Werk

Titel: Mathematische Annalen

Verlag: Springer

Jahr: 1989

Kollektion: Mathematica

Werk Id: PPN235181684_0283

PURL: http://resolver.sub.uni-goettingen.de/purl?PID=PPN235181684_0283|LOG_0012

List of Symbols

I. Ω_X^m
$\mathscr{C}_X$
$\mathscr{F}_{X,\mathbb{R}}$
$\mathscr{F}(U,\mathbb{R})$

1.1. $\mathscr{C}_X^\infty$
A_X^m
$A_X^{k,l}$
PH_X

1.2. $[\varphi]$
$[\mathscr{C}_X^\infty]$

2.1. P_X^0
SP_X^0
P_X^∞
SP_X^∞
$[P_X^\infty]$
$[SP_X^\infty]$

2.4. $SP^{0,\infty}(U,V)$

2.6. SP_π^∞

3.1. $\mathrm{Sym}^k(X)$
$\sum\{x_j\}$

3.2. $\mathbf{B}_m(X)$

3.3. $\mathbf{D}_m(X)$
$c(Y)$

3.4. $F_\varphi(c)$
$(c\cdot\xi)$

3.5. $\pi_*\varphi$
$\mathbf{B}_m(X)^{(0)}$

II.

1.1. $\mathscr{K}_X^1$
$\mathscr{K}_{X,\mathbb{R}}^1$
$\partial\bar\partial\kappa$

1.2. $\hat c_1$
c_1
$\tilde c_1$

3.1. $\mathrm{Tr}^{(c)}_{X/X'}$
$\mathrm{Tr}^{(h)}_{X/X'}$

4.1. $\mathscr{W}_X$
WPH_X
$\hat X$

III.

1.1. $\underline{X}$
$F:\underline{X}\to\underline{Y}$
$\underline{U}\ll\underline{X}$
$\underline{U}_1\cap\underline{U}_2$
ε
δ
$\varphi|\underline{U}$
T

1.2. $\varphi\cdot\psi$

2.1. $\check{C}^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot)$
Δ
$\check{Z}^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot)$
$\check{H}^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot)$

3.1. $\mathscr{L}_m^r$
D

3.3 φ^*
$\mathscr{L}_{m,\mathbb{R}}^r$

3.4. μ

3.5. $\mathscr{G}_m^q$
$\hat d$
β
γ

3.6. $\mathscr{E}_m^q(\underline{X})$
$\mathscr{E}_m^q(\underline{X},[\mathbb{R}])$
$\mathscr{E}_m^q(\underline{X},\mathbb{R})$

4.2. $\Phi_1(f,\varphi)$

4.3. $\mathscr{K}^m(\underline{X})$, $\mathscr{K}(\underline{X})$
$\mathscr{K}^m(\underline{X},[\mathbb{R}])$, $\mathscr{K}(\underline{X},[\mathbb{R}])$
$\mathscr{K}^m(\underline{X},\mathbb{R})$, $\mathscr{K}(\underline{X},\mathbb{R})$

4.4. $\Phi\times\Psi$

5.4. $\mathscr{D}_m^q(\underline{X})$
$\hat\Delta$

IV.

3.1. $\mathscr{C}$
$\mathscr{C}^*$

I. Preliminaries

X will always denote a complex space, not necessarily reduced unless explicitly stated. $X_{\mathrm{red}}\to X$ denotes the reduction of X. $\mathcal{O}_X=\Omega_X^0$ is the structure sheaf of X and Ω_X^m the sheaf of holomorphic m-forms on X. $\mathscr{C}_X$ is the sheaf of continuous functions on the topological space underlying to X. If $\mathscr{F}=\mathscr{F}_X$ is any sheaf on X, $\mathscr{F}(U)$ will denote $\Gamma(U,\mathscr{F}_X)$. If $\mathscr{F}_X$ is a sheaf of $\mathbb{C}$-vector spaces with a natural $\mathbb{C}$-antilinear involution, $\mathscr{F}_{X,\mathbb{R}}$ will denote the subsheaf of elements left fixed by the involution and $\mathscr{F}(U,\mathbb{R}):=\Gamma(U,\mathscr{F}_{X,\mathbb{R}})$. We always assume X countable at infinity.

1. $\mathscr{C}^\infty$ Forms and Functions on Complex Spaces

There are two inequivalent definitions of $\mathscr{C}_X^\infty$ in the literature. The first, which we call the "old" one [5, 10, 21] defines $\mathscr{C}_X^\infty$ as the subsheaf of $\mathscr{C}_X$ consisting of local restrictions of $\mathscr{C}^\infty$ functions under smooth embeddings. So $\mathscr{C}_X^\infty = \mathscr{C}_{X_{\text{red}}}^\infty$ in this sense. The second which we will call thee "modern" one [8, 12] is the one we give below.

1.1. Definitions. We define on X the sheaves $\mathscr{C}_X^\omega$ of real-analytic functions, PH_X of pluriharmonic functions, $\mathscr{C}_X^\infty = A_X^0$ of $\mathscr{C}^\infty$ functions, A_X^m (resp. $A_X^{k,l}$) of $\mathscr{C}^\infty$ m-forms [resp. (k, l)-forms] as follows: For smooth X, they are well defined. Now suppose $X \to D$ is an embedding of X in a domain D of $\mathbb{C}^n$ and $\mathscr{I}_X \subset \mathcal{O}_D$ is the corresponding coherent ideal sheaf. Set

$$\mathscr{I}_X^\omega := (\mathscr{I}_X + \bar{\mathscr{I}}_X)\mathscr{C}_D^\omega, \qquad \mathscr{I}_X^\infty := (\mathscr{I}_X + \bar{\mathscr{I}}_X)\mathscr{C}_D^\infty$$

and

$$\mathscr{C}_X^\omega := \mathscr{C}_D^\omega / \mathscr{I}_X^\omega, \qquad \mathscr{C}_X^\infty := \mathscr{C}_D^\infty / \mathscr{I}_X^\infty, \qquad A_X^m := A_D^m / (\mathscr{I}_X^\infty A_D^m + d\mathscr{I}_X^\infty A_D^{m-1})$$

$$A_X^{k,l} := \text{the image of } A_D^{k,l} \text{ under the canonical morphism } \quad A_D^{k+l} \to A_X^{k+l}.$$

It is clear that these sheaves are independent of the choice of the embedding $X \to D$ so they extend to arbitrary X. There are canonical morphisms

$$\mathcal{O}_X \to \mathscr{C}_X^\omega \to \mathscr{C}_X^\infty \to \mathscr{C}_X.$$

1.2. Elementary Properties and Conventions. (i) The canonical morphisms $\mathcal{O}_X \to \mathscr{C}_X^\omega$ and $\mathscr{C}_X^\omega \to \mathscr{C}_X^\infty$ are injective. (The first is elementary and the second is a consequence of the fact that $\mathscr{C}_D^\infty$ is a faithfully flat $\mathscr{C}_D^\omega$-module by Malgrange [19, Chap. VI, Corollary 1.12].) They will be considered as inclusions

$$\mathcal{O}_X \subset \mathscr{C}_X^\omega \subset \mathscr{C}_X^\infty$$

and so we may define $PH_X := \mathcal{O}_X + \bar{\mathcal{O}}_X \subset \mathscr{C}_X^\omega$.

(ii) In $\mathscr{C}_X^\omega$ we have $\mathcal{O}_X \cap \bar{\mathcal{O}}_X = \mathbb{C}$ and there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{O}_X & \xrightarrow{-2\,\mathrm{Im}} & PH_{X,\mathbb{R}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\lambda} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{O}_X \oplus \bar{\mathcal{O}}_X & \xrightarrow{(i\ \ -i)} & PH_X & \longrightarrow & 0,
\end{array}$$

where $\lambda(f) = (f, \bar{f})$ and the unspecified morphisms are the canonical inclusions.

(iii) The canonical morphism $\varrho : \mathscr{C}_X^\infty \to \mathscr{C}_X$ is not injective in general even for X reduced; for $fg = 0$ in $\mathcal{O}_X$ does not imply $f\bar{g} = 0$ in $\mathscr{C}_X^\omega$. However, for X *reduced and locally irreducible*, ϱ is injective. (It is elementary that the restriction of ϱ to $\mathscr{C}_X^\omega$ is injective; we deduce that ϱ is injective by Malgrange [19, Chap. VI, Theorem 3.10].)

We write $[\varphi] := \varrho(\varphi)$, $[\mathscr{C}_X^\infty] := \varrho(\mathscr{C}_X^\infty)$. So $[\mathscr{C}_X^\infty]$ is the $\mathscr{C}^\infty$ sheaf of the "old" theory. For normal X the two theories coincide, by the above remark.

(iv) The kernel of the canonical morphism $PH_X \to \mathscr{C}_X$ is $\mathscr{N}_X + \bar{\mathscr{N}}_X$ where $\mathscr{N}_X$ is the sheaf of nilpotent sections of $\mathcal{O}_X$. In particular, for reduced X, PH_X may be considered as a subsheaf of $\mathscr{C}_X$.

(v) If $f : X \to Y$ is a morphism of complex spaces, $\varphi \in \mathscr{C}(Y)$ and $\psi \in \mathscr{C}^\infty(Y)$, write $\varphi \circ f \in \mathscr{C}(X)$ and $f^*\psi \in \mathscr{C}^\infty(X)$ for the corresponding induced elements. Write $\psi \circ f$ instead of $[\psi] \circ f$, so that $[f^*\psi] = \psi \circ f$ in $\mathscr{C}(X)$.

(vi) The canonical morphisms $\Omega_X^m \to A_X^{m,o}$ are injective and will be considered as inclusions.

(vii) The inclusions $A_X^{k,l} \subset A_X^{k+l}$ give a direct sum decomposition $A_X^m = \bigoplus_{k+l=m} A_X^{k,l}$ and $A_X^{\cdot}$ is a bigraded algebra with respect to the wedge product. The natural involution $\varphi \mapsto \bar{\varphi}$ applies $A_X^{k,l}$ on to $A_X^{l,k}$.

(viii) There is a canonical morphism $d = \partial + \bar{\partial} : A_X^m \to A_X^{m+1}$ satisfying the usual identities $d^2 = \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. However, none of the resulting complexes (Dolbeault, De Rham, etc. ...) is an exact sequence of sheaves in general.

(ix) Any morphism $f: X \to Y$ of complex spaces gives rise to a linear $f^*: A^m(Y) \to A^m(X)$ which is compatible with the wedge product, bigraduation and the operators $d, \partial, \bar{\partial}$. We have $(fg)^* = g^* f^*$.

2. *Strongly Plurisubharmonic Functions*

We write p.s.h. for plurisubharmonic.

2.1. Definitions. We define on X the sheaves of real convex cones P_X^0 (resp. SP_X^0) of continuous p.s.h. (resp. strongly p.s.h.) functions, P_X^∞ (resp. SP_X^∞) of $\mathscr{C}^\infty$ p.s.h. (resp. strongly p.s.h.) functions as the subsheaves of $\mathscr{C}_{X,\mathbb{R}}$ (resp. $\mathscr{C}_{X,\mathbb{R}}^\infty$) consisting of elements induced by corresponding functions on open sets of $\mathbb{C}^n$ under local embeddings. Also define $[P_X^\infty] := \varrho(P_X^\infty)$, $[SP_X^\infty] := \varrho(SP_X^\infty)$ where $\varrho: \mathscr{C}_X^\infty \to \mathscr{C}_X$ is the canonical morphism.

2.2. Examples. (i) On the subspace X of $\mathbb{C}^2$ defined by $z_1 z_2 = z_2^2 = 0$, set $\varphi_t(z_1, z_2) := z_1 \bar{z}_1 + t z_2 \bar{z}_2$ for real t. Then $[\varphi_t] \in SP^0(X)$ is independent of t, $\varphi_t \in \mathscr{C}^\infty(X, \mathbb{R})$ for all t but $\varphi_t \in P^\infty(X)$ only for $t \geqq 0$ and $\varphi_t \in SP^\infty(X)$ only for $t > 0$.

(ii) On the subspace X of $\mathbb{C}^2$ defined by $z_1 z_2 = 0$, set (for real t) $\varphi_t(z_1, z_2) := z_1 \bar{z}_1 + t(z_1 \bar{z}_2 + z_2 \bar{z}_1) + z_2 \bar{z}_2$. Then $[\varphi_t] \in SP^0(X)$ is independent of t, $\varphi_t \in \mathscr{C}^\infty(X, \mathbb{R})$ but $\varphi_t \in P^\infty(X)$ only for $|t| \leqq 1$ and $\varphi_t \in SP^\infty(X)$ only for $|t| < 1$.

(iii) On $\mathbb{C}^n$ set $\varphi(z_1, \ldots, z_n) := \sum_{j=1}^n |t_j|^2$ where $t_1, \ldots, t_n$ are the roots of $X^n - z_1 X^{n-1} + \ldots + (-1)^n z_n$. Then $\varphi \in SP^0(\mathbb{C}^n)$.

2.3. The Cone $SP^{0,\infty}(U, V)$. This is an auxiliary notion introduced to give a meaning to smoothing lemmas of strongly p.s.h. functions. For U, V open in X, $SP^{0,\infty}(U, V)$ is defined as the set of pairs $\varphi = (\varphi^0, \varphi^\infty) \in SP^0(U) \times SP^\infty(U \cap V)$ such that $[\varphi^\infty] = \varphi^0{}_{|U \cap V}$. We set $[\varphi] := \varphi^0$. The following are obvious

(i) $SP^{0,\infty}(U, V) = SP^{0,\infty}(U, U \cap V)$.

(ii) $SP^{0,\infty}(U, \emptyset) \cong SP^0(U)$ canonically.

(iii) $SP^{0,\infty}(U, X) \cong SP^\infty(U)$ canonically.

(iv) For fixed V, $U \mapsto SP^{0,\infty}(U, V)$ is a sheaf on X.

(v) For $\varphi = (\varphi^0, \varphi^\infty) \in SP^{0,\infty}(U, V)$ and $h \in PH(U, \mathbb{R})$, the element $\varphi + h := (\varphi^0 + [h], \varphi^\infty + h_{|U \cap V})$ is in $SP^{0,\infty}(U, V)$.

The following is a slight improvement of a result of Richberg [21, Satz 4.1]. For $X = \mathbb{C}^n$, a complete proof is in [23].

2.4. Richberg's Lemma. *Let U, V, W be open in X with $U \subset\subset W$. Let $\varphi \in SP^{0,\infty}(W, V)$. Then there is a compact S such that $U \subset S \subset W$ and an ele-*