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2. Theorem 1

2.1. Statement. Let X be a complex space. Suppose it admits an open covering $(U_\alpha)_{\alpha\in A}$ and a system of *continuous* strongly p.s.h. functions $\varphi_\alpha \in SP^0(U_\alpha)$ together with pluriharmonic functions $h_{\alpha\beta}\in PH(U_\alpha \cap U_\beta, \mathbb{R})$ such that

$$(2.1.1)\qquad \begin{aligned} &\text{(i)}\ \varphi_\alpha - \varphi_\beta = [h_{\alpha\beta}] \quad \text{in} \quad \mathscr{C}(U_\alpha\cap U_\beta, \mathbb{R}),\\ &\text{(ii)}\ h_{\alpha\beta} - h_{\alpha\gamma} + h_{\beta\gamma} = 0 \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma . \end{aligned}$$

Then there are elements $\psi_\alpha \in SP^\infty(U_\alpha)$ such that

$$(2.1.2)\qquad \psi_\alpha - \psi_\beta = h_{\alpha\beta} \quad \text{in} \mathscr{C}^\infty(U_\alpha\cap U_\beta, \mathbb{R}).$$

In particular, X is a Kähler space.

2.2. Remark. By Lemma 1.2(iv) of Chap. I, the cocycle condition (ii) is redundant for X reduced. For smooth X, Theorem 1 is proven in [23] and the proof we give there is valid for X reduced and locally irreducible. We will use the conventions stated in 2.4 of Chap. I.

2.3. Proof. Since X is paracompact, it admits two locally finite open coverings (V_k), (W_k) $(k\in\mathbb{N})$ such that $V_0=\emptyset$ and $V_k \subset\subset W_k \subset U_{\alpha_k}$ for each k. Set $T^k_{\alpha\beta} := U_\alpha \cap U_\beta \cap (V_1\cup\ldots\cup V_k)$.

We will define inductively elements

$$\varphi^k_\alpha \in SP^{0,\infty}(U_\alpha, V_1\cup\ldots\cup V_k)$$

such that

(i) For some compact S_k, $V_k\subset S_k \subset W_k$,

$$\varphi^k_\alpha|_{U_\alpha\setminus S_k} = \varphi^{k-1}_\alpha|_{U_\alpha\setminus S_k}$$

in $SP^{0,\infty}(U_\alpha\setminus S_k, V_1\cup\ldots\cup V_k) = SP^{0,\infty}(U_\alpha\setminus S_k, V_1\cup\ldots\cup V_{k-1})$

$$(2.3.1)\qquad \begin{aligned} &\text{(ii)}\ [\varphi^k_\alpha] - [\varphi^k_\beta] = [h_{\alpha\beta}] \quad \text{in} \quad \mathscr{C}(U_\alpha\cap U_\beta, \mathbb{R}),\\ &\text{(iii)}\ (\varphi^k_\alpha - \varphi^k_\beta)|_{T^k_{\alpha\beta}} = h_{\alpha\beta}|_{T^k_{\alpha\beta}} \quad \text{in} \quad \mathscr{C}^\infty(T^k_{\alpha\beta}, \mathbb{R}). \end{aligned}$$

We start by taking $\varphi^0_\alpha := \varphi_\alpha$ the initial data.

Suppose φ^{k-1}_α is defined for all α.

Apply Richberg's lemma to $X = W_k$,

$$U = V_k, \qquad V = V_1\cup\ldots\cup V_{k-1}, \qquad \varphi = \varphi^{k-1}_{\alpha_k}|_{W_k}.$$

We obtain an element

$\psi \in SP^{0,\infty}(W_k, V_1\cup\ldots\cup V_k)$

and a compact S_k, $V_k\subset S_k\subset W_k$ such that

$$\psi|_{W_k\setminus S_k} = \varphi^{k-1}_{\alpha_k}|_{W_k\setminus S_k}.$$

Now we set

$$(2.3.2)\qquad \varphi^k_\alpha := \begin{cases} \varphi^{k-1}_\alpha & \text{on} \quad U_\alpha\setminus S_k \\ \psi + h_{\alpha\alpha_k} & \text{on} \quad U_\alpha \cap W_k , \end{cases}$$

where the last expression is defined in 2.4(v) of Chap. I.

By the induction hypothesis, (2.3.1) is valid for the rank $k-1$, hence definition (2.3.2) is consistent. But this implies (2.3.1) for the rank k as well. Indeed, (i) is obvious. (ii) and (iii) can be easily checked on W_k by the cocycle condition (2.1.1)(ii) and outside S_k by the induction hypothesis. So (2.3.1) is valid.

Now since $S_k \subset W_k$, (S_k) is locally finite and, for fixed α, $(\varphi_\alpha^k)_{k\in\mathbb{N}}$ is locally stationary. We may set

$$\psi_\alpha := \lim_{k\to\infty} \varphi_\alpha^k \in SP^\infty(U_\alpha)$$

and the conclusion of Theorem 1 is satisfied.

2.4. Corollary. *The "old" and "modern" definition of a reduced Kähler space coincide.*

Proof. By 1.2(iv) of Chap. I, if X is reduced, PH_X can be identified to a subsheaf of $\mathscr{C}_X$. A Kähler metric in the "old" sense is a section of $\mathscr{C}_{X,\mathbb{R}}/PH_{X,\mathbb{R}}$ represented locally by sections of $[SP_X^\infty]$. Since $[SP_X^\infty] \subset SP_X^0$, Theorem 1 applies.

3. *Application to Finite Morphisms*

Theorem 1 implies that images of Kähler spaces under certain finite morphisms are Kähler. This solves a problem raised by Lieberman at the end of [18].

3.1. *Traces of Continuous and Holomorphic Functions*

3.1.1. Definitions. If X is reduced, $k \geqq 1$ an integer and $\varphi \in \mathscr{C}(X)$, then

$$\tilde{\varphi}: \sum_{j=1}^{k} \{x_j\} \mapsto \sum_{j=1}^{k} \varphi(x_j)$$

defines a continuous function on $\mathrm{Sym}^k(X)$. On the other hand, for arbitrary X we have

$$\mathbf{B}_0(X) = \coprod_{k\geqq 1} \mathrm{Sym}^k(X_{\mathrm{red}}).$$

Now suppose $\pi: X \to X'$ is a finite open surjective morphism with connected base X'. We examine the following two situations:

(1) X' is reduced and π is geometrically flat;

(2) X' is arbitrary and π is flat.

In the first case, there is an integer $k = k_\pi \geqq 1$ called the (geometric) *degree of* π such that the classifying morphism $H: X' \to \mathbf{B}_0(X)$ factors through $\mathrm{Sym}^k(X_{\mathrm{red}})$. We have for generic $x' \in X'$ (on the points of flatness of π)

$$H(x') = \sum_{x\in\pi^{-1}(x')} \{x\},$$

where the sum takes account of multiplicities.

Define a *continuous trace* morphism

$$\mathrm{Tr}_{X/X'}^{(c)}: \pi_* \mathscr{C}_X \to \mathscr{C}_{X'}$$

by $\varphi \mapsto \tilde{\varphi} \circ H$.