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By the induction hypothesis, (2.3.1) is valid for the rank  $k-1$ , hence definition (2.3.2) is consistent. But this implies (2.3.1) for the rank  $k$  as well. Indeed, (i) is obvious. (ii) and (iii) can be easily checked on  $W_k$  by the cocycle condition (2.1.1)(ii) and outside  $S_k$  by the induction hypothesis. So (2.3.1) is valid.

Now since  $S_k \subset W_k$ ,  $(S_k)$  is locally finite and, for fixed  $\alpha$ ,  $(\varphi_\alpha^k)_{k \in \mathbb{N}}$  is locally stationary. We may set

$$\psi_\alpha := \lim_{k \rightarrow \infty} \varphi_\alpha^k \in SP^\infty(U_\alpha)$$

and the conclusion of Theorem 1 is satisfied.

**2.4. Corollary.** *The “old” and “modern” definition of a reduced Kähler space coincide.*

*Proof.* By 1.2(iv) of Chap. I, if  $X$  is reduced,  $PH_X$  can be identified to a subsheaf of  $\mathcal{C}_X$ . A Kähler metric in the “old” sense is a section of  $\mathcal{C}_{X, \mathbb{R}}/PH_{X, \mathbb{R}}$  represented locally by sections of  $[SP_X^\infty]$ . Since  $[SP_X^\infty] \subset SP_X^0$ , Theorem 1 applies.

### 3. Application to Finite Morphisms

Theorem 1 implies that images of Kähler spaces under certain finite morphisms are Kähler. This solves a problem raised by Lieberman at the end of [18].

#### 3.1. Traces of Continuous and Holomorphic Functions

**3.1.1. Definitions.** If  $X$  is reduced,  $k \geq 1$  an integer and  $\varphi \in \mathcal{C}(X)$ , then

$$\tilde{\varphi}: \sum_{j=1}^k \{x_j\} \mapsto \sum_{j=1}^k \varphi(x_j)$$

defines a continuous function on  $\text{Sym}^k(X)$ . On the other hand, for arbitrary  $X$  we have

$$\mathbf{B}_0(X) = \coprod_{k \geq 1} \text{Sym}^k(X_{\text{red}}).$$

Now suppose  $\pi: X \rightarrow X'$  is a finite open surjective morphism with connected base  $X'$ . We examine the following two situations:

- (1)  $X'$  is reduced and  $\pi$  is geometrically flat;
- (2)  $X'$  is arbitrary and  $\pi$  is flat.

In the first case, there is an integer  $k = k_\pi \geq 1$  called the (geometric) *degree* of  $\pi$  such that the classifying morphism  $H: X' \rightarrow \mathbf{B}_0(X)$  factors through  $\text{Sym}^k(X_{\text{red}})$ . We have for generic  $x' \in X'$  (on the points of flatness of  $\pi$ )

$$H(x') = \sum_{x \in \pi^{-1}(x')} \{x\},$$

where the sum takes account of multiplicities.

Define a *continuous trace* morphism

$$\text{Tr}_{X/X'}^{(c)}: \pi_* \mathcal{C}_X \rightarrow \mathcal{C}_{X'}$$

by  $\varphi \mapsto \tilde{\varphi} \circ H$ .

In the second case, there is an integer  $r = r_\pi \geq 1$  called the (algebraic) *degree* of  $\pi$  such that  $\pi_* \mathcal{O}_X$  is a locally free  $\mathcal{O}_{X'}$ -module of rank  $r$ . Define the *holomorphic trace morphism*

$$\mathrm{Tr}_{X'/X'}^{(h)} : \pi_* \mathcal{O}_X \rightarrow \mathcal{O}_{X'}$$

by  $f \mapsto$  trace of the linear map  $\{g \mapsto fg\}$ .

$r_\pi$  is preserved by base change and, if  $X'$  is reduced, coincides with  $k_\pi$ . For general  $X'$ , define  $\bar{\pi}, Y, \bar{Y}$  by the cartesian diagram

$$\begin{array}{ccc} X & \longleftarrow & Y := X \times_{X'} Y' \\ \pi \downarrow & & \downarrow \bar{\pi} \\ X' & \longleftarrow & Y' := X'_{\mathrm{red}} \end{array}$$

Then we have  $r_\pi = r_{\bar{\pi}} = k_{\bar{\pi}}$ . We define

$$\mathrm{Tr}_{X'/X'}^{(c)} := \mathrm{Tr}_{Y'/Y'}^{(c)}$$

since  $\mathcal{C}_X = \mathcal{C}_Y$  and  $\mathcal{C}_{X'} = \mathcal{C}_{Y'}$ . The two trace morphisms so defined are compatible, i.e. the diagram

$$\begin{array}{ccc} \pi_* \mathcal{O}_X & \xrightarrow{q} & \pi_* \mathcal{C}_X \\ \mathrm{Tr}_{X'/X'}^{(h)} \downarrow & & \downarrow \mathrm{Tr}_{X'/X'}^{(c)} \\ \mathcal{O}_{X'} & \xrightarrow{q} & \mathcal{C}_{X'} \end{array}$$

is commutative, where  $q: f \mapsto [f]$  is the canonical morphism. The holomorphic trace morphism is obviously extended to  $\pi_* PH_X \rightarrow PH_{X'}$ .

We write  $\pi_* \varphi$  for  $\mathrm{Tr}_{X'/X'}^{(c)} \varphi$  or  $\mathrm{Tr}_{X'/X'}^{(h)} \varphi$  indifferently.

**3.1.2. Lemma** [5, 23]. *If  $\varphi$  is p.s.h., strongly p.s.h., holomorphic or pluriharmonic on  $X$ , then  $\tilde{\varphi}$  (resp.  $\pi_* \varphi$ ) has the corresponding properties on  $\mathrm{Sym}^k(X_{\mathrm{red}})$  (resp.  $X'$ ).*

**3.1.3. Remark.** (i) For the ‘‘p.s.h.’’ part of the above lemma, the Fornaess-Narasimhan theorem is needed.

(ii) It is not true in general that  $\pi_* \varphi$  is  $\mathcal{C}^\infty$  if  $\varphi$  is  $\mathcal{C}^\infty$  even if  $X$  and  $X'$  are smooth.

**3.2. Theorem.** *Let  $X$  be a Kähler space and  $\pi: X \rightarrow X'$  a finite open surjective morphism such that either*

- (i)  $X'$  is reduced and  $\pi$  is geometrically flat or
- (ii)  $\pi$  is flat.

*Then  $X'$  is Kähler.*

*Proof.* It results from 3.1.2 and Theorem 1 (exactly as Proposition 2.1 of [23]).

**3.2.1. Corollary.** *If  $X$  is a reduced Kähler space and  $G$  a finite group of automorphisms of  $X$ , then  $X/G$  is Kähler. In particular  $\mathrm{Sym}^k(X)$  is Kähler.*

*Proof.* It is clear that the canonical projection  $X \rightarrow X/G$  is geometrically flat. For  $X$  smooth and  $G$  having isolated fixed points, this is shown by Fujiki [13, Proposition 1].

**3.2.2. Corollary.** *If  $\pi: X \rightarrow X'$  is finite surjective with  $X$  Kähler and  $X'$  normal then  $X'$  is Kähler.*