1.3. m-Complete and m-Admissible Čech Open Sets. We extend the notion of m-admissible open sets (3.5.3 of Chap. I) to Čech open sets.

1.3.1. Definitions. A Čech space $\underline{X}$ is said to be *m-complete* if for any coherent analytic sheaf $\mathscr{F}$ on X and any $q>m$, we have $H^q(\underline{X},\mathscr{F})=0$.

A sufficient condition for this is that the underlying space X be m-complete and the elementary open sets of $\underline{X}$ be Stein.

If $\underline{U}\ll\underline{X}$ is a Čech open set of $\underline{X}$, we will say that $\underline{U}$ is *m-admissible in X* if

(i) $\underline{U}$ is m-complete.

(ii) There is a Čech open set $\underline{V}$ such that $\underline{U}\ll\underline{V}\ll\underline{X}$ and $H^k(\underline{V},\mathbb{R})=0$ for all $k>2m$.

Of course, if the above are satisfied, then the canonical morphism $H^k(\underline{X},\mathbb{R})\to H^k(\underline{U},\mathbb{R})$ vanishes for $k>2m$, since it factors through $H^k(\underline{V},\mathbb{R})=0$.

This may be expressed as follows:

1.3.2. Lemma. *If $\underline{U}\ll\underline{X}$ is m-admissible, $k>2m$ and $a\in Z^k(\underline{X},\mathbb{R})$, then there is an element $b\in C^{k-1}(\underline{U},\mathbb{R})$ such that $a|_{\underline{U}}=\delta b$.*

1.3.3. Proposition. *Let $\underline{X}$ be a Čech space and $U\subset X$ an open set (in the ordinary sense) that is m-admissible. Then U is underlying to some m-admissible Čech open set $\underline{U}\ll\underline{X}$.*

Proof. By definition U is m-complete and there is an open V such that $U\subset V\subset X$ and $H^k(V,\mathbb{R})=0$ for all $k>2m$. If we take a sufficiently fine Leray open covering of V with respect to the constant sheaf such that $\underline{V}\ll\underline{X}$ and then a sufficiently fine Stein open covering of U such that $\underline{U}\ll\underline{V}$, it is clear that $\underline{U}\ll\underline{X}$ is m-admissible.

2. *Čech Transform of a Complex of Sheaves*

2.1. Definitions. Let $\underline{X}$ be a Čech space and

$$0\to\mathscr{F}\xrightarrow{j}\mathscr{L}^0\xrightarrow{D}\mathscr{L}^1\xrightarrow{D}\dots\xrightarrow{D}\mathscr{L}^m\xrightarrow{D}\dots \tag{2.1.1}$$

a complex of sheaves of abelian groups on the underlying space X. We do not suppose it to be an exact sequence of sheaves.

The *Čech transform* of the complex (2.1.1) over $\underline{X}$ will be the single complex associated to the double complex

$$\begin{array}{ccccccc}
H^0(X,\mathscr{F}) & \to & H^0(X,\mathscr{L}^0) & \to & H^0(X,\mathscr{L}^1) & \to & \dots\\
\downarrow & & \downarrow & & \downarrow & & \\
C^0(\underline{X},\mathscr{F}) & \to & C^0(\underline{X},\mathscr{L}^0) & \to & C^0(\underline{X},\mathscr{L}^1) & \to & \dots\\
\downarrow & & \downarrow & & \downarrow & & \\
C^1(\underline{X},\mathscr{F}) & \to & C^1(\underline{X},\mathscr{L}^0) & \to & C^1(\underline{X},\mathscr{L}^1) & \to & \dots\\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

More precisely, we define for $q\geqq 0$

$$\check{C}^q(\underline{X};\mathscr{F},\mathscr{L}^\cdot):=C^q(\underline{X},\mathscr{F})\oplus\left\{\bigoplus_{k=1}^{q}C^{q-k}(\underline{X},\mathscr{L}^{k-1})\right\}\oplus H^0(X,\mathscr{L}^q). \tag{2.1.2}$$

An element of $\check{C}^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot)$ has the form

$$\Phi = (f; \varphi^0, \ldots, \varphi^{q-1}; \eta^q),$$

where

$$f \in C^q(\underline{X}, \mathscr{F}),$$

$$\varphi^{k-1} \in C^{q-k}(\underline{X}, \mathscr{L}^{k-1}) \quad \text{for} \quad k = 1, \ldots, q,$$

$$\eta^q \in H^0(X, \mathscr{L}^q).$$

We will call f the *head* of Φ, φ^{k-1} the *k-th component* of Φ and η^q the *tail* of Φ. Define the differential

$$\Delta : \check{C}^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot) \to \check{C}^{q+1}(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot)$$

by

$$\Delta := \delta + (-1)^{q+1} D \tag{2.1.3}$$

where

$$\delta\Phi := (\delta f; \delta\varphi^0, \ldots, \delta\varphi^{q-1}, \varepsilon\eta^q; 0),$$

$$D\Phi := (0; jf, D\varphi^0, \ldots, D\varphi^{q-1}; D\eta^q).$$

Sometimes we will change the sign convention

$$\Delta = \delta + (-1)^{q+1} D \quad \text{to} \quad \Delta = \delta + (-1)^q D.$$

We then define

$$\check{Z}^q(\underline{X}, \mathscr{F}, \mathscr{L}^\cdot) := \operatorname{Ker}\{\check{C}^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot) \xrightarrow{\Delta} \check{C}^{q+1}(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot)\} \tag{2.1.4}$$

and the Čech hypercohomology groups

$$\check{H}^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot) := \check{Z}^q(X; \mathscr{F}, \mathscr{L}^\cdot) / \Delta\check{C}^{q-1}(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot). \tag{2.1.5}$$

We will use the following.

2.2. Lemma. *Let $r^q : H^q(\underline{X}; \mathscr{F}, \mathscr{L}^\cdot) \to H^q(\underline{X}, \mathscr{F})$ be the canonical morphism.*
(i) *If $H^{q-k}(\underline{X}, \mathscr{L}^{k-1}) = 0$ for $k = 1, \ldots, q-1$ then r^q is injective.*
(ii) *If $H^{q-k}(\underline{X}, \mathscr{L}^k) = 0$ for $k = 0, \ldots, q-1$ then r^q is surjective.*

Proof. It is an immediate consequence of the following elementary property of double complexes: If $M^\cdot$ is the single complex associated to a double complex $K^{\cdot,\cdot} = (K^{i,j})_{i,j \geqq 0}$, then the canonical morphism $H^q(M^\cdot) \to H^q(K^{\cdot,0})$ is injective if $H^{q-j}(K^{\cdot,j}) = 0$ for $j = 1, \ldots, q-1$ and surjective if $H^{q-j}(K^{\cdot,j+1}) = 0$ for $j = 0, \ldots, q-1$. This is to be applied for

$$K^{i,j} = \begin{cases} 0 & \text{if} \quad i = j = 0 \\ H^0(X, \mathscr{L}^{j-1}) & \text{if} \quad j > i = 0 \\ C^{i-1}(\underline{X}, \mathscr{F}) & \text{if} \quad i > j = 0 \\ C^{i-1}(\underline{X}, \mathscr{L}^{j-1}) & \text{if} \quad i, j > 0. \end{cases}$$

Part (i) of the above lemma is equivalent to

2.3. Corollary. *If a cocycle $\Phi \in \check{Z}^q(X; \mathscr{F}, \mathscr{L}^{\cdot})$ has a head that is δ-exact and if $H^{q-k}(\underline{X}, \mathscr{L}^{k-1}) = 0$ for $k = 1, \ldots, q-1$, then Φ is Δ-exact and, in particular, the tail of Φ is D-exact.*

2.4. Remark. Definition 3.4.3 of Chap. I can be restated as follows: A $\bar{\partial}$-closed form $\tau \in A^{k,l}(X)$ is said to represent an element of $H^l(X, \Omega^k)$ if there is a cocycle of degree l

$$c \in \check{Z}^l(\underline{X}; \Omega^k, A^{k,\cdot})$$

of the Čech transform of the Dolbeault complex whose tail is τ, for some open covering of X.

3. *The $\partial\bar{\partial}$-Complex $\mathscr{L}^{\cdot}_m$*

Let X be a complex space. For any pair (p, q) of natural integers, there is a complex of sheaves on X of the form

$$\begin{array}{ccccccccccc} 0 \longrightarrow \mathbb{C} \xrightarrow{\lambda} \mathscr{L}^0_{p,q} \xrightarrow{D} \cdots \xrightarrow{D} & \mathscr{L}^{p+q-1}_{p,q} & \xrightarrow{D} & \mathscr{L}^{p+q}_{p,q} & \xrightarrow{D} \cdots \\ & \| & & \| & \\ & A_X^{p-1,q-1} & \xrightarrow{\partial\bar{\partial}} & A_X^{p,q} & \end{array}$$

defined in [7].

We will deal exclusively with the case $p = q$, so we write $\mathscr{L}^r_m$ for $\mathscr{L}^r_{m,m}$. The complex $\mathscr{L}^{\cdot}_m$ defined as follows (the suffix X will be omitted).

3.1. Definitions.

$$\mathscr{L}^r_m := \begin{cases} \Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r & \text{if } r < m \\ A^{m-1,r-m} \oplus \ldots \oplus A^{r-m,m-1} & \text{if } m \leqq r < 2m \\ A^{r-m,m} \oplus \ldots \oplus A^{m,r-m} & \text{if } r \geqq 2m. \end{cases} \tag{3.1}$$

Define $j : \mathbb{C} \xrightarrow{\binom{1}{1}} \Omega^0 \oplus \bar{\Omega}^0 = \mathscr{L}^0_m$ and

(i) For $0 \leqq r < m-1$,

$$\begin{array}{ccc} \mathscr{L}^r_m & \xrightarrow{\quad D \quad} & \mathscr{L}^{r+1}_m \\ \| & & \| \\ \Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r & \xrightarrow{\begin{pmatrix} d & 0 & 0 \\ (-1)^{r+1} & d & (-1)^r \\ 0 & 0 & d \end{pmatrix}} & \Omega^{r+1} \oplus A^r \oplus \bar{\Omega}^{r+1}. \end{array}$$

(ii) For $r = m-1$,

$$\begin{array}{ccc} \mathscr{L}^{m-1}_m & \xrightarrow{\quad D \quad} & \mathscr{L}^m_m \\ \| & & \| \\ \Omega^{m-1} \oplus A^{m-2} \oplus \bar{\Omega}^{m-1} & \xrightarrow{((-1)^m \;\; d \;\; (-1)^{m-1})} & A^{m-1}. \end{array}$$