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ment $\psi \in SP^{0,\infty}(W, U \cup V)$ *such that* $\varphi|_{W \setminus S} = \psi|_{W \setminus S}$ *in* $SP^{0,\infty}(W \setminus S, V) = SP^{0,\infty}(W \setminus S, U \cup V)$.

Sketch of Proof. Take a finite number of open sets $U_k \subset\subset V_k \subset\subset W_k$ $(1 \leqq k \leqq m)$ such that $U = \bigcup_{k=1}^{m} U_k$ and each W_k is embedded in an open subset D_k^0 of $\mathbb{C}^{n_k}$ such that $[\varphi]|_{W_k}$ is induced by an element of $SP^0(D_k)$. Using the method of [23] one can construct inductively elements $\varphi_k \in SP^{0,\infty}(W, U_1 \cup \ldots \cup U_k \cup V)$ such that $\varphi_k|_{W \setminus \bar{V}_k} = \varphi_{k-1}|_{W \setminus \bar{V}_k}$. Then set $S = \bar{V}_1 \cup \ldots \cup \bar{V}_m$ and $\psi = \varphi_m$.

2.5. The Fornaess-Narasimhan Theorem [10, Theorem 5.3.1]. *Let* $\varphi \in \mathscr{C}(X, \mathbb{R})$. *Suppose that for any holomorphic* $f: \Delta \to X$, *where* Δ *is the unit disc of* $\mathbb{C}$, $\varphi \circ f$ *is subharmonic on* Δ. *Then* $\varphi \in P^0(X)$.

2.6. The Cone $SP_\pi^\infty(X)$. Let $\pi: X \to Y$ be a morphism of complex spaces. Let $\varphi \in \mathscr{C}^\infty(X, \mathbb{R})$. We say that φ is *strongly p.s.h. relatively to* π and write $\varphi \in SP_\pi^\infty(X)$ if for any $x \in X$ there are open subsets $U \subset X$, $V \subset Y$ and $\psi \in SP^\infty(V)$ such that $x \in U \subset \pi^{-1}(V)$ and $(\varphi + \pi^* \psi)|_U \in SP^\infty(U)$.

3. Barlet's Space of Analytic Cycles

3.1. Symmetric Powers of Complex Spaces. If $k \geqq 1$ is an integer, let $\mathrm{Sym}^k(X) := X^k / \mathscr{S}_k$ be the quotient of X^k under the action of the symmetric group permuting components. Denote by $\sum_{j=1}^{k} \{x_j\}$ the image of $(x_1, \ldots, x_k)$ in $\mathrm{Sym}^k(X)$ under the canonical projection.

3.2. Analytic Families of Complex Cycles. $\mathbf{B}_m(X)$. Let X be reduced and $m \geqq 0$ an integer. A *compact complex-analytic m-cycle* (or briefly *m-cycle*) of X is a formal finite sum

$$c = \sum_{i \in I} n_i Y_i,$$

where $n_1 \geqq 1$ are integers and Y_i are compact irreducible m-dimensional complex-analytic subsets of X. $|c| := \bigcup_{i \in I} Y_i$ is called the *support* of c.

Let c be as above and $\sigma: V \to U \times B$ an embedding of an open set $V \subset X$ into a connected open set $U \times B$ of $\mathbb{C}^N = \mathbb{C}^m \times \mathbb{C}^{N-m}$. We say that $\mathscr{V} = (\sigma, V, U \times B)$ is a *well-adapted chart with respect to c* if σ extends to an embedding $\sigma_1: V_1 \to U_1 \times B_1$ such that $V \subset\subset V_1 \subset X$, $U \subset\subset U_1 \subset \mathbb{C}^m$, $B \subset\subset B_1 \subset \mathbb{C}^{N-m}$ and $\sigma_1(|c|) \cap (\bar{U} \times \partial B) = \emptyset$.

If we set $Z_i := \sigma(V \subset Y_i) \subset U \times B$, then the projection $U \times B \to U$ restricted to each Z_i is a branched covering $\pi_i: Z_i \to U$ of finite degree k_i and defines as such a morphism $\psi_i: U \to \mathrm{Sym}^{k_i}(B)$. Set $k := \sum n_i k_i$, $\psi := \sum n_i \psi_i: U \to \mathrm{Sym}^k(B)$, $\deg(c, \mathscr{V}) := k$.

Now let S be a reduced complex space and $(c_s)_{s \in S}$ a family of m-cycles of X parametrized by S. We say that (c_s) is an *analytic family of cycles* if for any $s_0 \in S$ and for *any* well-adapted chart $\mathscr{V}$ with respect to c_{s_0}, there is a neighborhood T of s_0 in S such that

(i) $\mathscr{V}$ is well-adapted with respect to c_s for all $s \in T$.
(ii) $\deg(c_s, \mathscr{V}) = k$ is independent of $s \in T$.
(iii) The resulting map $\psi: U \times T \to \mathrm{Sym}^k(B)$ is holomorphic.

The Barlet space $\mathbf{B}_m(X)$ of m-cycles of X is a reduced complex space, constructed in [3], whose points are the m-cycles of X forming a tautological analytic family and such that for any analytic family $(c_s)_{s\in S}$ of m-cycles of X, there is a unique morphism of complex spaces $H: S \to \mathbf{B}_m(X)$ such that

$$H(s) = c_s \quad \text{for all} \quad s \in S.$$

For X not necessarily reduced, we set

$$\mathbf{B}_m(X) := \mathbf{B}_m(X_{\mathrm{red}}).$$

3.3. Proper Open Morphism. Geometric Flatness. Let $\mathbf{D}_m(X)$ be the Douady space [9] of compact subspaces of pure dimension m of X. In [3, Chap. 5], Barlet constructed a canonical morphism

$$c: (\mathbf{D}_m(X))_{\mathrm{red}} \to \mathbf{B}_m(X).$$

If Y is a point of $\mathbf{D}_m(X)$ (a subspace of X) then $c(Y) = \sum n_i Y_i$ where Y_i are the irreducible components of Y_{red} and $n_i \geqq 1$ integers called multiplicities. If Y is generically reduced, all n_i are equal to 1.

Now suppose that $\pi: X \to X'$ is a morphism of complex spaces such that, for some fixed $m \geqq 0$

$$\begin{array}{ll} & \text{(i) } \pi \text{ is proper open and surjective,} \\ (3.3) & \text{(ii) all fibers of } \pi \text{ are of pure dimension } m, \\ & \text{(iii) } X' \text{ is reduced.} \end{array}$$

[If X, X' are pure dimensional, then (i) implies (ii).]

We will say that π is *geometrically flat* if there is a morphism of complex spaces

$$H: X' \to \mathbf{B}_m(X)$$

such that $H(x') = c(\pi^{-1}(x'))$ generically on X'. We call H the *classifying morphism of* π. The domain of validity of the equality $H(x') = c(\pi^{-1}(x'))$ is the dense Zariski open set U' of points of flatness of π (Frisch [11]).

3.3.1. Proposition. *Suppose* $\pi: X \to X'$ *satisfies* (3.3). *Then:*
(i) *If* π *is flat, then it is geometrically flat.*
(ii) *If* X' *is normal, then* π *is geometrically flat.*
(iii) *If* π *is geometrically flat, then* H *defines an isomorphism of* X' *onto a subspace of* $\mathbf{B}_m(X)$.

Proof. (i) If π is flat, then there is a morphism $X' \to \mathbf{D}_m(X)$, factoring through $(\mathbf{D}_m(X))_{\mathrm{red}}$ since X' is reduced, taking the value $\pi^{-1}(x')$ at x'. Composing with $c: (\mathbf{D}_m(X))_{\mathrm{red}} \to \mathbf{B}_m(X)$, we obtain the required H.
(ii) This is part of Theorem 1 of [3].
(iii) This is shown in [24, Appendix, p. 259].

3.3.2. Examples. (i) Let X be the union of two planes defined by $z_1z_2 = z_1z_4 = z_2z_3 = z_3z_4 = 0$ in $\mathbb{C}^4$, Y' the union of two lines defined by $x_1x_2 = 0$ in $\mathbb{C}^2$, $\pi: X \to X' = \mathbb{C}^2$ and $\varrho: Y' \to X'$ defined by $\pi(z_1, z_2, z_3, z_4) = (z_1 + z_2, z_3 + z_4)$ and $\varrho(x_1, x_2) = (x_1, 0)$

$$\begin{array}{ccc} X & \longleftarrow & Y = X \times_{X'} Y' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ X' & \xleftarrow{\varrho} & Y' \end{array}$$

Then π is geometrically flat be 3.3.1 (ii) but π_1 is not since Y consists of one triple line over one branch of Y' and two single lines over the other. π is not flat.

(ii) Let X be the union of two single lines and one double line defined by $z_1z_2 = z_2^2 - z_3^2 = 0$ in $\mathbb{C}^2$ and X' the union of two lines $z_1z_2 = 0$ (as Y' above).

If $\pi(z_1, z_2, z_3) = (z_1, z_2)$, then $\pi: X \to X'$ is flat, X' is reduced but if $r: X_{\mathrm{red}} \to X$ is the reduction of X then $\pi r: X_{\mathrm{red}} \to X'$ is not geometrically flat.

3.4. Integration of Differential Forms. If $\varphi \in A^{m,m}(X)$ and $c = \sum n_i Y_i \in \mathbf{B}_m(X)$, define

$$F_\varphi(c) := \int_c \varphi = \sum n_i \int_{Y_i} \varphi .$$

If $\pi: X \to X'$ is geometrically flat with m-dimensional fibers and φ is a above, define

$$\pi_* \varphi := F_\varphi \circ H_\pi .$$

We have the following:

3.4.1. Proposition [4, 5, 23]. *With the above notations.*

(i) F_φ *(resp.* $\pi_*\varphi$*) is continuous on* $\mathbf{B}_m(X)$ *(resp.* X'*).*

(ii) *If* $d\varphi = 0$, *then* F_φ *and* $\pi_*\varphi$ *are locally constant.*

(iii) *If* $\varphi = \bar\varphi$ *and* $i\partial\bar\partial\varphi \geqq 0$ *then* F_φ *and* $\pi_*\varphi$ *are p.s.h.*

(iv) *If* $\varphi = \bar\varphi$ *and* $i\partial\bar\partial\varphi \gg 0$ *then* F_φ *and* $\pi_*\varphi$ *are strongly p.s.h.*

(v) *If* $\bar\partial\varphi = 0$ *then* F_φ *and* $\pi_*\varphi$ *are weakly holomorphic; if moreover* X *is smooth, they are holomorphic.*

3.4.2. Remark. Case (iii) above needs the Fornaess-Narasimhan theorem if we look at the proof of Proposition 1 of [5].

3.4.3. Definition. A $\bar\partial$-closed $\tau \in A^{m,m}(X)$ is said to *represent* an element $\xi \in H^m(X, \Omega_X^m)$ (or to be a *$\bar\partial$-closed representative of* ξ) if the class of τ in $H_{\bar\partial}^{m,m}(X)$ is the image of ξ under the canonical morphism $H^m(X, \Omega_X^m) \to H_{\bar\partial}^{m,m}(X)$. In that case we define $F_\xi(c) := F_\tau(c)$ for $c \in \mathbf{B}_m(X)$ and also write $(c \cdot \xi)$ for $F_\xi(c)$ (since it depends on ξ alone).

3.5. m-Complete and m-Admissible Neighborhoods. By the Andreotti-Grauert theorem [1], if X is a m-complete complex space, then for any coherent analytic sheaf $\mathscr{F}$ on X and any $q > m$ we have $H^q(X, \mathscr{F}) = 0$. We will use

3.5.1. Proposition. *Let Y be a compact m-dimensional complex-analytic subset of X. Then*

(i) *Y admits in X a fundamental system of m-complete neighborhoods (Barlet [6]).*

(ii) *Y admits in X a fundamental system of neighborhoods V such that $H^k(V, \mathbb{R})=0$ for $k>2m$* [23, *Lemma 3.5*].

3.5.2. Definition. An open $U \subset X$ is said to be *m-admissible* if
(i) U is m-complete.
(ii) There is an open V such that $U \subset V \subset X$ and $H^k(V, \mathbb{R})=0$ for all $k>2m$.

3.5.3. Remark. If X is a Kähler manifold with a Kähler form ω and $U \subset X$ is 0-admissible, then one easily sees that $\omega|_U = i\partial\bar{\partial}\varphi$ for some $\varphi \in SP^{\infty}(U)$. This is the most trivial particular case of our Theorem 2.

3.5.3. Proposition. (i) *If $U \subset X$ is m-admissible and $k>2m$, then the canonical morphism $H^k(X, \mathbb{R}) \to H^k(U, \mathbb{R})$ is zero.*
(ii) *Any compact m-dimensional complex-analytic subset of X admits a fundamental system of m-admissible neighborhoods.*

Proof. (i) Is obvious by the definitions and (ii) is a restatement of 3.5.1.

3.5.4. Proposition. *Let $\mathbf{B}_m(X)^{(o)}$ be the open set of $\mathbf{B}_m(X)$ consisting of cycles whose support admits in X a smoothly embeddable neighborhood. Let $\xi \in H^m(X, \Omega_X^m)$. Then F_ξ is holomorphic on $\mathbf{B}_m(X)^{(o)}$.*

Sketch of Proof. For $c \in \mathbf{B}_m(X)^{(o)}$, $|c|$ admits a smoothly embeddable neighborhood V therefore by 3.5.1 a neighborhood U with an embedding $\sigma: U \to U_1$ in a smooth m-complete U_1.

If $\mathcal{N}$ is the coherent sheaf on U_1 defined by the exact sequence

$$0 \to \mathcal{N} \to \Omega_{U_1}^m \to \sigma_* \Omega_U^m \to 0,$$

then $H^{m+1}(U_1, \mathcal{N})=0$ and hence $\xi|_U$ is induced by some $\xi_1 \in H^m(U_1, \Omega_{U_1}^m)$. By 3.4.1(v), F_{ξ_1} is holomorphic on $\mathbf{B}_m(U_1)$ so F_ξ is holomorphic near c.

3.5.5. Corollary. *If $\pi: X \to X'$ is geometrically flat with m-dimensional fibers and U' is the set of $x' \in X'$ such that $\pi^{-1}(x')$ admits in X smoothly embeddable neighborhoods then for any $\xi \in H^m(X, \Omega_X^m)$, $\pi_* \xi|_{U'}$ is holomorphic.*

3.6. Note Added in Proof. After having submitted the manuscript, the author together with D. Barlet solved problem D of the Introduction. Proposition 3.5.4 and Corollary 3.5.5 above are now true with $\mathbf{B}_m(X)$ instead of $\mathbf{B}_m(X)^{(0)}$. The notion of a weakly Kähler space loses its importance and Theorems 3 and 4 below (Ch. IV) become

Theorem 3′. *If $\pi: X \to X'$ is geometrically flat with X Kähler and X' reduced, then X' is Kähler.*

Theorem 4′. *If X is Kähler then $\mathbf{B}_m(X)$ is Kähler.*

II. Theorem 1 and its First Consequences

1. Kähler Spaces and Kähler Metrics

Let X be a complex space.