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2.3. Corollary. *If a cocycle $\Phi \in \check{Z}^q(\underline{X}; \mathcal{F}, \mathcal{L}')$ has a head that is δ -exact and if $H^{q-k}(\underline{X}, \mathcal{L}^{k-1}) = 0$ for $k = 1, \dots, q-1$, then Φ is Δ -exact and, in particular, the tail of Φ is D -exact.*

2.4. Remark. Definition 3.4.3 of Chap. I can be restated as follows: A $\bar{\partial}$ -closed form $\tau \in A^{k,l}(X)$ is said to represent an element of $H^l(X, \Omega^k)$ if there is a cocycle of degree l

$$c \in \check{Z}^l(\underline{X}; \Omega^k, A^{k,\cdot})$$

of the Čech transform of the Dolbeault complex whose tail is τ , for some open covering of X .

3. The $\partial\bar{\partial}$ -Complex \mathcal{L}_m^{\cdot}

Let X be a complex space. For any pair (p, q) of natural integers, there is a complex of sheaves on X of the form

$$0 \longrightarrow \mathbb{C} \xrightarrow{\lambda} \mathcal{L}_{p,q}^0 \xrightarrow{D} \dots \xrightarrow{D} \mathcal{L}_{p,q}^{p+q-1} \xrightarrow{D} \mathcal{L}_{p,q}^{p+q} \xrightarrow{D} \dots$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$A_X^{p-1, q-1} \xrightarrow{\partial\bar{\partial}} A_X^{p,q}$$

defined in [7].

We will deal exclusively with the case $p=q$, so we write \mathcal{L}_m^r for $\mathcal{L}_{m,m}^r$. The complex \mathcal{L}_m^{\cdot} defined as follows (the suffix X will be omitted).

3.1. Definitions.

$$(3.1) \quad \mathcal{L}_m^r := \begin{cases} \Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r & \text{if } r < m \\ A^{m-1, r-m} \oplus \dots \oplus A^{r-m, m-1} & \text{if } m \leq r < 2m \\ A^{r-m, m} \oplus \dots \oplus A^{m, r-m} & \text{if } r \geq 2m. \end{cases}$$

Define $j: \mathbb{C} \xrightarrow{(1)} \Omega^0 \oplus \bar{\Omega}^0 = \mathcal{L}_m^0$ and

(i) For $0 \leq r < m-1$,

$$\mathcal{L}_m^r \xrightarrow{D} \mathcal{L}_m^{r+1}$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$\Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r \xrightarrow{\begin{pmatrix} d & 0 & 0 \\ (-1)^{r+1} & d & (-1)^r \\ 0 & 0 & d \end{pmatrix}} \Omega^{r+1} \oplus A^r \oplus \bar{\Omega}^{r+1}.$$

(ii) For $r = m-1$,

$$\mathcal{L}_m^{m-1} \xrightarrow{D} \mathcal{L}_m^m$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$\Omega^{m-1} \oplus A^{m-2} \oplus \bar{\Omega}^{m-1} \xrightarrow{((-1)^m \ d \ (-1)^{m-1})} A^{m-1}.$$

(iii) For $m \leq r < 2m-1$,

$$\begin{array}{ccc} \mathcal{L}_m^r & \xrightarrow{D} & \mathcal{L}_m^{r+1} \\ \parallel & & \parallel \\ A^{m-1, r-m} \oplus \dots \oplus A^{r-m, m-1} & \xrightarrow{\begin{pmatrix} \partial & \bar{\partial} & 0 \\ 0 & \partial & \bar{\partial} \\ 0 & 0 & \partial \end{pmatrix}} & A^{m-1, r-m+1} \oplus \dots \oplus A^{r-m+1, m-1} \end{array}$$

(iv) For $r = 2m-1$,

$$\begin{array}{ccc} \mathcal{L}_m^{2m-1} & \xrightarrow{D} & \mathcal{L}_m^{2m} \\ \parallel & & \parallel \\ A^{m-1, m-1} & \xrightarrow{\partial\bar{\partial}} & A^{m, m} \end{array}$$

(v) For $r \geq 2m$,

$$\begin{array}{ccc} \mathcal{L}_m^r & \xrightarrow{D} & \mathcal{L}_m^{r+1} \\ \parallel & & \parallel \\ A^{r-m, m} \oplus \dots \oplus A^{m, r-m} & \xrightarrow{d} & A^{r-m+1, m} \oplus \dots \oplus A^{m, r-m+1} \end{array}$$

Actually, the part $\mathbb{C} \rightarrow \mathcal{L}_m^0 \rightarrow \dots \rightarrow \mathcal{L}_m^{2m-1}$ is the single complex associated to the truncated double complex

$$\begin{array}{ccccccc} \mathbb{C} & \xrightarrow{1} & \Omega^0 & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{m-1} \\ 1 \downarrow & & -1 \downarrow & & & & (-1)^m \downarrow \\ \bar{\Omega}^0 & \xrightarrow{1} & A^{0,0} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & A^{m-1,0} \\ d \downarrow & & \bar{\partial} \downarrow & & & & \bar{\partial} \downarrow \\ \vdots & & \vdots & & & & \vdots \\ d \downarrow & & \bar{\partial} \downarrow & & & & \bar{\partial} \downarrow \\ \bar{\Omega}^{m-1} & \xrightarrow{(-1)^{m-1}} & A^{0, m-1} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & A^{m-1, m-1} \end{array}$$

with the indicated sign conventions, and a similar observation may serve to define

$$\mathbb{C} \rightarrow \mathcal{L}_{p,q}^0 \rightarrow \dots \rightarrow \mathcal{L}_{p,q}^{p+q-1}.$$

3.2. Proposition (Bigolin [7]). *For smooth X , $(\mathcal{L}_{p,q}, D)$ is an exact sequence of sheaves.*

3.3. The Involution on \mathcal{L}_m^r . A \mathbb{C} -antilinear involution $\varphi \mapsto \varphi^*$ is defined on \mathcal{L}_m^r as follows:

(i) For $(g^r, \psi^{r-1}, \bar{h}^r)$ in $\mathcal{L}_m^r = \Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r$ ($r < m$)

$$(g^r, \psi^{r-1}, \bar{h}^r)^* := (h^r, -\bar{\psi}^{r-1}, \bar{g}^r).$$

(ii) For ψ^{r-1} in $\mathcal{L}_m^r \subset A^{r-1}$, $(\psi^{r-1})^* := -\bar{\psi}^{r-1}$ ($m \leq r < 2m$).

(iii) For ψ^r in $\mathcal{L}_m^r \subset A^r$, $(\psi^r)^* := \bar{\psi}^r$ ($r \geq 2m$).

It is obvious that $(D\varphi)^* = D(\varphi^*)$.

We denote by $\mathcal{L}_{m,\mathbb{R}}$ the sub-complex of \mathcal{L}_m^r of fixed points under $(\cdot)^*$. We set $\text{Re } \varphi := \frac{1}{2}(\varphi + \varphi^*)$. Note that a self-conjugate element of \mathcal{L}_m^r , for $r < 2m$ has pure imaginary \mathcal{C}^∞ components.

3.4. *The Morphism $\mu: \mathcal{L}_{m+1}^r \rightarrow \mathcal{L}_m^r$.* A morphism $\mu = \mu_m^r: \mathcal{L}_{m+1}^r \rightarrow \mathcal{L}_m^r$ is defined by

(i) For $r < m$, $\mathcal{L}_{m+1}^r = \Omega^r \oplus A^{r-1} \oplus \Omega^r = \mathcal{L}_m^r$ and $\mu_m^r := \text{id}$. We define $\mu = \text{id}$ on \mathbb{C} as well.

(ii) For $m \leq r < 2m$, \mathcal{L}_m^r is a direct summand of \mathcal{L}_{m+1}^r and μ_m^r is defined as the canonical projection.

(iii) For $r = 2m$,

$$\begin{array}{ccc} \mathcal{L}_{m+1}^{2m} & \xrightarrow{\mu_m^{2m}} & \mathcal{L}_m^{2m} \\ \parallel & & \parallel \\ A^{m,m-1} \oplus A^{m-1,m} & \xrightarrow{\frac{1}{2}(-\partial, \bar{\partial})} & A^{m,m}. \end{array}$$

(iv) For $r = 2m+1$,

$$\begin{array}{ccc} \mathcal{L}_{m+1}^{2m+1} & \xrightarrow{\mu_m^{2m+1}} & \mathcal{L}_m^{2m+1} \\ \parallel & & \parallel \\ A^{m,m} & \xrightarrow{\frac{1}{2}(-\partial, \bar{\partial})} & A^{m+1,m} \oplus A^{m,m+1}. \end{array}$$

(v) For $r > 2m+1$, \mathcal{L}_{m+1}^r is a direct summand of \mathcal{L}_m^r and μ_m^r is defined as the canonical inclusion.

3.4.1. Lemma. *The above morphism μ commutes with D and the involution $(\cdot)^*$.*

3.5. *Relation with the $(\bar{\partial} \oplus \partial)$ -Complex.* The $(\bar{\partial} \oplus \partial)$ -complex $(\mathcal{G}_m^{\cdot}, \bar{\partial})$ is the direct sum of the Dolbeault complex and its conjugate

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{G}_m^{-1} & \xrightarrow{j} & \mathcal{G}_m^0 & \xrightarrow{\bar{\partial}} \dots & \xrightarrow{\bar{\partial}} & \dots \\ & \parallel & & \parallel & & \parallel & \\ & \Omega^m \oplus \bar{\Omega}^m & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} & A^{m,0} \oplus A^{0,m} & \xrightarrow{\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}} \dots & \xrightarrow{\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}} & A^{m,q} \oplus A^{q,m} \xrightarrow{\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}} \dots \end{array}$$

We define on \mathcal{G}_m^{\cdot} the involution $(\varphi, \psi) \mapsto (\varphi, \psi)^* := (\bar{\psi}, \bar{\varphi})$. It is related to the $\partial\bar{\partial}$ -complex by a homotopy operator $\beta: \mathcal{L}_{m+1}^{m+q+1} \rightarrow \mathcal{G}_m^q$ and a morphism of complexes $\gamma: \mathcal{L}_m^{m+q} \rightarrow \mathcal{G}_m^q$.

3.5.1. *The Homotopy Operator $\beta: \mathcal{L}_{m+1}^{m+1} \rightarrow \mathcal{G}_m^1$.* It is defined by

(i) For $q = -1$,

$$\begin{array}{ccc} \mathcal{L}_{m+1}^m & \xrightarrow{\beta} & \mathcal{G}_m^{-1} \\ \parallel & & \parallel \\ \Omega^m \oplus A^{m-1} \oplus \bar{\Omega}^m & \xrightarrow{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & \Omega^m \oplus \bar{\Omega}^m. \end{array}$$

(ii) For $0 \leq q < m$,

$$\begin{array}{ccc} \mathcal{L}_{m+1}^{m+q+1} & \xrightarrow{\beta} & \mathcal{G}_m^q \\ \parallel & & \parallel \\ A^{m,q} \oplus \dots \oplus A^{q,m} & \xrightarrow{(-1)^{m-q} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}} & A^{m,q} \oplus A^{q,m}. \end{array}$$

(iii) For $q=m$,

$$\begin{array}{ccc} \mathcal{L}_{m+1}^{2m+1} & \xrightarrow{\beta} & \mathcal{G}_m^m \\ \parallel & & \parallel \\ A^{m,m} & \xrightarrow{\frac{1}{2}(1)} & A^{m,m} \oplus A^{m,m}. \end{array}$$

(iv) For $q > m$, $\beta: \mathcal{L}_{m+1}^{m+q+1} \rightarrow \mathcal{G}_m^q$ is defined by 0.

3.5.2. *The Morphism of Complexes* $\gamma: \mathcal{L}_m^{m+1} \rightarrow \mathcal{G}_m^1$. It is defined by

(i) For $q = -1$,

$$\begin{array}{ccc} \mathcal{L}_m^{m-1} & \xrightarrow{\gamma} & \mathcal{G}_m^{-1} \\ \parallel & & \parallel \\ A^{m-1} \oplus A^{m-2} \oplus \bar{Q}^{m-1} & \xrightarrow{\begin{pmatrix} d & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}} & Q^m \oplus \bar{Q}^m. \end{array}$$

(ii) For $0 \leq q < m$,

$$\begin{array}{ccc} \mathcal{L}_m^{m+q} & \xrightarrow{\gamma} & \mathcal{G}_m^q \\ \parallel & & \parallel \\ A^{m-1,q} \oplus \dots \oplus A^{q,m-1} & \xrightarrow{(-1)^{m-q} \begin{pmatrix} \bar{d} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \bar{d} \end{pmatrix}} & A^{m,q} \oplus A^{q,m}. \end{array}$$

(iii) For $q \geq m$,

$$\begin{array}{ccc} \mathcal{L}_m^{m+q} & \xrightarrow{\gamma} & \mathcal{G}_m^q \\ \parallel & & \parallel \\ A^{q,m} \oplus \dots \oplus A^{m,q} & \xrightarrow{\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & \dots & 0 & 0 \end{pmatrix}} & A^{m,q} \oplus A^{q,m}. \end{array}$$

In particular, for $q = m$, $\gamma(\alpha^{m,m}) = (\alpha^{m,m}, -\alpha^{m,m})$.

The following can be easily checked.

3.5.3. **Lemma.** (i) $\hat{d}\beta + \beta D = \gamma\mu$.

(ii) $\hat{d}\gamma = \gamma D$.

(iii) If $\eta^{m,m}$ and $\zeta^{m,m}$ are (m,m) -forms, then $\beta(\eta^{m,m}) + \gamma(\zeta^{m,m}) = (\varrho^{m,m}, \sigma^{m,m})$ where $\varrho^{m,m} + \sigma^{m,m} = \eta^{m,m}$.

(iv) β and γ anticommute with the involutions $(\cdot)^*$

$$\begin{array}{ccccc} & & \mathcal{L}_m^{m+q-1} & & \\ & & \downarrow & & \\ \mathcal{L}_{m+1}^{m+q} & \xrightarrow{\beta} & \mathcal{G}_m^{q-1} & & \\ \downarrow \mu & & \downarrow D & & \downarrow \hat{d} \\ \mathcal{L}_m^{m+q} & & & & \mathcal{G}_m^q \\ \downarrow D & & & & \downarrow \beta \\ \mathcal{L}_{m+1}^{m+q+1} & \xrightarrow{\beta} & \mathcal{G}_m^q & & \end{array}$$

3.6. *The Čech Transform of the $\partial\bar{\partial}$ -Complex.* For any Čech space \underline{X} , we denote by $\mathcal{E}_m^q(\underline{X})$, $\mathcal{E}_m^q(\underline{X}, [\mathbb{R}])$ and $\mathcal{E}_m^q(\underline{X}, \mathbb{R})$ the Čech transforms of the complexes

$$\begin{aligned} 0 \rightarrow \mathbb{C} \rightarrow \mathcal{L}_m^0 \rightarrow \mathcal{L}_m^1 \rightarrow \dots, \\ 0 \rightarrow \mathbb{R} \rightarrow \mathcal{L}_m^0 \rightarrow \mathcal{L}_m^1 \rightarrow \dots, \\ 0 \rightarrow \mathbb{R} \rightarrow \mathcal{L}_{m,\mathbb{R}}^0 \rightarrow \mathcal{L}_{m,\mathbb{R}}^1 \rightarrow \dots, \end{aligned}$$

respectively. So we set

$$(3.6.1) \quad \begin{aligned} \text{(i)} \quad \mathcal{E}_m^q(\underline{X}) &:= \check{C}^q(\underline{X}; \mathbb{C}, \mathcal{L}_m^{\cdot}), \\ \text{(ii)} \quad \mathcal{E}_m^q(\underline{X}, [\mathbb{R}]) &:= \check{C}^q(\underline{X}; \mathbb{R}, \mathcal{L}_m^{\cdot}), \\ \text{(iii)} \quad \mathcal{E}_m^q(\underline{X}, \mathbb{R}) &:= \check{C}^q(\underline{X}; \mathbb{R}, \mathcal{L}_{m,\mathbb{R}}^{\cdot}). \end{aligned}$$

Of course, $\mathcal{E}_m^q(\underline{X}, \mathbb{R}) \subset \mathcal{E}_m^q(\underline{X}, [\mathbb{R}]) \subset \mathcal{E}_m^q(\underline{X})$. Elements of $\mathcal{E}_m^q(\underline{X})$ will be written in a matrix form. For example an element of $\mathcal{E}_m^{2m}(\underline{X})$ will be written as

$$(3.6.2) \quad \Phi = \begin{array}{|c|ccc|} \hline a & g^0 & \dots & g^{m-1} \\ \hline \bar{h}^0 & \varphi^{0,0} & \dots & \varphi^{m-1,0} \\ \vdots & \vdots & & \vdots \\ \bar{h}^{m-1} & \varphi^{0,m-1} & \dots & \varphi^{m-1,m-1} \\ \hline & & & \eta^{m,m} \\ \end{array}$$

where

$$\begin{aligned} a &\in C^{2m}(\underline{X}, \mathbb{C}) \\ g^k &\in C^{2m-k-1}(\underline{X}, \Omega^k) \\ \bar{h}^l &\in C^{2m-l-1}(\underline{X}, \bar{\Omega}^l) \\ \varphi^{k,l} &\in C^{2m-k-l-2}(\underline{X}, A^{k,l}) \\ \eta^{m,m} &\in H^0(X, A^{m,m}) \end{aligned}$$

a is the head and $\eta^{m,m}$ the tail of Φ .

$$\begin{aligned} \Phi \in \mathcal{E}_m^{2m}(\underline{X}, [\mathbb{R}]) &\quad \text{iff} \quad a = \bar{a} \in C^{2m}(\underline{X}, \mathbb{R}) \\ \Phi \in \mathcal{E}_m^{2m}(\underline{X}, \mathbb{R}) &\quad \text{iff} \quad a = \bar{a}, \end{aligned}$$

$$g^k = h^k, \varphi^{k,l} + \bar{\varphi}^{l,k} = 0 \text{ and } \eta^{m,m} = \bar{\eta}^{m,m}.$$

If we apply $\Delta: \mathcal{E}_m^{2m}(\underline{X}) \rightarrow \mathcal{E}_m^{2m+1}(\underline{X})$ we obtain

$$\Delta \Phi = \begin{array}{|c|ccc|} \hline b & u^0 & \dots & u^{m-1} \\ \hline \bar{v}^0 & \psi^{0,0} & \dots & \psi^{m-1,0} \\ \vdots & \vdots & & \vdots \\ \bar{v}^{m-1} & \psi^{0,m-1} & \dots & \psi^{m-1,m-1} \\ \hline & & & \psi^{m,m} & \lambda^{m+1,m} \\ & & & \lambda^{m,m+1} & \\ \end{array}$$

where

- (i) $b = \delta a$
- (ii) $u^0 = \delta g^0 - a$
- (iii) $u^k = \delta g^k - dg^{k-1}$ for $1 \leq k < m$
- (iv) $\bar{v}^0 = \delta \bar{h}^0 - a$
- (v) $\bar{v}^l = \delta \bar{h}^l - d\bar{h}^{l-1}$ for $1 \leq l < m$
- (vi) $\psi^{0,0} = \delta \varphi^{0,0} + g^0 - \bar{h}^0$
- (vii) $\psi^{k,0} = \delta \varphi^{k,0} + (-1)^k g^k - \delta \varphi^{k-1,0}$ for $1 \leq k < m$
- (viii) $\psi^{0,l} = \delta \varphi^{0,l} - \bar{\delta} \varphi^{0,l-1} + (-1)^{l-1} \bar{h}^l$ for $1 \leq l < m$
- (ix) $\psi^{k,l} = \delta \varphi^{k,l} - \bar{\delta} \varphi^{k,l-1} - \delta \varphi^{k-1,l}$ for $1 \leq k, l < m$
- (x) $\psi^{m,m} = \varepsilon(\eta^{m,m}) - \delta \bar{\delta} \varphi^{m-1,m-1}$
- (xi) $\lambda^{m+1,m} = \partial \eta^{m,m}$
- (xii) $\lambda^{m,m+1} = \bar{\delta} \eta^{m,m}$.

In the next section we construct, for any Kähler space (X, ω) , an open covering \mathcal{X} such that on the resulting Čech space \underline{X} and for any integer $m > 0$, there is a cocycle in $\mathcal{E}_m^{2m}(\underline{X}, [\mathbb{R}])$ whose tail is ω^m .

4. The Čech Cochains Associated to a Kähler Metric

We first note that, if X is a Kähler space, it admits by definition an open covering (U_α) such that there are elements $\varphi_\alpha \in SP^\infty(U_\alpha)$ such that $\varphi_\alpha - \varphi_\beta$ is locally the real part of a holomorphic function on $U_\alpha \cap U_\beta$. We show that “locally” can be omitted.

4.1. Covering Lemma. *Let X be a paracompact topological space and $(U_\alpha)_{\alpha \in A}$ an open covering of X such that, for every $\alpha, \beta \in A$, $(U_{\alpha\beta}^j)_{j \in J_{\alpha\beta}}$ is an open covering of $U_\alpha \cap U_\beta$. Let $J = \bigcup_{\alpha, \beta} J_{\alpha\beta}$.*

Then there exists a refinement

$$\mathcal{X} = (X_\lambda)_{\lambda \in \Lambda}$$

of (U_α) together with two maps

$$\begin{aligned} \alpha: A &\rightarrow \Lambda \\ j: A \times A &\rightarrow J \end{aligned}$$

such that

- (i) $X_\lambda \subset U_{\alpha(\lambda)}$
- (ii) $X_\lambda \cap X_\mu \subset U_{\alpha(\lambda)\alpha(\mu)}^{j(\lambda, \mu)}$.