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1.3. *m-Complete and m-Admissible Čech Open Sets.* We extend the notion of *m*-admissible open sets (3.5.3 of Chap. I) to Čech open sets.

1.3.1. Definitions. A Čech space \underline{X} is said to be *m-complete* if for any coherent analytic sheaf \mathcal{F} on X and any $q > m$, we have $H^q(\underline{X}, \mathcal{F}) = 0$.

A sufficient condition for this is that the underlying space X be *m*-complete and the elementary open sets of \underline{X} be Stein.

If $\underline{U} \ll \underline{X}$ is a Čech open set of \underline{X} , we will say that \underline{U} is *m-admissible in X* if

- (i) \underline{U} is *m*-complete.
- (ii) There is a Čech open set \underline{V} such that $\underline{U} \ll \underline{V} \ll \underline{X}$ and $H^k(\underline{V}, \mathbb{R}) = 0$ for all $k > 2m$.

Of course, if the above are satisfied, then the canonical morphism $H^k(\underline{X}, \mathbb{R}) \rightarrow H^k(\underline{U}, \mathbb{R})$ vanishes for $k > 2m$, since it factors through $H^k(\underline{V}, \mathbb{R}) = 0$.

This may be expressed as follows:

1.3.2. Lemma. *If $\underline{U} \ll \underline{X}$ is m-admissible, $k > 2m$ and $a \in Z^k(\underline{X}, \mathbb{R})$, then there is an element $b \in C^{k-1}(\underline{U}, \mathbb{R})$ such that $a|_{\underline{U}} = \delta b$.*

1.3.3. Proposition. *Let \underline{X} be a Čech space and $U \subset X$ an open set (in the ordinary sense) that is m-admissible. Then U is underlying to some m-admissible Čech open set $\underline{U} \ll \underline{X}$.*

Proof. By definition U is *m*-complete and there is an open V such that $U \subset V \subset X$ and $H^k(V, \mathbb{R}) = 0$ for all $k > 2m$. If we take a sufficiently fine Leray open covering of V with respect to the constant sheaf such that $\underline{V} \ll \underline{X}$ and then a sufficiently fine Stein open covering of U such that $\underline{U} \ll \underline{V}$, it is clear that $\underline{U} \ll \underline{X}$ is *m*-admissible.

2. Čech Transform of a Complex of Sheaves

2.1. Definitions. Let \underline{X} be a Čech space and

$$(2.1.1) \quad 0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{D} \mathcal{L}^1 \xrightarrow{D} \dots \xrightarrow{D} \mathcal{L}^m \xrightarrow{D} \dots$$

a complex of sheaves of abelian groups on the underlying space X . We do not suppose it to be an exact sequence of sheaves.

The Čech transform of the complex (2.1.1) over \underline{X} will be the single complex associated to the double complex

$$\begin{array}{ccccccc} H^0(X, \mathcal{F}) & \rightarrow & H^0(X, \mathcal{L}^0) & \rightarrow & H^0(X, \mathcal{L}^1) & \rightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ C^0(\underline{X}, \mathcal{F}) & \rightarrow & C^0(\underline{X}, \mathcal{L}^0) & \rightarrow & C^0(\underline{X}, \mathcal{L}^1) & \rightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ C^1(\underline{X}, \mathcal{F}) & \rightarrow & C^1(\underline{X}, \mathcal{L}^0) & \rightarrow & C^1(\underline{X}, \mathcal{L}^1) & \rightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \vdots & & \end{array}$$

More precisely, we define for $q \geq 0$

$$(2.1.2) \quad \check{C}^q(\underline{X}; \mathcal{F}, \mathcal{L}^\bullet) := C^q(\underline{X}, \mathcal{F}) \oplus \left\{ \bigoplus_{k=1}^q C^{q-k}(\underline{X}, \mathcal{L}^{k-1}) \right\} \oplus H^0(X, \mathcal{L}^q).$$

An element of $\check{C}^q(\underline{X}; \mathcal{F}, \mathcal{L}')$ has the form

$$\Phi = (f; \varphi^0, \dots, \varphi^{q-1}; \eta^q),$$

where

$$\begin{aligned} f &\in C^q(\underline{X}, \mathcal{F}), \\ \varphi^{k-1} &\in C^{q-k}(\underline{X}, \mathcal{L}^{k-1}) \quad \text{for } k=1, \dots, q, \\ \eta^q &\in H^0(X, \mathcal{L}^q). \end{aligned}$$

We will call f the *head* of Φ , φ^{k-1} the k -th *component* of Φ and η^q the *tail* of Φ . Define the differential

$$\Delta: \check{C}^q(\underline{X}; \mathcal{F}, \mathcal{L}') \rightarrow \check{C}^{q+1}(\underline{X}; \mathcal{F}, \mathcal{L}')$$

by

$$(2.1.3) \quad \Delta := \delta + (-1)^{q+1} D$$

where

$$\begin{aligned} \delta\Phi &:= (\delta f; \delta\varphi^0, \dots, \delta\varphi^{q-1}, \varepsilon\eta^q; 0), \\ D\Phi &:= (0; jf, D\varphi^0, \dots, D\varphi^{q-1}; D\eta^q). \end{aligned}$$

Sometimes we will change the sign convention

$$\Delta = \delta + (-1)^{q+1} D \quad \text{to} \quad \Delta = \delta + (-1)^q D.$$

We then define

$$(2.1.4) \quad \check{Z}^q(\underline{X}, \mathcal{F}, \mathcal{L}') := \text{Ker} \{ \check{C}^q(\underline{X}; \mathcal{F}, \mathcal{L}') \xrightarrow{\Delta} \check{C}^{q+1}(\underline{X}; \mathcal{F}, \mathcal{L}') \}$$

and the Čech hypercohomology groups

$$(2.1.5) \quad \check{H}^q(\underline{X}; \mathcal{F}, \mathcal{L}') := \check{Z}^q(\underline{X}; \mathcal{F}, \mathcal{L}') / \Delta \check{C}^{q-1}(\underline{X}; \mathcal{F}, \mathcal{L}').$$

We will use the following.

2.2. Lemma. *Let $r^q: H^q(\underline{X}; \mathcal{F}, \mathcal{L}') \rightarrow H^q(\underline{X}, \mathcal{F})$ be the canonical morphism.*

(i) *If $H^{q-k}(\underline{X}, \mathcal{L}^{k-1}) = 0$ for $k=1, \dots, q-1$ then r^q is injective.*

(ii) *If $H^{q-k}(\underline{X}, \mathcal{L}^k) = 0$ for $k=0, \dots, q-1$ then r^q is surjective.*

Proof. It is an immediate consequence of the following elementary property of double complexes: If M' is the single complex associated to a double complex $K^{\cdot, \cdot} = (K^{i, j})_{i, j \geq 0}$, then the canonical morphism $H^q(M') \rightarrow H^q(K^{\cdot, 0})$ is injective if $H^{q-j}(K^{\cdot, j}) = 0$ for $j=1, \dots, q-1$ and surjective if $H^{q-j}(K^{\cdot, j+1}) = 0$ for $j=0, \dots, q-1$. This is to be applied for

$$K^{i, j} = \begin{cases} 0 & \text{if } i=j=0 \\ H^0(X, \mathcal{L}^{j-1}) & \text{if } j>i=0 \\ C^{i-1}(\underline{X}, \mathcal{F}) & \text{if } i>j=0 \\ C^{i-1}(X, \mathcal{L}^{j-1}) & \text{if } i, j>0. \end{cases}$$

Part (i) of the above lemma is equivalent to

2.3. Corollary. *If a cocycle $\Phi \in \check{Z}^q(X; \mathcal{F}, \mathcal{L}')$ has a head that is δ -exact and if $H^{q-k}(\underline{X}, \mathcal{L}^{k-1}) = 0$ for $k = 1, \dots, q-1$, then Φ is Δ -exact and, in particular, the tail of Φ is D -exact.*

2.4. Remark. Definition 3.4.3 of Chap. I can be restated as follows: A $\bar{\partial}$ -closed form $\tau \in A^{k,l}(X)$ is said to represent an element of $H^l(X, \Omega^k)$ if there is a cocycle of degree l

$$c \in \check{Z}^l(\underline{X}; \Omega^k, A^{k,\cdot})$$

of the Čech transform of the Dolbeault complex whose tail is τ , for some open covering of X .

3. The $\partial\bar{\partial}$ -Complex \mathcal{L}'_m

Let X be a complex space. For any pair (p, q) of natural integers, there is a complex of sheaves on X of the form

$$0 \longrightarrow \mathbf{C} \xrightarrow{\lambda} \mathcal{L}_{p,q}^0 \xrightarrow{D} \dots \xrightarrow{D} \mathcal{L}_{p,q}^{p+q-1} \xrightarrow{D} \mathcal{L}_{p,q}^{p+q} \xrightarrow{D} \dots$$

$$\begin{array}{ccc} & & \parallel \\ & & A_X^{p-1, q-1} \xrightarrow{\partial\bar{\partial}} A_X^{p,q} \\ & & \parallel \end{array}$$

defined in [7].

We will deal exclusively with the case $p=q$, so we write \mathcal{L}'_m for $\mathcal{L}_{m,m}^r$. The complex \mathcal{L}'_m defined as follows (the suffix X will be omitted).

3.1. Definitions.

$$(3.1) \quad \mathcal{L}'_m := \begin{cases} \Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r & \text{if } r < m \\ A^{m-1, r-m} \oplus \dots \oplus A^{r-m, m-1} & \text{if } m \leq r < 2m \\ A^{r-m, m} \oplus \dots \oplus A^{m, r-m} & \text{if } r \geq 2m. \end{cases}$$

Define $j: \mathbf{C} \xrightarrow{\binom{1}{1}} \Omega^0 \oplus \bar{\Omega}^0 = \mathcal{L}'_m{}^0$ and

(i) For $0 \leq r < m-1$,

$$\begin{array}{ccc} \mathcal{L}'_m & \xrightarrow{D} & \mathcal{L}'_m{}^{r+1} \\ \parallel & & \parallel \\ \Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r & \xrightarrow{\begin{pmatrix} d & 0 & 0 \\ (-1)^{r+1} d & (-1)^r & \\ 0 & 0 & d \end{pmatrix}} & \Omega^{r+1} \oplus A^r \oplus \bar{\Omega}^{r+1}. \end{array}$$

(ii) For $r = m-1$,

$$\begin{array}{ccc} \mathcal{L}'_m{}^{m-1} & \xrightarrow{D} & \mathcal{L}'_m{}^m \\ \parallel & & \parallel \\ \Omega^{m-1} \oplus A^{m-2} \oplus \bar{\Omega}^{m-1} & \xrightarrow{\begin{pmatrix} (-1)^m d & (-1)^{m-1} \end{pmatrix}} & A^{m-1}. \end{array}$$