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4. Weakly Kähler Metrics

Because of the impossibility to solve (for the moment) problem 3.6 (Chap. I) we are forced to introduce the notion of weakly Kähler spaces.

4.1. Definitions. If X, Y are reduced spaces, a function $f: X \rightarrow Y$ is *weakly holomorphic* if it is *continuous* and generically holomorphic. Let \mathcal{W}_X be the sheaf of weakly holomorphic complex-valued functions on X . Define the sheaf WPH_X of *weakly pluriharmonic* functions by $WPH_X := \mathcal{W}_X + \overline{\mathcal{W}_X}$. X is *weakly normal* iff $\mathcal{W}_X = \mathcal{O}_X$. The *weak normalization* of X is a weakly normal space \hat{X} [2] together with a holomorphic homeomorphism $n: \hat{X} \rightarrow X$ such that $n_* \mathcal{O}_{\hat{X}} = \mathcal{W}_X$. If X is not reduced, define the weak normalization $\hat{X} \rightarrow X$ as that of X_{red} followed by the reduction $X_{\text{red}} \rightarrow X$.

A *weakly Kähler metric* on X is a section of the quotient sheaf $\mathcal{C}_{X, \mathbb{R}}/WPH_{X, \mathbb{R}}$ represented by a system of sections of SP_X^0 . X is *weakly Kähler* if X_{red} admits a weakly Kähler metric. We have (for X, Y, Z reduced spaces):

4.1.1. Lemma. (i) If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are weakly holomorphic, then $g \circ f: X \rightarrow Z$ is weakly holomorphic.

- (ii) If $f: X \rightarrow Y$ is weakly holomorphic and $h \in WPH(Y)$, then $h \circ f \in WPH(X)$.
- (iii) X is weakly normal iff every local irreducible component of X is normal.

The Fornaess-Narasimhan theorem implies:

4.1.2. Lemma. (i) If $f: X \rightarrow Y$ is weakly holomorphic and $\varphi \in P^0(Y)$, then $\varphi \circ f \in P^0(X)$.

- (ii) $WPH_X \subset P_X^0$ (weakly pluriharmonic functions are p.s.h.).
- (iii) $WPH_X SP_X^0 \subset SP_X^0$ [a consequence of (ii)].

4.1.3. Lemma. Let $n: X \rightarrow \hat{X}$ be the weak normalization of X . For $\varphi \in \mathcal{C}(\hat{X})$, set $n_* \varphi := \varphi \circ n^{-1} \in \mathcal{C}(X)$. Then

- (i) If $\varphi \in P^0(\hat{X})$, then $n_* \varphi \in P^0(X)$.
- (ii) If $\varphi \in SP^0(\hat{X})$, then $n_* \varphi \in SP^0(X)$.
- (iii) If $\varphi \in \mathcal{O}(\hat{X})$, then $n_* \varphi \in \mathcal{W}(X)$.
- (iv) If $\varphi \in PH(\hat{X})$, then $n_* \varphi \in WPH(X)$.

4.2. Relation with Kähler Metrics

4.2.1. Lemma. If X is weakly Kähler and weakly normal, then X is Kähler.

Proof. Since $WPH_X = PH_X$, Theorem 1 applies.

4.2.2. Lemma. If $\pi: X \rightarrow Y$ is a Kähler morphism and Y a weakly Kähler space, then any open $U \subset X$ is weakly Kähler.

Proof. By an elementary argument similar to 1.3.1(vi).

4.2.3. Proposition. Let X be a complex space and $n: \hat{X} \rightarrow X$ its weakly normalization. Then

- (i) If \hat{X} is Kähler, then X is weakly Kähler.
- (ii) If X is weakly Kähler, then every open $U \subset \hat{X}$ is Kähler.

Proof. (i) Is a consequence of Lemma 4.1.3 above.

(ii) Since n is finite, it is Kähler morphism by 1.3.1(v). We apply 4.2.2 and 4.2.1 to conclude.

4.2.4. Corollary. *If X is compact, then \hat{X} is Kähler iff X is weakly Kähler.*

III. Theorem 2

1. Čech Spaces and Čech Open Sets

1.1. Definitions. A (topological or complex-analytic) Čech space will be by definition a pair

$$\underline{X} = (X, \mathcal{X}),$$

where X is a (topological or complex) space and \mathcal{X} an open covering of X . We call X the *space underlying to \underline{X}* and always denote both by the same letter. We will deal only with complex-analytic Čech spaces. If $\mathcal{X} = (X_\lambda)_{\lambda \in A}$, the X_λ will be called the *elementary open sets of \underline{X}* .

Suppose $\underline{X} = (X, (X_\lambda)_{\lambda \in A})$ and $\underline{Y} = (Y, (Y_\mu)_{\mu \in M})$ are two Čech spaces. A morphism

$$F: \underline{X} \rightarrow \underline{Y}$$

will be a pair $F = (f, \mu)$ where $f: X \rightarrow Y$ is a morphism in the ordinary sense and $\mu: A \rightarrow M$ a map such that

$$(1.1.1) \quad X_\lambda \subset f^{-1}(Y_{\mu(\lambda)})$$

for all $\lambda \in A$. We call f the *morphism underlying to F* . We will say that F is an *open inclusion* if f is one.

A Čech open set $\underline{U} \ll \underline{X}$ will be a Čech space whose underlying space is an open subset of X together with an open inclusion

$$j: \underline{U} \rightarrow \underline{X}.$$

Of course, j is not uniquely determined by \underline{U} .

If $\underline{U}_1 = (U_1, (U_{1,\alpha})_{\alpha \in A_1})$ and $\underline{U}_2 = (U_2, (U_{2,\beta})_{\beta \in A_2})$ are two Čech open sets of \underline{X} , define

$$(1.1.2) \quad \underline{U}_1 \cap \underline{U}_2 := (U_1 \cap U_2, (U_{1,\alpha} \cap U_{2,\beta})_{(\alpha,\beta) \in A_1 \times A_2}).$$

Notice that there are two open inclusions

$$j_1, j_2: \underline{U}_1 \cap \underline{U}_2 \rightarrow \underline{X}$$

each factoring through \underline{U}_1 and \underline{U}_2 , respectively.

If $\underline{X} = (X, \mathcal{X})$ is a Čech space and \mathcal{F} a sheaf of abelian groups on X , write

$$C^q(\underline{X}, \mathcal{F}), Z^q(\underline{X}, \mathcal{F}), H^q(\underline{X}, \mathcal{F})$$

for the groups of Čech cochains, cocycles and cohomology classes of degree q of the covering \mathcal{X} with coefficients in \mathcal{F} . Denote by

$$(1.1.3) \quad \varepsilon: H^0(X, \mathcal{F}) \rightarrow C^0(\underline{X}, \mathcal{F})$$