Kähler.

(iv) A compact space is weakly Kähler iff its weak normalization is Kähler.

Now we may enumerate our main results:

(i) If $\pi : X \to X'$ is geometrically flat with m-dimensional fibers, then problem B has a solution for π if problem D has a solution for X, m. Otherwise all we can say is that X' is weakly Kähler. But this is enough to ensure that X' is Kähler if it is normal (Chap. IV, Theorem 3–Corollary 1.2).

(ii) Problem C has a solution for X, m if problem D has a solution for X, m. Otherwise all we can say is that $\mathbf{B}_m(X)$ is weakly Kähler. But this is enough to ensure that, if X is compact, the weak normalization of $\mathbf{B}_m(X)$ is Kähler (Chap. IV, Theorem 4–Corollary 2.2).

(iii) The solution of problem B for normal X' implies that any reduced compact complex space in Fujiki's class $\mathscr{C}$ (holomorphic image of a compact Kähler space) is bimeromorphically equivalent to a compact Kähler space

(Chap. IV, Theorem 5). An alternative proof of this was given in [24] using the solution of problem C for smooth X.

Our paper is organized as follows:

In Chap. I we give a rapid discussion of the sheaf $\mathscr{C}_X^\infty$ in the sense we choose for a complex space X. $\mathscr{C}_X^\infty$ is *not* a subsheaf of the sheaf $\mathscr{C}_X$ of continuous complex-valued functions on X; there is only a canonical morphism $\varphi \mapsto [\varphi]$ from $\mathscr{C}_X^\infty$ to $\mathscr{C}_X$. This is important for the formulation of a smoothing lemma (2.5) for continuous strongly plurisubharmonic (p.s.h.) functions which is essentially due to Richberg [21]. We also remind some of the main properties of the Barlet space $\mathbf{B}_m(X)$ which we will use. Geometric flatness is defined in 3.3.

In Chap. II we define the notions of Kähler metrics, classes, spaces, and morphisms and prove *Theorem 1* (valid on any complex space) according to which, a space is Kähler if it admits an open covering $\mathscr{U}$ with 0-cochain $\varphi=(\varphi_\alpha)$ of *continuous* strongly p.s.h. functions and a 1-cocycle $h=(h_{\alpha\beta})$ of pluriharmonic functions such that $\delta\varphi=[h]$ in $C^1(\mathscr{U},\mathscr{C}_X)$. (The cocycle condition on h is redundant only for X reduced). As a consequence we solve problem B for *finite* $\pi: X\to X'$ such that either π is flat and X' arbitrary (not necessarily reduced) or π is geometrically flat and X' reduced. If X is a Kähler space and G a finite group of automorphisms of $X, X/G$ is Kähler. In particular, $\mathrm{Sym}^k(X)$ is Kähler for any $k \geqq 1$ (Corollary 3.2.1). Finally we define weakly Kähler spaces in 4.1.

Chapter III is entirely devoted to the proof of *Theorem 2*: if X is a complex space and $m \geqq 0$ an integer, then there are open sets $U_\alpha \subset X$ and $U^j_{\alpha\beta} \subset U_\alpha \cap U_\beta$ such that any compact m-dimensional complex-analytic subset of X (resp. $U_\alpha \cap U_\beta$) is contained in some U_α (resp. $U^j_{\alpha\beta}$). Moreover, if ω is a Kähler form on X, then there are (m,m)-forms $\chi_\alpha=\bar{\chi}_\alpha$ on U_α, $\tau^j_{\alpha\beta}$ on $U^j_{\alpha\beta}$ such that $\omega^{m+1}|U_\alpha=i\partial\bar{\partial}\chi_\alpha$, $\bar{\partial}\tau^j_{\alpha\beta}=0$, $(\chi_\alpha-\chi_\beta)|U^j_{\alpha\beta}=\tau^j_{\alpha\beta}+\bar{\tau}^j_{\alpha\beta}$ and the $\bar{\partial}$-cohomology class of $\tau^j_{\alpha\beta}$ lies in the image of the canonical morphism $H^m(U^j_{\alpha\beta},\Omega^m)\to H^{m,m}_{\bar{\partial}}(U^j_{\alpha\beta})$.

Theorem 2 is the main original element of this paper. It relies on Barlet's result [6] according to which m-dimensional compact complex-analytic subsets admit m-complete neighborhoods. For smooth X, Theorem 2 can be easily deduced from this [23, Lemma 3.6] and [24, 2.8] using the Dolbeault isomorphism. For singular X, this is considerably more difficult. Our method can be described as follows: When a complex of sheaves $(\mathscr{L}^\cdot, D)$ fails to be exact, we replace it by the single complex associated to the double complex (δ, D) where δ is the Čech differential with respect to some open covering. We call this new complex the *Čech transform* of $(\mathscr{L}^\cdot, D)$ and apply it to the $\partial\bar{\partial}$-complex $\mathscr{L}^\cdot_m$ (defined in 3.1). The key step is the existence of a cocycle Φ_{m+1} of degree $2m+2$ (defined in 4.3) of the Čech transform of the complex $\mathscr{L}^\cdot_{m+1}$ whose final component is ω^{m+1}. Using an elementary lemma of algebra (Lemma 2.2) we prove that Φ_{m+1} bounds near every m-dimensional compact complex-analytic subset of X, so ω^{m+1} is $\partial\bar{\partial}$-exact there. The last part of Theorem 2 relies on two morphisms β and γ (defined in 3.5) connecting the $\partial\bar{\partial}$-complex to the direct sum of the Dolbeault complex and its conjugate. Chapter III is self-contained.

Finally Chap. IV proves the main results we obtain as consequences of Theorems 1 and 2, namely Theorems 3–5 and corollaries.

List of Symbols

I. Ω_X^m
$\mathscr{C}_X$
$\mathscr{F}_{X,\mathbb{R}}$
$\mathscr{F}(U,\mathbb{R})$

1.1. $\mathscr{C}_X^\infty$
A_X^m
$A_X^{k,l}$
PH_X

1.2. $[\varphi]$
$[\mathscr{C}_X^\infty]$

2.1. P_X^0
SP_X^0
P_X^∞
SP_X^∞
$[P_X^\infty]$
$[SP_X^\infty]$

2.4. $SP^{0,\infty}(U,V)$

2.6. SP_π^∞

3.1. $\mathrm{Sym}^k(X)$
$\sum\{x_j\}$

3.2. $\mathbf{B}_m(X)$

3.3. $\mathbf{D}_m(X)$
$c(Y)$

3.4. $F_\varphi(c)$
$(c\cdot\xi)$

3.5. $\pi_*\varphi$
$\mathbf{B}_m(X)^{(0)}$

II.

1.1. $\mathscr{H}_X^1$
$\mathscr{H}_{X,\mathbb{R}}^1$
$\partial\bar\partial\kappa$

1.2. $\hat c_1$
c_1
$\tilde c_1$

3.1. $\mathrm{Tr}_{X/X'}^{(c)}$
$\mathrm{Tr}_{X/X'}^{(h)}$

4.1. $\mathscr{W}_X$
WPH_X
$\hat X$

III.

1.1. $\underline{X}$
$F:\underline{X}\to\underline{Y}$
$\underline{U}\ll\underline{X}$
$\underline{U}_1\cap\underline{U}_2$
ε
δ
$\varphi|\underline{U}$
T

1.2. $\varphi\cdot\psi$

2.1. $\check C^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot)$
$\varDelta$
$\check Z^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot)$
$\check H^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot)$

3.1. $\mathscr{L}_m^r$
D

3.3 φ^*
$\mathscr{L}_{m,\mathbb{R}}^r$

3.4. μ

3.5. $\mathscr{G}_m^q$
$\hat d$
β
γ

3.6. $\mathscr{E}_m^q(\underline{X})$
$\mathscr{E}_m^q(\underline{X},[\mathbb{R}])$
$\mathscr{E}_m^q(\underline{X},\mathbb{R})$

4.2. $\Phi_1(f,\varphi)$

4.3. $\mathscr{K}^m(\underline{X}),\ \mathscr{K}(\underline{X})$
$\mathscr{K}^m(\underline{X},[\mathbb{R}]),\ \mathscr{K}(\underline{X},[\mathbb{R}])$
$\mathscr{K}^m(\underline{X},\mathbb{R}),\ \mathscr{K}(\underline{X},\mathbb{R})$

4.4. $\Phi\times\Psi$

5.4. $\mathscr{D}_m^q(\underline{X})$
$\hat\varDelta$

IV.

3.1. $\mathscr{C}$
$\mathscr{C}^*$

I. Preliminaries

X will always denote a complex space, not necessarily reduced unless explicitly stated. $X_{\mathrm{red}}\to X$ denotes the reduction of X. $\mathscr{O}_X=\Omega_X^0$ is the structure sheaf of X and Ω_X^m the sheaf of holomorphic m-forms on X. $\mathscr{C}_X$ is the sheaf of continuous functions on the topological space underlying to X. If $\mathscr{F}=\mathscr{F}_X$ is any sheaf on X, $\mathscr{F}(U)$ will denote $\Gamma(U,\mathscr{F}_X)$. If $\mathscr{F}_X$ is a sheaf of $\mathbb{C}$-vector spaces with a natural $\mathbb{C}$-antilinear involution, $\mathscr{F}_{X,\mathbb{R}}$ will denote the subsheaf of elements left fixed by the involution and $\mathscr{F}(U,\mathbb{R}):=\Gamma(U,\mathscr{F}_{X,\mathbb{R}})$. We always assume X countable at infinity.

1. $\mathscr{C}^\infty$ Forms and Functions on Complex Spaces

There are two inequivalent definitions of $\mathscr{C}_X^\infty$ in the literature. The first, which we call the "old" one [5, 10, 21] defines $\mathscr{C}_X^\infty$ as the subsheaf of $\mathscr{C}_X$ consisting of local