2.3. Corollary. *If a cocycle* $\Phi \in \check{Z}^q(X; \mathscr{F}, \mathscr{L}^{\cdot})$ *has a head that is* δ*-exact and if* $H^{q-k}(\underline{X}, \mathscr{L}^{k-1}) = 0$ *for* $k = 1, \ldots, q-1$, *then* Φ *is* Δ*-exact and, in particular, the tail of* Φ *is D-exact.*

2.4. Remark. Definition 3.4.3 of Chap. I can be restated as follows: A $\bar{\partial}$-closed form $\tau \in A^{k,l}(X)$ is said to represent an element of $H^l(X, \Omega^k)$ if there is a cocycle of degree l

$$c \in \check{Z}^l(\underline{X}; \Omega^k, A^{k,\cdot})$$

of the Čech transform of the Dolbeault complex whose tail is τ, for some open covering of X.

3. The $\partial\bar{\partial}$*-Complex* $\mathscr{L}^{\cdot}_m$

Let X be a complex space. For any pair (p, q) of natural integers, there is a complex of sheaves on X of the form

$$\begin{array}{ccccccccccc} 0 \longrightarrow \mathbb{C} \xrightarrow{\lambda} \mathscr{L}^0_{p,q} \xrightarrow{D} \ldots \xrightarrow{D} & \mathscr{L}^{p+q-1}_{p,q} & \xrightarrow{D} & \mathscr{L}^{p+q}_{p,q} & \xrightarrow{D} \ldots \\ & \| & & \| & \\ & A_X^{p-1,q-1} & \xrightarrow{\partial\bar{\partial}} & A_X^{p,q} & \end{array}$$

defined in [7].

We will deal exclusively with the case $p = q$, so we write $\mathscr{L}^r_m$ for $\mathscr{L}^r_{m,m}$. The complex $\mathscr{L}^{\cdot}_m$ defined as follows (the suffix X will be omitted).

3.1. Definitions.

$$\mathscr{L}^r_m := \begin{cases} \Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r & \text{if } r < m \\ A^{m-1,r-m} \oplus \ldots \oplus A^{r-m,m-1} & \text{if } m \leqq r < 2m \\ A^{r-m,m} \oplus \ldots \oplus A^{m,r-m} & \text{if } r \geqq 2m. \end{cases} \tag{3.1}$$

Define $j : \mathbb{C} \xrightarrow{\binom{1}{1}} \Omega^0 \oplus \bar{\Omega}^0 = \mathscr{L}^0_m$ and

(i) For $0 \leqq r < m-1$,

$$\begin{array}{ccc} \mathscr{L}^r_m & \xrightarrow{\quad D \quad} & \mathscr{L}^{r+1}_m \\ \| & \begin{pmatrix} d & 0 & 0 \\ (-1)^{r+1} & d & (-1)^r \\ 0 & 0 & d \end{pmatrix} & \| \\ \Omega^r \oplus A^{r-1} \oplus \bar{\Omega}^r & \xrightarrow{\qquad\qquad} & \Omega^{r+1} \oplus A^r \oplus \bar{\Omega}^{r+1}. \end{array}$$

(ii) For $r = m-1$,

$$\begin{array}{ccc} \mathscr{L}^{m-1}_m & \xrightarrow{\quad D \quad} & \mathscr{L}^m_m \\ \| & & \| \\ \Omega^{m-1} \oplus A^{m-2} \oplus \bar{\Omega}^{m-1} & \xrightarrow{((-1)^m \;\; d \;\; (-1)^{m-1})} & A^{m-1}. \end{array}$$

(iii) For $m \leqq r < 2m-1$,

$$\begin{array}{ccc}
\mathscr{L}_m^r & \xrightarrow{\quad D \quad} & \mathscr{L}_m^{r+1} \\
\| & \begin{pmatrix} \partial & \bar\partial & & 0 \\ 0 & \partial & \ddots & \\ & & \ddots & \bar\partial \quad 0 \\ 0 & & \bar\partial & \partial \end{pmatrix} & \| \\
A^{m-1,r-m} \oplus \ldots \oplus A^{r-m,m-1} & \longrightarrow & A^{m-1,r-m+1} \oplus \ldots \oplus A^{r-m+1,m-1} .
\end{array}$$

(iv) For $r = 2m-1$,

$$\begin{array}{ccc}
\mathscr{L}_m^{2m-1} & \xrightarrow{D} & \mathscr{L}_m^{2m} \\
\| & & \| \\
A^{m-1,m-1} & \xrightarrow{\partial\bar\partial} & A^{m,m} .
\end{array}$$

(v) For $r \geqq 2m$,

$$\begin{array}{ccc}
\mathscr{L}_m^r & \xrightarrow{D} & \mathscr{L}_m^{r+1} \\
\| & & \| \\
A^{r-m,m} \oplus \ldots \oplus A^{m,r-m} & \xrightarrow{d} & A^{r-m+1,m} \oplus \ldots \oplus A^{m,r-m+1} .
\end{array}$$

Actually, the part $\mathbb{C} \to \mathscr{L}_m^0 \to \ldots \to \mathscr{L}_m^{2m-1}$ is the single complex associated to the truncated double complex

$$\begin{array}{ccccccc}
\mathbb{C} & \xrightarrow{\quad 1 \quad} & \Omega^0 & \xrightarrow{\quad d \quad} & \ldots & \xrightarrow{\quad d \quad} & \Omega^{m-1} \\
{\scriptstyle 1}\downarrow & & {\scriptstyle -1}\downarrow & & & & {\scriptstyle (-1)^m}\downarrow \\
\bar\Omega^0 & \xrightarrow{\quad 1 \quad} & A^{0,0} & \xrightarrow{\quad \partial \quad} & \ldots & \xrightarrow{\quad \partial \quad} & A^{m-1,0} \\
{\scriptstyle d}\downarrow & & {\scriptstyle \bar\partial}\downarrow & & & & {\scriptstyle \bar\partial}\downarrow \\
\vdots & & \vdots & & & & \vdots \\
{\scriptstyle d}\downarrow & & {\scriptstyle \bar\partial}\downarrow & & & & {\scriptstyle \bar\partial}\downarrow \\
\bar\Omega^{m-1} & \xrightarrow{(-1)^{m-1}} & A^{0,m-1} & \xrightarrow{\quad \partial \quad} & \ldots & \xrightarrow{\quad \partial \quad} & A^{m-1,m-1}
\end{array}$$

with the indicated sign conventions, and a similar observation may serve to define

$$\mathbb{C} \to \mathscr{L}_{p,q}^0 \to \ldots \mathscr{L}_{p,q}^{p+q-1} .$$

3.2. Proposition (Bigolin [7]). *For smooth X, $(\mathscr{L}_{p,q}^{\cdot}, D)$ is an exact sequence of sheaves.*

3.3. The Involution on $\mathscr{L}_m^{\cdot}$. A $\mathbb{C}$-antilinear involution $\varphi \mapsto \varphi^*$ is defined on $\mathscr{L}_m^{\cdot}$ as follows:

(i) For $(g^r, \psi^{r-1}, \bar h^r)$ in $\mathscr{L}_m^r = \Omega^r \oplus A^{r-1} \oplus \bar\Omega^r$ $(r < m)$

$$(g^r, \psi^{r-1}, \bar h^r)^* := (h^r, -\bar\psi^{r-1}, \bar g^r) .$$

(ii) For ψ^{r-1} in $\mathscr{L}_m^r \subset A^{r-1}$, $(\psi^{r-1})^* := -\bar\psi^{r-1}$ $(m \leqq r < 2m)$.

(iii) For ψ^r in $\mathscr{L}_m^r \subset A^r$, $(\psi^r)^* := \bar\psi^r$ $(r \geqq 2m)$.

It is obvious that $(D\varphi)^* = D(\varphi^*)$.

We denote by $\mathscr{L}_{m,\mathbb{R}}^{\cdot}$ the sub-complex of $\mathscr{L}_m^{\cdot}$ of fixed points under $(\cdot)^*$. We set $\mathrm{Re}\,\varphi := \frac{1}{2}(\varphi + \varphi^*)$. Note that a self-conjugate element of $\mathscr{L}_m^r$, for $r < 2m$ has pure imaginary $\mathscr{C}^\infty$ components.

3.4. The Morphism $\mu:\mathscr{L}^{\cdot}_{m+1}\to\mathscr{L}^{\cdot}_m$. A morphism $\mu=\mu^r_m:\mathscr{L}^r_{m+1}\to\mathscr{L}^r_m$ is defined by

(i) For $r<m$, $\mathscr{L}^r_{m+1}=\Omega^r\oplus A^{r-1}\oplus\bar{\Omega}^r=\mathscr{L}^r_m$ and $\mu^r_m:=\mathrm{id}$. We define $\mu=\mathrm{id}$ on $\mathbb{C}$ as well.

(ii) For $m\leqq r<2m$, $\mathscr{L}^r_m$ is a direct summand of $\mathscr{L}^r_{m+1}$ and μ^r_m is defined as the canonical projection.

(iii) For $r=2m$,

$$\begin{array}{ccc}
\mathscr{L}^{2m}_{m+1} & \xrightarrow{\quad\mu^{2m}_m\quad} & \mathscr{L}^{2m}_m \\
\Vert & & \Vert \\
A^{m,m-1}\oplus A^{m-1,m} & \xrightarrow{\frac{1}{2}(-\partial\ \ \bar{\partial})} & A^{m,m}.
\end{array}$$

(iv) For $r=2m+1$,

$$\begin{array}{ccc}
\mathscr{L}^{2m+1}_{m+1} & \xrightarrow{\mu^{2m+1}_m} & \mathscr{L}^{2m+1}_m \\
\Vert & & \Vert \\
A^{m,m} & \xrightarrow{\frac{1}{2}\begin{pmatrix}-\partial\\ \partial\end{pmatrix}} & A^{m+1,m}\oplus A^{m,m+1}.
\end{array}$$

(v) For $r>2m+1$, $\mathscr{L}^r_{m+1}$ is a direct summand of $\mathscr{L}^r_m$ and μ^r_m is defined as the canonical inclusion.

3.4.1. Lemma. *The above morphism μ commutes with D and the involution $(\cdot)^*$.*

3.5. Relation with the $(\bar{\partial}\oplus\partial)$*-Complex.* The $(\bar{\partial}\oplus\partial)$-complex $(\mathscr{G}^{\cdot}_m,\hat{d})$ is the direct sum of the Dolbeault complex and its conjugate

$$\begin{array}{ccccccccc}
0\longrightarrow & \mathscr{G}^{-1}_m & \xrightarrow{\quad j\quad} & \mathscr{G}^0_m & \xrightarrow{\quad\hat{d}\quad}\dots\xrightarrow{\quad\hat{d}\quad} & \mathscr{G}^q_m & \xrightarrow{\quad\hat{d}\quad}\dots \\
 & \Vert & & \Vert & & \Vert & \\
 & \Omega^m\oplus\bar{\Omega}^m & \xrightarrow{\begin{pmatrix}1&0\\0&1\end{pmatrix}} & A^{m,0}\oplus A^{0,m} & \xrightarrow{\begin{pmatrix}\bar{\partial}&0\\0&\partial\end{pmatrix}}\dots\xrightarrow{\begin{pmatrix}\bar{\partial}&0\\0&\partial\end{pmatrix}} & A^{m,q}\oplus A^{q,m} & \xrightarrow{\begin{pmatrix}\bar{\partial}&0\\0&\partial\end{pmatrix}}\dots
\end{array}$$

We define on $\mathscr{G}^{\cdot}_m$ the involution $(\varphi,\psi)\mapsto(\varphi,\psi)^*:=(\bar{\psi},\bar{\varphi})$. It is related to the $\partial\bar{\partial}$-complex by a homotopy operator $\beta:\mathscr{L}^{m+q+1}_{m+1}\to\mathscr{G}^q_m$ and a morphism of complexes $\gamma:\mathscr{L}^{m+q}_m\to\mathscr{G}^q_m$.

3.5.1. The Homotopy Operator $\beta:\mathscr{L}^{m+1+\cdot}_{m+1}\to\mathscr{G}^{\cdot}_m$. It is defined by

(i) For $q=-1$,

$$\begin{array}{ccc}
\mathscr{L}^m_{m+1} & \xrightarrow{\quad\beta\quad} & \mathscr{G}^{-1}_m \\
\Vert & & \Vert \\
\Omega^m\oplus A^{m-1}\oplus\bar{\Omega}^m & \xrightarrow{\begin{pmatrix}-1&0&0\\0&0&1\end{pmatrix}} & \Omega^m\oplus\bar{\Omega}^m.
\end{array}$$

(ii) For $0\leqq q<m$,

$$\begin{array}{ccc}
\mathscr{L}^{m+q+1}_{m+1} & \xrightarrow{\quad\beta\quad} & \mathscr{G}^q_m \\
\Vert & & \Vert \\
A^{m,q}\oplus\dots\oplus A^{q,m} & \xrightarrow{(-1)^{m-q}\begin{pmatrix}1&0\dots0&0\\0&0\dots0&1\end{pmatrix}} & A^{m,q}\oplus A^{q,m}.
\end{array}$$

(iii) For $q=m$,

$$\begin{array}{ccc} \mathscr{L}^{2m+1}_{m+1} & \xrightarrow{\beta} & \mathscr{G}^{m}_{m} \\ \| & & \| \\ A^{m,m} & \xrightarrow{\frac{1}{2}\binom{1}{1}} & A^{m,m}\oplus A^{m,m}. \end{array}$$

(iv) For $q>m$, $\beta: \mathscr{L}^{m+q+1}_{m+1}\to\mathscr{G}^{q}_{m}$ is defined by 0.

3.5.2. The Morphism of Complexes $\gamma: \mathscr{L}^{m+\cdot}_{m}\to\mathscr{G}^{\cdot}_{m}$. It is defined by

(i) For $q=-1$,

$$\begin{array}{ccc} \mathscr{L}^{m-1}_{m} & \xrightarrow{\gamma} & \mathscr{G}^{-1}_{m} \\ \| & & \| \\ \Omega^{m-1}\oplus A^{m-2}\oplus\bar{\Omega}^{m-1} & \xrightarrow{\begin{pmatrix} d & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}} & \Omega^{m}\oplus\bar{\Omega}^{m}. \end{array}$$

(ii) For $0\leqq q<m$,

$$\begin{array}{ccc} \mathscr{L}^{m+q}_{m} & \xrightarrow{\gamma} & \mathscr{G}^{q}_{m} \\ \| & & \| \\ A^{m-1,q}\oplus\ldots\oplus A^{q,m-1} & \xrightarrow{(-1)^{m-q}\begin{pmatrix} \partial & 0\ldots 0 & 0 \\ 0 & 0\ldots 0 & \bar{\partial} \end{pmatrix}} & A^{m,q}\oplus A^{q,m}. \end{array}$$

(iii) For $q\geqq m$,

$$\begin{array}{ccc} \mathscr{L}^{m+q}_{m} & \xrightarrow{\gamma} & \mathscr{G}^{q}_{m} \\ \| & & \| \\ A^{q,m}\oplus\ldots\oplus A^{m,q} & \xrightarrow{\begin{pmatrix} 0 & 0\ldots 0 & 1 \\ -1 & 0\ldots 0 & 0 \end{pmatrix}} & A^{m,q}\oplus A^{q,m}. \end{array}$$

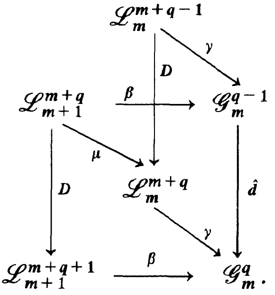
In particular, for $q=m$, $\gamma(\alpha^{m,m})=(\alpha^{m,m}, -\alpha^{m,m})$.
The following can be easily checked.

3.5.3. Lemma. (i) $\hat{d}\beta+\beta D=\gamma\mu$.

(ii) $\hat{d}\gamma=\gamma D$.

(iii) *If* $\eta^{m,m}$ *and* $\zeta^{m,m}$ *are* (m,m)*-forms, then* $\beta(\eta^{m,m})+\gamma(\zeta^{m,m})=(\varrho^{m,m},\sigma^{m,m})$ *where* $\varrho^{m,m}+\sigma^{m,m}=\eta^{m,m}$.

(iv) β *and* γ *anticommute with the involutions* $(\cdot)^*$



3.6. The Čech Transform of the $\partial\bar\partial$-Complex. For any Čech space $\underline{X}$, we denote by $\mathscr{E}_m^{\cdot}(\underline{X})$, $\mathscr{E}_m^{\cdot}(\underline{X}, [\mathbb{R}])$ and $\mathscr{E}_m^{\cdot}(\underline{X}, \mathbb{R})$ the Čech transforms of the complexes

$$0 \to \mathbb{C} \to \mathscr{L}_m^0 \to \mathscr{L}_m^1 \to \dots,$$
$$0 \to \mathbb{R} \to \mathscr{L}_m^0 \to \mathscr{L}_m^1 \to \dots,$$
$$0 \to \mathbb{R} \to \mathscr{L}_{m,\mathbb{R}}^0 \to \mathscr{L}_{m,\mathbb{R}}^1 \to \dots,$$

respectively. So we set

$$\text{(3.6.1)} \qquad \begin{aligned} &\text{(i)} && \mathscr{E}_m^q(\underline{X}) := \check{C}^q(\underline{X}; \mathbb{C}, \mathscr{L}_m^{\cdot}),\\ &\text{(ii)} && \mathscr{E}_m^q(\underline{X}, [\mathbb{R}]) := \check{C}^q(\underline{X}; \mathbb{R}, \mathscr{L}_m^{\cdot}),\\ &\text{(iii)} && \mathscr{E}_m^q(\underline{X}, \mathbb{R}) := \check{C}^q(\underline{X}; \mathbb{R}, \mathscr{L}_{m,\mathbb{R}}^{\cdot}). \end{aligned}$$

Of course, $\mathscr{E}_m^q(\underline{X}, \mathbb{R}) \subset \mathscr{E}_m^q(X, [\mathbb{R}]) \subset \mathscr{E}_m^q(X)$. Elements of $\mathscr{E}_m^q(\underline{X})$ will be written in a matrix form. For example an element of $\mathscr{E}_m^{2m}(\underline{X})$ will be written as

$$\text{(3.6.2)} \qquad \Phi = \begin{array}{|c|ccc|c} \hline a & g^0 & \dots & g^{m-1} & \\ \hline \bar h^0 & \varphi^{0,0} & \dots & \varphi^{m-1,0} & \\ \vdots & \vdots & & \vdots & \\ \bar h^{m-1} & \varphi^{0,m-1} & \dots & \varphi^{m-1,m-1} & \\ \hline \multicolumn{4}{c|}{} & \eta^{m,m} \\ \cline{5-5} \end{array}$$

where

$$a \in C^{2m}(\underline{X}, \mathbb{C})$$
$$g^k \in C^{2m-k-1}(\underline{X}, \Omega^k)$$
$$\bar h^l \in C^{2m-l-1}(\underline{X}, \bar\Omega^l)$$
$$\varphi^{k,l} \in C^{2m-k-l-2}(\underline{X}, A^{k,l})$$
$$\eta^{m,m} \in H^0(X, A^{m,m})$$

a is the head and $\eta^{m,m}$ the tail of Φ.

$$\Phi \in \mathscr{E}_m^{2m}(\underline{X}, [\mathbb{R}]) \quad \text{iff} \quad a = \bar a \in C^{2m}(\underline{X}, \mathbb{R})$$
$$\Phi \in \mathscr{E}_m^{2m}(\underline{X}, \mathbb{R}) \qquad \text{iff} \quad a = \bar a,$$

$g^k = h^k$, $\varphi^{k,l} + \bar\varphi^{l,k} = 0$ and $\eta^{m,m} = \bar\eta^{m,m}$.

If we apply $\Delta: \mathscr{E}_m^{2m}(\underline{X}) \to \mathscr{E}_m^{2m+1}(\underline{X})$ we obtain

$$\Delta\Phi = \begin{array}{|c|ccc|c|c|} \hline b & u^0 & \dots & u^{m-1} & \multicolumn{2}{c}{} \\ \cline{1-4} \bar v^0 & \psi^{0,0} & \dots & \psi^{m-1,0} & \multicolumn{2}{c}{} \\ \vdots & \vdots & & \vdots & \multicolumn{2}{c}{} \\ \bar v^{m-1} & \psi^{0,m-1} & \dots & \psi^{m-1,m-1} & \multicolumn{2}{c}{} \\ \hline \multicolumn{4}{c|}{} & \psi^{m,m} & \lambda^{m+1,m} \\ \cline{5-6} \multicolumn{4}{c|}{} & \lambda^{m,m+1} & \multicolumn{1}{c}{} \\ \cline{5-5} \end{array}$$

where

$$(3.6.3)\quad \begin{aligned} &\text{(i)} && b=\delta a\\ &\text{(ii)} && u^0=\delta g^0-a\\ &\text{(iii)} && u^k=\delta g^k-dg^{k-1} \quad \text{for} \quad 1\leqq k<m\\ &\text{(iv)} && \bar{v}^0=\delta\bar{h}^0-a\\ &\text{(v)} && \bar{v}^l=\delta\bar{h}^l-d\bar{h}^{l-1} \quad \text{for} \quad 1\leqq l<m\\ &\text{(vi)} && \psi^{0,0}=\delta\varphi^{0,0}+g^0-\bar{h}^0\\ &\text{(vii)} && \psi^{k,0}=\delta\varphi^{k,0}+(-1)^k g^k-\partial\varphi^{k-1,0} \quad \text{for} \quad 1\leqq k<m\\ &\text{(viii)} && \psi^{0,l}=\delta\varphi^{0,l}-\bar{\partial}\varphi^{0,l-1}+(-1)^{l-1}\bar{h}^l \quad \text{for} \quad 1\leqq l<m\\ &\text{(ix)} && \psi^{k,l}=\delta\varphi^{k,l}-\bar{\partial}\varphi^{k,l-1}-\partial\varphi^{k-1,l} \quad \text{for} \quad 1\leqq k,l<m\\ &\text{(x)} && \psi^{m,m}=\varepsilon(\eta^{m,m})-\partial\bar{\partial}\varphi^{m-1,m-1}\\ &\text{(xi)} && \lambda^{m+1,m}=\partial\eta^{m,m}\\ &\text{(xii)} && \lambda^{m,m+1}=\bar{\partial}\eta^{m,m}. \end{aligned}$$

In the next section we construct, for any Kähler space (X,ω), an open covering $\mathscr{X}$ such that on the resulting Čech space $\underline{X}$ and for any integer $m>0$, there is a cocycle in $\mathscr{E}_m^{2m}(\underline{X},[\mathbb{R}])$ whose tail is ω^m.

4. The Čech Cochains Associated to a Kähler Metric

We first note that, if X is a Kähler space, it admits by definition an open covering (U_α) such that there are elements $\varphi_\alpha\in SP^\infty(U_\alpha)$ such that $\varphi_\alpha-\varphi_\beta$ is *locally* the real part of a holomorphic function on $U_\alpha\cap U_\beta$. We show that "locally" can be omitted.

4.1. Covering Lemma. *Let X be a paracompact topological space and $(U_\alpha)_{\alpha\in A}$ an open covering of X such that, for every $\alpha,\beta\in A$, $(U^j_{\alpha\beta})_{j\in J_{\alpha\beta}}$ is an open covering of $U_\alpha\cap U_\beta$. Let $J=\bigcup_{\alpha,\beta} J_{\alpha\beta}$.*

Then there exists a refinement

$$\mathscr{X}=(X_\lambda)_{\lambda\in\Lambda}$$

of (U_α) together with two maps

$$\alpha:\Lambda\to A$$

$$j:\Lambda\times\Lambda\to J$$

such that

$$(4.1.1)\quad \begin{aligned} &\text{(i)} && X_\lambda\subset U_{\alpha(\lambda)}\\ &\text{(ii)} && X_\lambda\cap X_\mu\subset U^{j(\lambda,\mu)}_{\alpha(\lambda)\alpha(\mu)}. \end{aligned}$$