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Proof. (i) Is a consequence of Lemma 4.1.3 above.

(ii) Since n is finite, it is Kähler morphism by 1.3.1(v). We apply 4.2.2 and 4.2.1 to conclude.

4.2.4. Corollary. *If X is compact, then \hat{X} is Kähler iff X is weakly Kähler.*

III. Theorem 2

1. Čech Spaces and Čech Open Sets

1.1. Definitions. A (topological or complex-analytic) Čech space will be by definition a pair

$$\underline{X} = (X, \mathcal{X}),$$

where X is a (topological or complex) space and \mathcal{X} an open covering of X . We call X the *space underlying to \underline{X}* and always denote both by the same letter. We will deal only with complex-analytic Čech spaces. If $\mathcal{X} = (X_\lambda)_{\lambda \in A}$, the X_λ will be called the *elementary open sets of \underline{X}* .

Suppose $\underline{X} = (X, (X_\lambda)_{\lambda \in A})$ and $\underline{Y} = (Y, (Y_\mu)_{\mu \in M})$ are two Čech spaces. A morphism

$$F: \underline{X} \rightarrow \underline{Y}$$

will be a pair $F = (f, \mu)$ where $f: X \rightarrow Y$ is a morphism in the ordinary sense and $\mu: A \rightarrow M$ a map such that

$$(1.1.1) \quad X_\lambda \subset f^{-1}(Y_{\mu(\lambda)})$$

for all $\lambda \in A$. We call f the *morphism underlying to F* . We will say that F is an *open inclusion* if f is one.

A Čech open set $\underline{U} \ll \underline{X}$ will be a Čech space whose underlying space is an open subset of X together with an open inclusion

$$j: \underline{U} \rightarrow \underline{X}.$$

Of course, j is not uniquely determined by \underline{U} .

If $\underline{U}_1 = (U_1, (U_{1,\alpha})_{\alpha \in A_1})$ and $\underline{U}_2 = (U_2, (U_{2,\beta})_{\beta \in A_2})$ are two Čech open sets of \underline{X} , define

$$(1.1.2) \quad \underline{U}_1 \cap \underline{U}_2 := (U_1 \cap U_2, (U_{1,\alpha} \cap U_{2,\beta})_{(\alpha,\beta) \in A_1 \times A_2}).$$

Notice that there are two open inclusions

$$j_1, j_2: \underline{U}_1 \cap \underline{U}_2 \rightarrow \underline{X}$$

each factoring through \underline{U}_1 and \underline{U}_2 , respectively.

If $\underline{X} = (X, \mathcal{X})$ is a Čech space and \mathcal{F} a sheaf of abelian groups on X , write

$$C^q(\underline{X}, \mathcal{F}), Z^q(\underline{X}, \mathcal{F}), H^q(\underline{X}, \mathcal{F})$$

for the groups of Čech cochains, cocycles and cohomology classes of degree q of the covering \mathcal{X} with coefficients in \mathcal{F} . Denote by

$$(1.1.3) \quad \varepsilon: H^0(X, \mathcal{F}) \rightarrow C^0(\underline{X}, \mathcal{F})$$

the canonical inclusion and by

$$\delta: C^{q-1}(\underline{X}, \mathcal{F}) \rightarrow C^q(\underline{X}, \mathcal{F})$$

the Čech differential given by the usual formula

$$(1.1.4) \quad (\delta\varphi)_{\lambda_0 \dots \lambda_q} := \sum_{r=0}^q (-1)^r \varphi_{\lambda_0 \dots \hat{\lambda}_r \dots \lambda_q} |_{X_{\lambda_0} \cap \dots \cap X_{\lambda_q}}.$$

If $\underline{U} \ll \underline{X}$ is a Čech open set with an open inclusion $j: \underline{U} \rightarrow \underline{X}$, denote by

$$j^*: C^q(\underline{X}, \mathcal{F}) \rightarrow C^q(\underline{U}, \mathcal{F})$$

the obvious morphism. We will write

$$(1.1.5) \quad \varphi|_{\underline{U}} := j^*(\varphi)$$

if there is no ambiguity about j .

Now suppose there are two open inclusions

$$j_1, j_2: \underline{U} \rightarrow \underline{X}$$

with $\underline{U} = (U, (U_\alpha)_{\alpha \in A})$, $\underline{X} = (X, (X_\lambda)_{\lambda \in \Lambda})$.

There is a homotopy operator

$$T: C^{q+1}(\underline{X}, \mathcal{F}) \rightarrow C^q(\underline{U}, \mathcal{F})$$

defined by

$$(1.1.6) \quad (T\varphi)_{\alpha_0 \dots \alpha_q} := \sum_{r=0}^q (-1)^r \varphi_{\lambda_0 \dots \lambda_r \mu_{r+1} \dots \mu_q} |_{U_{\alpha_0} \cap \dots \cap U_{\alpha_q}},$$

where

$$U_{\alpha_r} \subset X_{\lambda_r} \quad \text{by } j_1$$

and

$$U_{\alpha_r} \subset X_{\mu_r} \quad \text{by } j_2.$$

T is extended by 0 on $C^0(\underline{X}, \mathcal{F})$ and $H^0(X, \mathcal{F})$. The following is obvious

$$(1.1.7) \quad \delta T + T\delta = j_2^* - j_1^*.$$

1.2. Cup-Products of Čech Cochains. Now suppose that $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are sheaves of differential forms (\mathcal{C}^∞ , holomorphic or antiholomorphic) such that

$$\mathcal{F} \wedge \mathcal{G} \subset \mathcal{H}.$$

We define the cup-product

$$C^q(\underline{X}, \mathcal{F}) \times C^r(\underline{X}, \mathcal{G}) \rightarrow C^{q+r}(\underline{X}, \mathcal{H})$$

by the identity

$$(1.2.1) \quad (\varphi \cdot \psi)_{\alpha_0 \dots \alpha_{q+r}} := (\varphi_{\alpha_0 \dots \alpha_q} \wedge \psi_{\alpha_{q+1} \dots \alpha_{q+r}}) |_{X_{\lambda_0} \cap \dots \cap X_{\lambda_{q+r}}}.$$

As an immediate consequence we have

$$(1.2.2) \quad \delta(\varphi \cdot \psi) = (\delta\varphi) \cdot \psi + (-1)^q \varphi \cdot \delta\psi,$$

$$(1.2.3) \quad T(\varphi \cdot \psi) = (T\varphi) \cdot j_2^* \psi + (-1)^q (j_1^* \varphi) \cdot T\psi.$$

1.3. *m-Complete and m-Admissible Čech Open Sets.* We extend the notion of *m*-admissible open sets (3.5.3 of Chap. I) to Čech open sets.

1.3.1. Definitions. A Čech space \underline{X} is said to be *m-complete* if for any coherent analytic sheaf \mathcal{F} on X and any $q > m$, we have $H^q(\underline{X}, \mathcal{F}) = 0$.

A sufficient condition for this is that the underlying space X be *m*-complete and the elementary open sets of \underline{X} be Stein.

If $\underline{U} \ll \underline{X}$ is a Čech open set of \underline{X} , we will say that \underline{U} is *m-admissible in X* if

- (i) \underline{U} is *m*-complete.
- (ii) There is a Čech open set \underline{V} such that $\underline{U} \ll \underline{V} \ll \underline{X}$ and $H^k(\underline{V}, \mathbb{R}) = 0$ for all $k > 2m$.

Of course, if the above are satisfied, then the canonical morphism $H^k(\underline{X}, \mathbb{R}) \rightarrow H^k(\underline{U}, \mathbb{R})$ vanishes for $k > 2m$, since it factors through $H^k(\underline{V}, \mathbb{R}) = 0$.

This may be expressed as follows:

1.3.2. Lemma. *If $\underline{U} \ll \underline{X}$ is m-admissible, $k > 2m$ and $a \in Z^k(\underline{X}, \mathbb{R})$, then there is an element $b \in C^{k-1}(\underline{U}, \mathbb{R})$ such that $a|_{\underline{U}} = \delta b$.*

1.3.3. Proposition. *Let \underline{X} be a Čech space and $U \subset X$ an open set (in the ordinary sense) that is m-admissible. Then U is underlying to some m-admissible Čech open set $\underline{U} \ll \underline{X}$.*

Proof. By definition U is *m*-complete and there is an open V such that $U \subset V \subset X$ and $H^k(V, \mathbb{R}) = 0$ for all $k > 2m$. If we take a sufficiently fine Leray open covering of V with respect to the constant sheaf such that $\underline{V} \ll \underline{X}$ and then a sufficiently fine Stein open covering of U such that $\underline{U} \ll \underline{V}$, it is clear that $\underline{U} \ll \underline{X}$ is *m*-admissible.

2. Čech Transform of a Complex of Sheaves

2.1. Definitions. Let \underline{X} be a Čech space and

$$(2.1.1) \quad 0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{D} \mathcal{L}^1 \xrightarrow{D} \dots \xrightarrow{D} \mathcal{L}^m \xrightarrow{D} \dots$$

a complex of sheaves of abelian groups on the underlying space X . We do not suppose it to be an exact sequence of sheaves.

The Čech transform of the complex (2.1.1) over \underline{X} will be the single complex associated to the double complex

$$\begin{array}{ccccccc} H^0(X, \mathcal{F}) & \rightarrow & H^0(X, \mathcal{L}^0) & \rightarrow & H^0(X, \mathcal{L}^1) & \rightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ C^0(\underline{X}, \mathcal{F}) & \rightarrow & C^0(\underline{X}, \mathcal{L}^0) & \rightarrow & C^0(\underline{X}, \mathcal{L}^1) & \rightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ C^1(\underline{X}, \mathcal{F}) & \rightarrow & C^1(\underline{X}, \mathcal{L}^0) & \rightarrow & C^1(\underline{X}, \mathcal{L}^1) & \rightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \vdots & & \end{array}$$

More precisely, we define for $q \geq 0$

$$(2.1.2) \quad \check{C}^q(\underline{X}; \mathcal{F}, \mathcal{L}^\bullet) := C^q(\underline{X}, \mathcal{F}) \oplus \left\{ \bigoplus_{k=1}^q C^{q-k}(\underline{X}, \mathcal{L}^{k-1}) \right\} \oplus H^0(X, \mathcal{L}^q).$$