*Proof.* (i) Is a consequence of Lemma 4.1.3 above.

(ii) Since $n$ is finite, it is Kähler morphism by 1.3.1 (v). We apply 4.2.2 and 4.2.1 to conclude.

**4.2.4. Corollary.** *If $X$ is compact, then $\hat{X}$ is Kähler iff $X$ is weakly Kähler.*

## III. Theorem 2

### *1. Čech Spaces and Čech Open Sets*

**1.1. Definitions.** A (topological or complex-analytic) *Čech space* will be by definition a pair

$$\underline{X}=(X,\mathscr{X}),$$

where $X$ is a (topological or complex) space and $\mathscr{X}$ an open covering of $X$. We call $X$ the *space underlying to* $\underline{X}$ and always denote both by the same letter. We will deal only with complex-analytic Čech spaces. If $\mathscr{X}=(X_\lambda)_{\lambda\in\Lambda}$, the $X_\lambda$ will be called the *elementary* open sets of $\underline{X}$.

Suppose $\underline{X}=(X,(X_\lambda)_{\lambda\in\Lambda})$ and $\underline{Y}=(Y,(Y_\mu)_{\mu\in M})$ are two Čech spaces. A morphism

$$F:\underline{X}\to\underline{Y}$$

will be a pair $F=(f,\mu)$ where $f:X\to Y$ is a morphism in the ordinary sense and $\mu:\Lambda\to M$ a map such that

$$X_\lambda\subset f^{-1}(Y_{\mu(\lambda)}) \tag{1.1.1}$$

for all $\lambda\in\Lambda$. We call $f$ the *morphism underlying to* $F$. We will say that $F$ is an *open inclusion* if $f$ is one.

A *Čech open set* $\underline{U}\ll\underline{X}$ will be a Čech space chose underlying space is an open subset of $X$ together with an open inclusion

$$j:\underline{U}\to\underline{X}.$$

Or course, $j$ is not uniquely determined by $\underline{U}$.

If $\underline{U}_1=(U_1,(U_{1,\alpha})_{\alpha\in A_1})$ and $\underline{U}_2=(U_2,(U_{2,\beta})_{\beta\in A_2})$ are two Čech open sets of $\underline{X}$, define

$$\underline{U}_1\cap\underline{U}_2:=(U_1\cap U_2,(U_{1,\alpha}\cap U_{2,\beta})_{(\alpha,\beta)\in A_1\times A_2}). \tag{1.1.2}$$

Notice that there are two open inclusions

$$j_1,j_2:\underline{U}_1\cap\underline{U}_2\to\underline{X}$$

each factoring through $\underline{U}_1$ and $\underline{U}_2$, respectively.

If $\underline{X}=(X,\mathscr{X})$ is a Čech space and $\mathscr{F}$ a sheaf of abelian groups on $X$, write

$$C^q(\underline{X},\mathscr{F}),\ Z^q(\underline{X},\mathscr{F}),\ H^q(\underline{X},\mathscr{F})$$

for the groups of Čech contains, cocycles and cohomology classes of degree $q$ of the covering $\mathscr{X}$ with coefficients in $\mathscr{F}$. Denote by

$$\varepsilon:H^0(X,\mathscr{F})\to C^0(\underline{X},\mathscr{F}) \tag{1.1.3}$$

the canonical inclusion and by

$$\delta: C^{q-1}(\underline{X}, \mathscr{F}) \to C^q(\underline{X}, \mathscr{F})$$

the Čech differential given by the usual formula

$$(1.1.4) \qquad (\delta\varphi)_{\lambda_0 \ldots \lambda_q} := \sum_{r=0}^{q} (-1)^r \varphi_{\lambda_0 \ldots \hat{\lambda}_r \ldots \lambda_q}|_{X_{\lambda_0} \cap \ldots \cap X_{\lambda_q}}.$$

If $\underline{U} \ll \underline{X}$ is a Čech open set with an open inclusion $j: \underline{U} \to \underline{X}$, denote by

$$j^*: C^q(\underline{X}, \mathscr{F}) \to C^q(\underline{U}, \mathscr{F})$$

the obvious morphism. We will write

$$(1.1.5) \qquad \varphi|_{\underline{U}} := j^*(\varphi)$$

if there is no ambiguity about $j$.

Now suppose there are two open inclusions

$$j_1, j_2: \underline{U} \to \underline{X}$$

with $\underline{U} = (U, (U_\alpha)_{\alpha \in A})$, $\underline{X} = (X, (X_\lambda)_{\lambda \in \Lambda})$.

There is a homotopy operator

$$T: C^{q+1}(\underline{X}, \mathscr{F}) \to C^q(\underline{U}, \mathscr{F})$$

defined by

$$(1.1.6) \qquad (T\varphi)_{\alpha_0 \ldots \alpha_q} := \sum_{r=0}^{q} (-1)^r \varphi_{\lambda_0 \ldots \lambda_r \mu_r \ldots \mu_q}|_{U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}},$$

where

$$U_{\alpha_r} \subset X_{\lambda_r} \quad \text{by } j_1$$

and

$$U_{\alpha_r} \subset X_{\mu_r} \quad \text{by } j_2.$$

$T$ is extended by 0 on $C^0(\underline{X}, \mathscr{F})$ and $H^0(X, \mathscr{F})$. The following is obvious

$$(1.1.7) \qquad \delta T + T\delta = j_2^* - j_1^*.$$

*1.2. Cup-Products of Čech Cochains.* Now suppose that $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ are sheaves of differential forms ($\mathscr{C}^\infty$, holomorphic or antiholomorphic) such that

$$\mathscr{F} \wedge \mathscr{G} \subset \mathscr{H}.$$

We define the cup-product

$$C^q(\underline{X}, \mathscr{F}) \times C^r(\underline{X}, \mathscr{G}) \to C^{q+r}(\underline{X}, \mathscr{H})$$

by the identity

$$(1.2.1) \qquad (\varphi \cdot \psi)_{\alpha_0 \ldots \alpha_{q+r}} := (\varphi_{\alpha_0 \ldots \alpha_q} \wedge \psi_{\alpha_q \ldots \alpha_{q+r}})|_{X_{\lambda_0} \cap \ldots \cap X_{\lambda_{q+r}}}.$$

As an immediate consequence we have

$$(1.2.2) \qquad \delta(\varphi \cdot \psi) = (\delta\varphi) \cdot \psi + (-1)^q \varphi \cdot \delta\psi,$$

$$(1.2.3) \qquad T(\varphi \cdot \psi) = (T\varphi) \cdot j_2^* \psi + (-1)^q (j_1^* \varphi) \cdot T\psi.$$

*1.3. m-Complete and m-Admissible Čech Open Sets.* We extend the notion of $m$-admissible open sets (3.5.3 of Chap. I) to Čech open sets.

**1.3.1. Definitions.** A Čech space $\underline{X}$ is said to be *m-complete* if for any coherent analytic sheaf $\mathscr{F}$ on $X$ and any $q > m$, we have $H^q(\underline{X}, \mathscr{F}) = 0$.

A sufficient condition for this is that the underlying space $X$ be $m$-complete and the elementary open sets of $\underline{X}$ be Stein.

If $\underline{U} \ll \underline{X}$ is a Čech open set of $\underline{X}$, we will say that $\underline{U}$ is *m-admissible in X* if

(i) $\underline{U}$ is $m$-complete.

(ii) There is a Čech open set $\underline{V}$ such that $\underline{U} \ll \underline{V} \ll \underline{X}$ and $H^k(\underline{V}, \mathbb{R}) = 0$ for all $k > 2m$.

Of course, if the above are satisfied, then the canonical morphism $H^k(\underline{X}, \mathbb{R}) \to H^k(\underline{U}, \mathbb{R})$ vanishes for $k > 2m$, since it factors through $H^k(\underline{V}, \mathbb{R}) = 0$.

This may be expressed as follows:

**1.3.2. Lemma.** *If $\underline{U} \ll \underline{X}$ is m-admissible, $k > 2m$ and $a \in Z^k(\underline{X}, \mathbb{R})$, then there is an element $b \in C^{k-1}(\underline{U}, \mathbb{R})$ such that $a|_{\underline{U}} = \delta b$.*

**1.3.3. Proposition.** *Let $\underline{X}$ be a Čech space and $U \subset X$ an open set (in the ordinary sense) that is m-admissible. Then U is underlying to some m-admissible Čech open set $\underline{U} \ll \underline{X}$.*

*Proof.* By definition $U$ is $m$-complete and there is an open $V$ such that $U \subset V \subset X$ and $H^k(V, \mathbb{R}) = 0$ for all $k > 2m$. If we take a sufficiently fine Leray open covering of $V$ with respect to the constant sheaf such that $\underline{V} \ll \underline{X}$ and then a sufficiently fine Stein open covering of $U$ such that $\underline{U} \ll \underline{V}$, it is clear that $\underline{U} \ll \underline{X}$ is $m$-admissible.

## *2. Čech Transform of a Complex of Sheaves*

**2.1. Definitions.** Let $\underline{X}$ be a Čech space and

$$0 \to \mathscr{F} \xrightarrow{j} \mathscr{L}^0 \xrightarrow{D} \mathscr{L}^1 \xrightarrow{D} \dots \xrightarrow{D} \mathscr{L}^m \xrightarrow{D} \dots \tag{2.1.1}$$

a complex of sheaves of abelian groups on the underlying space $X$. We do not suppose it to be an exact sequence of sheaves.

The *Čech transform* of the complex (2.1.1) over $\underline{X}$ will be the single complex associated to the double complex

$$\begin{array}{ccccccc}
H^0(X, \mathscr{F}) & \to & H^0(X, \mathscr{L}^0) & \to & H^0(X, \mathscr{L}^1) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
C^0(\underline{X}, \mathscr{F}) & \to & C^0(\underline{X}, \mathscr{L}^0) & \to & C^0(\underline{X}, \mathscr{L}^1) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
C^1(\underline{X}, \mathscr{F}) & \to & C^1(\underline{X}, \mathscr{L}^0) & \to & C^1(\underline{X}, \mathscr{L}^1) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

More precisely, we define for $q \geqq 0$

$$\check{C}^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot) := C^q(\underline{X}, \mathscr{F}) \oplus \left\{ \bigoplus_{k=1}^{q} C^{q-k}(\underline{X}, \mathscr{L}^{k-1}) \right\} \oplus H^0(X, \mathscr{L}^q). \tag{2.1.2}$$