(ii) *Y admits in X a fundamental system of neighborhoods V such that $H^k(V, \mathbb{R}) = 0$ for $k > 2m$* [23, *Lemma 3.5*].

3.5.2. Definition. An open $U \subset X$ is said to be *m-admissible* if
(i) U is m-complete.
(ii) There is an open V such that $U \subset V \subset X$ and $H^k(V, \mathbb{R}) = 0$ for all $k > 2m$.

3.5.3. Remark. If X is a Kähler manifold with a Kähler form ω and $U \subset X$ is 0-admissible, then one easily sees that $\omega|_U = i\partial\bar{\partial}\varphi$ for some $\varphi \in SP^\infty(U)$. This is the most trivial particular case of our Theorem 2.

3.5.3. Proposition. (i) *If $U \subset X$ is m-admissible and $k > 2m$, then the canonical morphism $H^k(X, \mathbb{R}) \to H^k(U, \mathbb{R})$ is zero.*
(ii) *Any compact m-dimensional complex-analytic subset of X admits a fundamental system of m-admissible neighborhoods.*

Proof. (i) Is obvious by the definitions and (ii) is a restatement of 3.5.1.

3.5.4. Proposition. *Let $\mathbf{B}_m(X)^{(o)}$ be the open set of $\mathbf{B}_m(X)$ consisting of cycles whose support admits in X a smoothly embeddable neighborhood. Let $\xi \in H^m(X, \Omega_X^m)$. Then F_ξ is holomorphic on $\mathbf{B}_m(X)^{(o)}$.*

Sketch of Proof. For $c \in \mathbf{B}_m(X)^{(o)}$, $|c|$ admits a smoothly embeddable neighborhood V therefore by 3.5.1 a neighborhood U with an embedding $\sigma: U \to U_1$ in a smooth m-complete U_1.

If $\mathcal{N}$ is the coherent sheaf on U_1 defined by the exact sequence

$$0 \to \mathcal{N} \to \Omega_{U_1}^m \to \sigma_* \Omega_U^m \to 0,$$

then $H^{m+1}(U_1, \mathcal{N}) = 0$ and hence $\xi|_U$ is induced by some $\xi_1 \in H^m(U_1, \Omega_{U_1}^m)$. By 3.4.1(v), F_{ξ_1} is holomorphic on $\mathbf{B}_m(U_1)$ so F_ξ is holomorphic near c.

3.5.5. Corollary. *If $\pi: X \to X'$ is geometrically flat with m-dimensional fibers and U' is the set of $x' \in X'$ such that $\pi^{-1}(x')$ admits in X smoothly embeddable neighborhoods then for any $\xi \in H^m(X, \Omega_X^m)$, $\pi_* \xi|_{U'}$ is holomorphic.*

3.6. Note Added in Proof. After having submitted the manuscript, the author together with D. Barlet solved problem D of the Introduction. Proposition 3.5.4 and Corollary 3.5.5 above are now true with $\mathbf{B}_m(X)$ instead of $\mathbf{B}_m(X)^{(0)}$. The notion of a weakly Kähler space loses its importance and Theorems 3 and 4 below (Ch. IV) become

Theorem 3'. *If $\pi: X \to X'$ is geometrically flat with X Kähler and X' reduced, then X' is Kähler.*

Theorem 4'. *If X is Kähler then $\mathbf{B}_m(X)$ is Kähler.*

II. Theorem 1 and its First Consequences

1. Kähler Spaces and Kähler Metrics

Let X be a complex space.

1.1. The Sheaf $\mathscr{K}_X^1$. Define

$$\mathscr{K}_X^1 := \mathscr{C}_X^\infty / PH_X, \qquad \mathscr{K}_{X,\mathbb{R}}^1 := \mathscr{C}_{X,\mathbb{R}}^\infty / PH_{X,\mathbb{R}},$$
$$\mathscr{K}^1(X) := H^0(X, \mathscr{K}_X^1), \qquad \mathscr{K}^1(X,\mathbb{R}) := H^0(X, \mathscr{K}_{X,\mathbb{R}}^1).$$

A section $\kappa \in \mathscr{K}^1(X)$ corresponds by definition to an open covering (U_α) of X together with elements $\varphi_\alpha \in \mathscr{C}^\infty(U_\alpha)$ such that $\varphi_\alpha - \varphi_\beta \in PH(U_\alpha \cap U_\beta)$. We write $\kappa = \{(U_\alpha, \varphi_\alpha)\}$. We have

$$\{(U_\alpha, \varphi_\alpha)\} = \{(V_j, \psi_j)\} \quad \text{iff} \quad (\varphi_\alpha - \psi_j)|_{U_\alpha \cap V_j} \in PH(U_\alpha \cap V_j).$$

For such κ, we set $\partial\bar{\partial}\kappa := \omega \in A^{1,1}(X)$ where

$$\omega|_{U_\alpha} = \partial\bar{\partial}\varphi_\alpha .$$

Of course, ω is well-defined and $d\omega = 0$. We say that κ is *represented* by the φ_α.

1.2. Kähler Metrics, Kähler Classes. A *Kähler metric* on X is by definition an element $\kappa \in \mathscr{K}^1(X,\mathbb{R})$ represented by a system of sections of SP_X^∞. The *Kähler form* of (X,κ) is $\omega := i\partial\bar{\partial}\kappa$ $(i = \sqrt{-1})$. We will often write (X,ω) instead of (X,κ), although ω does not determine κ unless X is smooth.

Similarly, if $\pi: X \to Y$ is a morphism of complex spaces, a *relative Kähler metric for* π is an element κ_π of $\mathscr{K}^1(X,\mathbb{R})$ represented by sections of SP_π^∞.

To any element $\kappa \in \mathscr{K}^1(X)$ we associate three cohomology classes as follows:

From the exact sequence $0 \to PH_X \to \mathscr{C}_X^\infty \to \mathscr{K}_X^1 \to 0$, we deduce a canonical morphism

$$\hat{c}_1 : \mathscr{K}^1(X) \to H^1(X, PH_X) \tag{1.2.1}$$

which obviously sends $\mathscr{K}^1(X,\mathbb{R})$ into $H^1(X, PH_{X,\mathbb{R}})$. From the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r, "-2\mathrm{Im}"] \arrow[d] & PH_{X,\mathbb{R}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{O}_X \oplus \bar{\mathcal{O}}_X \arrow[r, "(i\ \ -i)"] \arrow[dr, "(d\ \ o)"'] & PH_X \arrow[r] \arrow[d, "-i\partial"] & 0 \\
 & & & d\mathcal{O}_X &
\end{tikzcd} \tag{1.2.2}$$

we deduce canonical morphisms $H^1(X, PH_X) \to H^2(X, \mathbb{C})$ and $H^1(X, PH_X) \to H^1(X, d\mathcal{O}_X)$ and, composing with $\hat{c}_1$, we obtain

$$\begin{aligned} c_1 &: \mathscr{K}^1(X) \to H^2(X, \mathbb{C}), \\ \tilde{c}_1 &: \mathscr{K}^1(X) \to H^1(X, d\mathcal{O}_X). \end{aligned} \tag{1.2.3}$$

Of course c_1 sends $\mathscr{K}^1(X,\mathbb{R})$ into $H^2(X,\mathbb{R})$. $d\mathcal{O}_X$ is the subsheaf of Ω_X^1 consisting of locally exact holomorphic 1-forms. Sometimes we will replace $\tilde{c}_1(\kappa)$ by its image in $H^1(X, \Omega_X^1)$.

So we have a diagram

$$
(1.2.4)\qquad
\begin{tikzcd}
H^{1,1}_{\bar\partial}(X) & & H^1(X,\Omega^1_X) \arrow[ll] & & H^1(X,d\mathcal{O}_X) \arrow[ll] \\
Z^{1,1}_d(X) \arrow[u] \arrow[dr] & & \mathscr{K}^1(X) \arrow[ll, "i\partial\bar\partial"'] \arrow[rr, "\hat{c}_1"] \arrow[urr, "\tilde{c}_1"] \arrow[dr, "c_1"] & & H_1(X,PH_X) \arrow[u] \arrow[dl] \\
& H^2_d(X) & & H^2(X,\mathbb{C}) \arrow[ll] &
\end{tikzcd}
$$

which is commutative (see 4.2 of Chap. III). This means that if κ is a Kähler metric on X and $\omega = i\partial\bar\partial\kappa$ the corresponding Kähler form, then ω is a d-closed represenattive of $c_1(\kappa)$ in $H^2(X,\mathbb{R})$ and also a $\bar\partial$-closed representative of $\tilde{c}_1(\kappa)$ in $H^1(X,\Omega^1_X)$.

In [15] Grauert proved that if κ is a Kähler metric on a normal compact space X such that $c_1(\kappa)$ lies in the canonical image of $H^2(X,\mathbb{Q})$ in $H^2(X,\mathbb{R})$, then X is a projective variety.

1.3. Kähler Spaces, Kähler Morphisms. X is said to be a *Kähler space* if there exists a Kähler metric on X.

A morphism $\pi: X \to Y$ is a *Kähler morphism* if there exists a relative Kähler metric κ_π for π.

We have the following elementary properties:

1.3.1. Proposition. (i) *Subspaces of Kähler spaces are Kähler.*

(ii) *Smooth Kähler spaces are Kähler manifolds in the usual sense.*

(iii) *$X \to \{y\}$ is a Kähler morphism iff X is a Kähler space.*

(iv) *Kähler morphisms are preserved by composition and base-change* [8].

(v) *Projective morphisms (for example: finite morphisms and blow-ups) are Kähler* [8, 12].

(vi) *If $\pi: X \to Y$ is a Kähler morphism, and Y a Kähler space then any open $U \subset\subset X$ is Kähler. More precisely: If κ_Y is a Kähler metric on Y and κ_π a relative Kähler metric for π, then for any $U \subset\subset X$ there is a constant $c_0 > 0$ such that for any $c > c_0$, $(\kappa_\pi + c\pi^*\kappa_Y)_{|U}$ is a Kähler metric on U* [8, 12].

On the other hand,

1.3.2. Proposition. (i) *It is not always true that a reduced compact space is Kähler if its normalization is Kähler.*

(ii) *It is not always true that a compact space X is Kähler if X_{red} is Kähler. A counterexample* [8, *II*] *is given by an infinitesimal neighborhood of a K3 surface in its space of moduli.*

(iii) *It is not always true that a normal compact space is Kähler if the complement of a point is Kähler* [15, 20].

(iv) *It is not always true that small deformations of compact Kähler spaces are Kähler* [20].

(v) *It is not always true that a normal compact space that is both Moĭezon and Kähler is projective* [20].

2. *Theorem 1*

2.1. Statement. Let X be a complex space. Suppose it admits an open covering $(U_\alpha)_{\alpha \in A}$ and a system of *continuous* strongly p.s.h. functions $\varphi_\alpha \in SP^0(U_\alpha)$ together with pluriharmonic functions $h_{\alpha\beta} \in PH(U_\alpha \cap U_\beta, \mathbb{R})$ such that

$$(2.1.1) \qquad \begin{aligned} &\text{(i) } \varphi_\alpha - \varphi_\beta = [h_{\alpha\beta}] \quad \text{in} \quad \mathscr{C}(U_\alpha \cap U_\beta, \mathbb{R}), \\ &\text{(ii) } h_{\alpha\beta} - h_{\alpha\gamma} + h_{\beta\gamma} = 0 \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma . \end{aligned}$$

Then there are elements $\psi_\alpha \in SP^\infty(U_\alpha)$ such that

$$(2.1.2) \qquad \psi_\alpha - \psi_\beta = h_{\alpha\beta} \quad \text{in} \mathscr{C}^\infty(U_\alpha \cap U_\beta, \mathbb{R}).$$

In particular, X is a Kähler space.

2.2. Remark. By Lemma 1.2(iv) of Chap. I, the cocycle condition (ii) is redundant for X reduced. For smooth X, Theorem 1 is proven in [23] and the proof we give there is valid for X reduced and locally irreducible. We will use the conventions stated in 2.4 of Chap. I.

2.3. Proof. Since X is paracompact, it admits two locally finite open coverings (V_k), (W_k) $(k \in \mathbb{N})$ such that $V_0 = \emptyset$ and $V_k \subset\subset W_k \subset U_{\alpha_k}$ for each k. Set $T^k_{\alpha\beta} := U_\alpha \cap U_\beta \cap (V_1 \cup \ldots \cup V_k)$.

We will define inductively elements

$$\varphi^k_\alpha \in SP^{0,\infty}(U_\alpha, V_1 \cup \ldots \cup V_k)$$

such that

(i) For some compact S_k, $V_k \subset S_k \subset W_k$,

$$\varphi^k_\alpha|_{U_\alpha \setminus S_k} = \varphi^{k-1}_\alpha|_{U_\alpha \setminus S_k}$$

in $SP^{0,\infty}(U_\alpha \setminus S_k, V_1 \cup \ldots \cup V_k) = SP^{0,\infty}(U_\alpha \setminus S_k, V_1 \cup \ldots \cup V_{k-1})$

$$(2.3.1) \qquad \begin{aligned} &\text{(ii) } [\varphi^k_\alpha] - [\varphi^k_\beta] = [h_{\alpha\beta}] \quad \text{in} \quad \mathscr{C}(U_\alpha \cap U_\beta, \mathbb{R}), \\ &\text{(iii) } (\varphi^k_\alpha - \varphi^k_\beta)|_{T^k_{\alpha\beta}} = h_{\alpha\beta}|_{T^k_{\alpha\beta}} \quad \text{in} \quad \mathscr{C}^\infty(T^k_{\alpha\beta}, \mathbb{R}). \end{aligned}$$

We start by taking $\varphi^0_\alpha := \varphi_\alpha$ the initial data.
Suppose φ^{k-1}_α is defined for all α.
Apply Richberg's lemma to $X = W_k$,

$$U = V_k, \qquad V = V_1 \cup \ldots \cup V_{k-1}, \qquad \varphi = \varphi^{k-1}_{\alpha_k}|_{W_k}.$$

We obtain an element

$\psi \in SP^{0,\infty}(W_k, V_1 \cup \ldots \cup V_k)$

and a compact S_k, $V_k \subset S_k \subset W_k$ such that

$$\psi|_{W_k \setminus S_k} = \varphi^{k-1}_{\alpha_k}|_{W_k \setminus S_k}.$$

Now we set

$$(2.3.2) \qquad \varphi^k_\alpha := \begin{cases} \varphi^{k-1}_\alpha & \text{on} \quad U_\alpha \setminus S_k \\ \psi + h_{\alpha\alpha_k} & \text{on} \quad U_\alpha \cap W_k , \end{cases}$$

where the last expression is defined in 2.4(v) of Chap. I.