By the induction hypothesis, (2.3.1) is valid for the rank $k-1$, hence definition (2.3.2) is consistent. But this implies (2.3.1) for the rank k as well. Indeed, (i) is obvious. (ii) and (iii) can be easily checked on W_k by the cocycle condition (2.1.1)(ii) and outside S_k by the induction hypothesis. So (2.3.1) is valid.

Now since $S_k \subset W_k$, (S_k) is locally finite and, for fixed α, $(\varphi_\alpha^k)_{k\in\mathbb{N}}$ is locally stationary. We may set

$$\psi_\alpha := \lim_{k\to\infty} \varphi_\alpha^k \in SP^\infty(U_\alpha)$$

and the conclusion of Theorem 1 is satisfied.

2.4. Corollary. *The "old" and "modern" definition of a reduced Kähler space coincide.*

Proof. By 1.2(iv) of Chap. I, if X is reduced, PH_X can be identified to a subsheaf of $\mathscr{C}_X$. A Kähler metric in the "old" sense is a section of $\mathscr{C}_{X,\mathbb{R}}/PH_{X,\mathbb{R}}$ represented locally by sections of $[SP_X^\infty]$. Since $[SP_X^\infty]\subset SP_X^0$, Theorem 1 applies.

3. Application to Finite Morphisms

Theorem 1 implies that images of Kähler spaces under certain finite morphisms are Kähler. This solves a problem raised by Lieberman at the end of [18].

3.1. Traces of Continuous and Holomorphic Functions

3.1.1. Definitions. If X is reduced, $k\geqq 1$ an integer and $\varphi\in\mathscr{C}(X)$, then

$$\tilde\varphi:\sum_{j=1}^{k}\{x_j\}\mapsto\sum_{j=1}^{k}\varphi(x_j)$$

defines a continuous function on $\operatorname{Sym}^k(X)$. On the other hand, for arbitrary X we have

$$\mathbf{B}_0(X)=\coprod_{k\geqq 1}\operatorname{Sym}^k(X_{\mathrm{red}}).$$

Now suppose $\pi: X\to X'$ is a finite open surjective morphism with connected base X'. We examine the following two situations:

(1) X' is reduced and π is geometrically flat;

(2) X' is arbitrary and π is flat.

In the first case, there is an integer $k=k_\pi\geqq 1$ called the (geometric) *degree of* π such that the classifying morphism $H: X'\to\mathbf{B}_0(X)$ factors through $\operatorname{Sym}^k(X_{\mathrm{red}})$. We have for generic $x'\in X'$ (on the points of flatness of π)

$$H(x')=\sum_{x\in\pi^{-1}(x')}\{x\},$$

where the sum takes account of multiplicities.

Define a *continuous trace* morphism

$$\mathrm{Tr}^{(c)}_{X/X'}:\pi_*\mathscr{C}_X\to\mathscr{C}_{X'}$$

by $\varphi\mapsto\tilde\varphi\circ H$.

In the second case, there is an integer $r=r_\pi \geqq 1$ called the (algebraic) *degree of* π such that $\pi_* \mathcal{O}_X$ is a locally free $\mathcal{O}_{X'}$-module of rank r. Define the *holomorphic trace* morphism

$$\mathrm{Tr}^{(h)}_{X/X'}: \pi_* \mathcal{O}_X \to \mathcal{O}_{X'}$$

by $f \mapsto$ trace of the linear map $\{g \mapsto fg\}$.

r_π is preserved by base change and, if X' is reduced, coincides with k_π. For general X', define $\bar{\pi}$, Y, $\bar{Y}$ by the cartesian diagram

$$\begin{array}{ccc} X & \longleftarrow & Y := X \times_{X'} Y' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ X' & \longleftarrow & Y' := X'_{\mathrm{red}} . \end{array}$$

Then we have $r_\pi = r_{\bar{\pi}} = k_{\bar{\pi}}$. We define

$$\mathrm{Tr}^{(c)}_{X/X'} := \mathrm{Tr}^{(c)}_{Y/Y'}$$

since $\mathscr{C}_X = \mathscr{C}_Y$ and $\mathscr{C}_{X'} = \mathscr{C}_{Y'}$. The two trace morphisms so defined are compatible, i.e. the diagram

$$\begin{array}{ccc} \pi_* \mathcal{O}_X & \xrightarrow{\varrho} & \pi_* \mathscr{C}_X \\ {\scriptstyle \mathrm{Tr}^{(h)}_{X/X'}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}^{(c)}_{X/X'}} \\ \mathcal{O}_{X'} & \xrightarrow{\varrho} & \mathscr{C}_{X'} \end{array}$$

is commutative, where $\varrho: f \mapsto [f]$ is the canonical morphism. The holomorphic trace morphism is obviously extended to $\pi_* PH_X \to PH_{X'}$.

We write $\pi_* \varphi$ for $\mathrm{Tr}^{(c)}_{X/X'} \varphi$ or $\mathrm{Tr}^{(h)}_{X/X'} \varphi$ indifferently.

3.1.2. Lemma [5, 23]. *If φ is p.s.h., strongly p.s.h., holomorphic or pluriharmonic on X, then $\tilde{\varphi}$ (resp. $\pi_* \varphi$) has the corresponding properties on* $\mathrm{Sym}^k(X_{\mathrm{red}})$ *(resp. X').*

3.1.3. Remark. (i) For the "p.s.h." part of the above lemma, the Fornaess-Narasimhan theorem is needed.

(ii) It is not true in general that $\pi_* \varphi$ is $\mathscr{C}^\infty$ if φ is $\mathscr{C}^\infty$ even if X and X' are smooth.

3.2. Theorem. *Let X be a Kähler space and $\pi: X \to X'$ a finite open surjective morphism such that either*

(i) *X' is reduced and π is geometrically flat or*

(ii) *π is flat.*

Then X' is Kähler.

Proof. It results from 3.1.2 and Theorem 1 (exactly as Proposition 2.1 of [23]).

3.2.1. Corollary. *If X is a reduced Kähler space and G a finite group of automorphisms of X, then X/G is Kähler. In particular* $\mathrm{Sym}^k(X)$ *is Kähler.*

Proof. It is clear that the canonical projection $X \to X/G$ is geometrically flat. For X smooth and G having isolated fixed points, this is shown by Fujiki [13, Proposition 1].

3.2.2. Corollary. *If $\pi: X \to X'$ is finite surjective with X Kähler and X' normal then X' is Kähler.*