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4. *Weakly Kähler Metrics*

Because of the impossibility to solve (for the moment) problem 3.6 (Chap. I) we are forced to introduce the notion of weakly Kähler spaces.

4.1. Definitions. If X, Y are reduced spaces, a function $f: X \to Y$ is *weakly holomorphic* if it is *continuous* and generically holomorphic. Let $\mathscr{W}_X$ be the sheaf of weakly holomorphic complex-valued functions on X. Define the sheaf WPH_X of *weakly pluriharmonic* functions by $WPH_X := \mathscr{W}_X + \overline{\mathscr{W}_X}$. X is *weakly normal* iff $\mathscr{W}_X = \mathcal{O}_X$. The *weak normalization* of X is a weakly normal space $\hat{X}$ [2] together with a holomorphic homeomorphism $n: \hat{X} \to X$ such that $n_* \mathcal{O}_{\hat{X}} = \mathscr{W}_X$. If X is not reduced, define the weak normalization $\hat{X} \to X$ as that of X_{red} followed by the reduction $X_{\text{red}} \to X$.

A *weakly Kähler metric* on X is a section of the quotient sheaf $\mathscr{C}_{X,\mathbb{R}}/WPH_{X,\mathbb{R}}$ represented by a system of sections of SP_X^0. X is *weakly Kähler* if X_{red} admits a weakly Kähler metric. We have (for X, Y, Z reduced spaces):

4.1.1. Lemma. (i) *If $f: X \to Y$ and $g: Y \to Z$ are weakly holomorphic, then $g \circ f: X \to Z$ is weakly holomorphic.*

(ii) *If $f: X \to Y$ is weakly holomorphic and $h \in WPH(Y)$, then $h \circ f \in WPH(X)$.*

(iii) *X is weakly normal, iff every local irreducible component of X is normal.*

The Fornaess-Narasimhan theorem implies:

4.1.2. Lemma. (i) *If $f: X \to Y$ is weakly holomorphic and $\varphi \in P^0(Y)$, then $\varphi \circ f \in P^0(X)$.*

(ii) *$WPH_X \subset P_X^0$ (weakly pluriharmonic functions are p.s.h.).*

(iii) *$WPH_X SP_X^0 \subset SP_X^0$ [a consequence of (ii)].*

4.1.3. Lemma. *Let $n: X \to \hat{X}$ be the weak normalization of X. For $\varphi \in \mathscr{C}(\hat{X})$, set $n_* \varphi := \varphi \circ n^{-1} \in \mathscr{C}(X)$. Then*

(i) *If $\varphi \in P^0(\hat{X})$, then $n_* \varphi \in P^0(X)$.*

(ii) *If $\varphi \in SP^0(\hat{X})$, then $n_* \varphi \in SP^0(X)$.*

(iii) *If $\varphi \in \mathcal{O}(\hat{X})$, then $n_* \varphi \in \mathscr{W}(X)$.*

(iv) *If $\varphi \in PH(\hat{X})$, then $n_* \varphi \in WPH(X)$.*

4.2. *Relation with Kähler Metrics*

4.2.1. Lemma. *If X is weakly Kähler and weakly normal, then X is Kähler.*

Proof. Since $WPH_X = PH_X$, Theorem 1 applies.

4.2.2. Lemma. *If $\pi: X \to Y$ is a Kähler morphism and Y a weakly Kähler space, then any open $U \subset\subset X$ is weakly Kähler.*

Proof. By an elementary argument similar to 1.3.1 (vi).

4.2.3. Proposition. *Let X be a complex space and $n: \hat{X} \to X$ its weakly normalization. Then*

(i) *If $\hat{X}$ is Kähler, then X is weakly Kähler.*

(ii) *If X is weakly Kähler, then every open $U \subset\subset \hat{X}$ is Kähler.*

Proof. (i) Is a consequence of Lemma 4.1.3 above.

(ii) Since n is finite, it is Kähler morphism by 1.3.1 (v). We apply 4.2.2 and 4.2.1 to conclude.

4.2.4. Corollary. *If X is compact, then $\hat{X}$ is Kähler iff X is weakly Kähler.*

III. Theorem 2

1. Čech Spaces and Čech Open Sets

1.1. Definitions. A (topological or complex-analytic) *Čech space* will be by definition a pair

$$\underline{X}=(X,\mathscr{X}),$$

where X is a (topological or complex) space and $\mathscr{X}$ an open covering of X. We call X the *space underlying to* $\underline{X}$ and always denote both by the same letter. We will deal only with complex-analytic Čech spaces. If $\mathscr{X}=(X_\lambda)_{\lambda\in\Lambda}$, the X_λ will be called the *elementary* open sets of $\underline{X}$.

Suppose $\underline{X}=(X,(X_\lambda)_{\lambda\in\Lambda})$ and $\underline{Y}=(Y,(Y_\mu)_{\mu\in M})$ are two Čech spaces. A morphism

$$F:\underline{X}\to\underline{Y}$$

will be a pair $F=(f,\mu)$ where $f:X\to Y$ is a morphism in the ordinary sense and $\mu:\Lambda\to M$ a map such that

$$X_\lambda\subset f^{-1}(Y_{\mu(\lambda)}) \tag{1.1.1}$$

for all $\lambda\in\Lambda$. We call f the *morphism underlying to* F. We will say that F is an *open inclusion* if f is one.

A *Čech open set* $\underline{U}\ll\underline{X}$ will be a Čech space chose underlying space is an open subset of X together with an open inclusion

$$j:\underline{U}\to\underline{X}.$$

Or course, j is not uniquely determined by $\underline{U}$.

If $\underline{U}_1=(U_1,(U_{1,\alpha})_{\alpha\in A_1})$ and $\underline{U}_2=(U_2,(U_{2,\beta})_{\beta\in A_2})$ are two Čech open sets of $\underline{X}$, define

$$\underline{U}_1\cap\underline{U}_2:=(U_1\cap U_2,(U_{1,\alpha}\cap U_{2,\beta})_{(\alpha,\beta)\in A_1\times A_2}). \tag{1.1.2}$$

Notice that there are two open inclusions

$$j_1,j_2:\underline{U}_1\cap\underline{U}_2\to\underline{X}$$

each factoring through $\underline{U}_1$ and $\underline{U}_2$, respectively.

If $\underline{X}=(X,\mathscr{X})$ is a Čech space and $\mathscr{F}$ a sheaf of abelian groups on X, write

$$C^q(\underline{X},\mathscr{F}),\ Z^q(\underline{X},\mathscr{F}),\ H^q(\underline{X},\mathscr{F})$$

for the groups of Čech contains, cocycles and cohomology classes of degree q of the covering $\mathscr{X}$ with coefficients in $\mathscr{F}$. Denote by

$$\varepsilon:H^0(X,\mathscr{F})\to C^0(\underline{X},\mathscr{F}) \tag{1.1.3}$$