(vi) The canonical morphisms $\Omega_X^m \to A_X^{m,o}$ are injective and will be considered as inclusions.

(vii) The inclusions $A_X^{k,l} \subset A_X^{k+l}$ give a direct sum decomposition $A_X^m = \bigoplus_{k+l=m} A_X^{k,l}$ and $A_X^{\cdot}$ is a bigraded algebra with respect to the wedge product. The natural involution $\varphi \mapsto \bar{\varphi}$ applies $A_X^{k,l}$ on to $A_X^{l,k}$.

(viii) There is a canonical morphism $d = \partial + \bar{\partial}: A_X^m \to A_X^{m+1}$ satisfying the usual identities $d^2 = \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. However, none of the resulting complexes (Dolbeault, De Rham, etc. ...) is an exact sequence of sheaves in general.

(ix) Any morphism $f: X \to Y$ of complex spaces gives rise to a linear $f^*: A^m(Y) \to A^m(X)$ which is compatible with the wedge product, bigraduation and the operators $d, \partial, \bar{\partial}$. We have $(fg)^* = g^* f^*$.

2. *Strongly Plurisubharmonic Functions*

We write p.s.h. for plurisubharmonic.

2.1. Definitions. We define on X the sheaves of real convex cones P_X^0 (resp. SP_X^0) of continuous p.s.h. (resp. strongly p.s.h.) functions, P_X^∞ (resp. SP_X^∞) of $\mathscr{C}^\infty$ p.s.h. (resp. strongly p.s.h.) functions as the subsheaves of $\mathscr{C}_{X,\mathbb{R}}$ (resp. $\mathscr{C}_{X,\mathbb{R}}^\infty$) consisting of elements induced by corresponding functions on open sets of $\mathbb{C}^n$ under local embeddings. Also define $[P_X^\infty] := \varrho(P_X^\infty)$, $[SP_X^\infty] := \varrho(SP_X^\infty)$ where $\varrho: \mathscr{C}_X^\infty \to \mathscr{C}_X$ is the canonical morphism.

2.2. Examples. (i) On the subspace X of $\mathbb{C}^2$ defined by $z_1 z_2 = z_2^2 = 0$, set $\varphi_t(z_1, z_2) := z_1\bar{z}_1 + t z_2 \bar{z}_2$ for real t. Then $[\varphi_t] \in SP^0(X)$ is independent of t, $\varphi_t \in \mathscr{C}^\infty(X, \mathbb{R})$ for all t but $\varphi_t \in P^\infty(X)$ only for $t \geqq 0$ and $\varphi_t \in SP^\infty(X)$ only for $t > 0$.

(ii) On the subspace X of $\mathbb{C}^2$ defined by $z_1 z_2 = 0$, set (for real t) $\varphi_t(z_1, z_2) := z_1\bar{z}_1 + t(z_1\bar{z}_2 + z_2\bar{z}_1) + z_2\bar{z}_2$. Then $[\varphi_t] \in SP^0(X)$ is independent of t, $\varphi_t \in \mathscr{C}^\infty(X, \mathbb{R})$ but $\varphi_t \in P^\infty(X)$ only for $|t| \leqq 1$ and $\varphi_t \in SP^\infty(X)$ only for $|t| < 1$.

(iii) On $\mathbb{C}^n$ set $\varphi(z_1, \ldots, z_n) := \sum_{j=1}^n |t_j|^2$ where $t_1, \ldots, t_n$ are the roots of $X^n - z_1 X^{n-1} + \ldots + (-1)^n z_n$. Then $\varphi \in SP^0(\mathbb{C}^n)$.

2.3. The Cone $SP^{0,\infty}(U, V)$. This is an auxiliary notion introduced to give a meaning to smoothing lemmas of strongly p.s.h. functions. For U, V open in X, $SP^{0,\infty}(U, V)$ is defined as the set of pairs $\varphi = (\varphi^0, \varphi^\infty) \in SP^0(U) \times SP^\infty(U \cap V)$ such that $[\varphi^\infty] = \varphi^0|_{U \cap V}$. We set $[\varphi] := \varphi^0$. The following are obvious

(i) $SP^{0,\infty}(U, V) = SP^{0,\infty}(U, U \cap V)$.

(ii) $SP^{0,\infty}(U, \emptyset) \cong SP^0(U)$ canonically.

(iii) $SP^{0,\infty}(U, X) \cong SP^\infty(U)$ canonically.

(iv) For fixed V, $U \mapsto SP^{0,\infty}(U, V)$ is a sheaf on X.

(v) For $\varphi = (\varphi^0, \varphi^\infty) \in SP^{0,\infty}(U, V)$ and $h \in PH(U, \mathbb{R})$, the element $\varphi + h := (\varphi^0 + [h], \varphi^\infty + h_{|U \cap V})$ is in $SP^{0,\infty}(U, V)$.

The following is a slight improvement of a result of Richberg [21, Satz 4.1]. For $X = \mathbb{C}^n$, a complete proof is in [23].

2.4. Richberg's Lemma. *Let U, V, W be open in X with $U \subset\subset W$. Let $\varphi \in SP^{0,\infty}(W, V)$. Then there is a compact S such that $U \subset S \subset W$ and an ele-*

ment $\psi \in SP^{0,\infty}(W, U \cup V)$ *such that* $\varphi|_{W \setminus S} = \psi|_{W \setminus S}$ *in* $SP^{0,\infty}(W \setminus S, V) = SP^{0,\infty}(W \setminus S, U \cup V)$.

Sketch of Proof. Take a finite number of open sets $U_k \subset\subset V_k \subset\subset W_k$ $(1 \leqq k \leqq m)$ such that $U = \bigcup_{k=1}^{m} U_k$ and each W_k is embedded in an open subset D_k^0 of $\mathbb{C}^{n_k}$ such that $[\varphi]|_{W_k}$ is induced by an element of $SP^0(D_k)$. Using the method of [23] one can construct inductively elements $\varphi_k \in SP^{0,\infty}(W, U_1 \cup \ldots \cup U_k \cup V)$ such that $\varphi_k|_{W \setminus \bar{V}_k} = \varphi_{k-1}|_{W \setminus \bar{V}_k}$. Then set $S = \bar{V}_1 \cup \ldots \cup \bar{V}_m$ and $\psi = \varphi_m$.

2.5. The Fornaess-Narasimhan Theorem [10, Theorem 5.3.1]. *Let* $\varphi \in \mathscr{C}(X, \mathbb{R})$. *Suppose that for any holomorphic* $f: \Delta \to X$, *where* Δ *is the unit disc of* $\mathbb{C}$, $\varphi \circ f$ *is subharmonic on* Δ. *Then* $\varphi \in P^0(X)$.

2.6. The Cone $SP_\pi^\infty(X)$. Let $\pi: X \to Y$ be a morphism of complex spaces. Let $\varphi \in \mathscr{C}^\infty(X, \mathbb{R})$. We say that φ is *strongly p.s.h. relatively to* π and write $\varphi \in SP_\pi^\infty(X)$ if for any $x \in X$ there are open subsets $U \subset X$, $V \subset Y$ and $\psi \in SP^\infty(V)$ such that $x \in U \subset \pi^{-1}(V)$ and $(\varphi + \pi^* \psi)|_U \in SP^\infty(U)$.

3. Barlet's Space of Analytic Cycles

3.1. Symmetric Powers of Complex Spaces. If $k \geqq 1$ is an integer, let $\mathrm{Sym}^k(X) := X^k / \mathscr{S}_k$ be the quotient of X^k under the action of the symmetric group permuting components. Denote by $\sum_{j=1}^{k} \{x_j\}$ the image of $(x_1, \ldots, x_k)$ in $\mathrm{Sym}^k(X)$ under the canonical projection.

3.2. Analytic Families of Complex Cycles. $\mathbf{B}_m(X)$. Let X be reduced and $m \geqq 0$ an integer. A *compact complex-analytic m-cycle* (or briefly *m-cycle*) of X is a formal finite sum

$$c = \sum_{i \in I} n_i Y_i,$$

where $n_1 \geqq 1$ are integers and Y_i are compact irreducible m-dimensional complex-analytic subsets of X. $|c| := \bigcup_{i \in I} Y_i$ is called the *support* of c.

Let c be as above and $\sigma: V \to U \times B$ an embedding of an open set $V \subset X$ into a connected open set $U \times B$ of $\mathbb{C}^N = \mathbb{C}^m \times \mathbb{C}^{N-m}$. We say that $\mathscr{V} = (\sigma, V, U \times B)$ is a *well-adapted chart with respect to c* if σ extends to an embedding $\sigma_1: V_1 \to U_1 \times B_1$ such that $V \subset\subset V_1 \subset X$, $U \subset\subset U_1 \subset \mathbb{C}^m$, $B \subset\subset B_1 \subset \mathbb{C}^{N-m}$ and $\sigma_1(|c|) \cap (\bar{U} \times \partial B) = \emptyset$.

If we set $Z_i := \sigma(V \subset Y_i) \subset U \times B$, then the projection $U \times B \to U$ restricted to each Z_i is a branched covering $\pi_i: Z_i \to U$ of finite degree k_i and defines as such a morphism $\psi_i: U \to \mathrm{Sym}^{k_i}(B)$. Set $k := \sum n_i k_i$, $\psi := \sum n_i \psi_i: U \to \mathrm{Sym}^k(B)$, $\deg(c, \mathscr{V}) := k$.

Now let S be a reduced complex space and $(c_s)_{s \in S}$ a family of m-cycles of X parametrized by S. We say that (c_s) is an *analytic family of cycles* if for any $s_0 \in S$ and for *any* well-adapted chart $\mathscr{V}$ with respect to c_{s_0}, there is a neighborhood T of s_0 in S such that