where

$$(3.6.3)\quad \begin{array}{rl}
\text{(i)} & b=\delta a \\
\text{(ii)} & u^0=\delta g^0-a \\
\text{(iii)} & u^k=\delta g^k-dg^{k-1} \quad \text{for} \quad 1\leqq k<m \\
\text{(iv)} & \bar{v}^0=\delta \bar{h}^0-a \\
\text{(v)} & \bar{v}^l=\delta \bar{h}^l-d\bar{h}^{l-1} \quad \text{for} \quad 1\leqq l<m \\
\text{(vi)} & \psi^{0,0}=\delta\varphi^{0,0}+g^0-\bar{h}^0 \\
\text{(vii)} & \psi^{k,0}=\delta\varphi^{k,0}+(-1)^k g^k-\partial\varphi^{k-1,0} \quad \text{for} \quad 1\leqq k<m \\
\text{(viii)} & \psi^{0,l}=\delta\varphi^{0,l}-\bar{\partial}\varphi^{0,l-1}+(-1)^{l-1}\bar{h}^l \quad \text{for} \quad 1\leqq l<m \\
\text{(ix)} & \psi^{k,l}=\delta\varphi^{k,l}-\bar{\partial}\varphi^{k,l-1}-\partial\varphi^{k-1,l} \quad \text{for} \quad 1\leqq k,l<m \\
\text{(x)} & \psi^{m,m}=\varepsilon(\eta^{m,m})-\partial\bar{\partial}\varphi^{m-1,m-1} \\
\text{(xi)} & \lambda^{m+1,m}=\partial\eta^{m,m} \\
\text{(xii)} & \lambda^{m,m+1}=\bar{\partial}\eta^{m,m}.
\end{array}$$

In the next section we construct, for any Kähler space (X, ω), an open covering $\mathscr{X}$ such that on the resulting Čech space $\underline{X}$ and for any integer $m>0$, there is a cocycle in $\mathscr{E}_m^{2m}(\underline{X}, [\mathbb{R}])$ whose tail is ω^m.

4. *The Čech Cochains Associated to a Kähler Metric*

We first note that, if X is a Kähler space, it admits by definition an open covering (U_α) such that there are elements $\varphi_\alpha \in SP^\infty(U_\alpha)$ such that $\varphi_\alpha-\varphi_\beta$ is *locally* the real part of a holomorphic function on $U_\alpha \cap U_\beta$. We show that "locally" can be omitted.

4.1. Covering Lemma. *Let X be a paracompact topological space and $(U_\alpha)_{\alpha\in A}$ an open covering of X such that, for every $\alpha, \beta \in A$, $(U^j_{\alpha\beta})_{j\in J_{\alpha\beta}}$ is an open covering of $U_\alpha \cap U_\beta$. Let $J=\bigcup_{\alpha,\beta} J_{\alpha\beta}$.*

Then there exists a refinement

$$\mathscr{X}=(X_\lambda)_{\lambda\in\Lambda}$$

of (U_α) together with two maps

$$\alpha:\Lambda\to A$$

$$j:\Lambda\times\Lambda\to J$$

such that

$$(4.1.1)\quad \begin{array}{rl}
\text{(i)} & X_\lambda \subset U_{\alpha(\lambda)} \\
\text{(ii)} & X_\lambda\cap X_\mu \subset U^{j(\lambda,\mu)}_{\alpha(\lambda)\alpha(\mu)}.
\end{array}$$

Proof. Since X paracompact, (U_α) admits a refinement $(\bar{V}_\alpha)_{\alpha\in A}$ indexed by the same set A such that $\bar{V}_\alpha \subset U_\alpha$ and $(\bar{V}_\alpha)$ is locally finite. Let Λ be the set of all multi-indices

$$\lambda = (\alpha_0, \ldots, \alpha_s; j_0, \ldots, j_s) \quad (s \in \mathbb{N}) \tag{4.1.2}$$

such that the α_r are pairwise distinct elements of A and $j_r \in J_{\alpha_0\alpha_r}$ for $0 \leqq r \leqq s$. Set

$$X_\lambda := V_{\alpha_0} \cap \bigcap_{r=0}^{s} U^{j_r}_{\alpha_0\alpha_r} \Big\backslash \bigcup_{\beta \neq \alpha_0, \ldots, \alpha_s} \bar{V}_\beta . \tag{4.1.3}$$

X_λ is open since $(\bar{V}_\beta)$ is locally finite.

Define $\alpha(\lambda) := \alpha_0$. Then obviously $X_\lambda \subset U_{\alpha(\lambda)}$.

Now suppose that

$$\mu = (\beta_0, \ldots, \beta_t; k_0, \ldots, k_t)$$

is a multi-index in Λ such that $X_\lambda \cap X_\mu \neq \emptyset$. Then β_0 must be equal to one (and only one) of the α_r for otherwise $\bar{V}_{\beta_0} \cap X_\lambda$ would be empty by construction of X_λ. If $\beta_0 = \alpha_r$, set

$$j(\lambda, \mu) := j_r .$$

It is clear that

$$X_\lambda \cap X_\mu \subset U^{j_r}_{\alpha_0\alpha_r} = U^{j_r}_{\alpha_0\beta_0} = U^{j(\lambda,\mu)}_{\alpha(\lambda)\alpha(\mu)}$$

as required. Finally it is true that the X_λ $(\lambda \in \Lambda)$ cover X; for if $x \in X$ is arbitrary, take $\alpha \in A$ such that $x \in V_\alpha$. The set S of $\beta \in A$ such that $x \in \bar{V}_\beta$ is finite containing α [since $(\bar{V}_\beta)$ is locally finite]; let

$$S = \{\alpha_0, \ldots, \alpha_s\} \quad \text{with} \quad \alpha_0 = \alpha .$$

For all

$$r \in \{0, \ldots, s\}, \qquad x \in V_{\alpha_0} \cap \bar{V}_\alpha \subset U_{\alpha_0} \cap U_{\alpha_r}$$

hence $x \in U^{j_r}_{\alpha_0\alpha_r}$ for some $j_r \in J_{\alpha_0\alpha_r}$. So we obtain a multi-index $\lambda \in \Lambda$ with $x \in X_\lambda$. Since $x \in X$ was arbitrary, the proof is complete.

4.1.1. Corollary. *Let X be a Kähler space with a fixed Kähler metric κ. Then X admits an open covering $\mathscr{X} = (X_\lambda)$ in which is represented by elements*

$$\varphi_\lambda \in SP^\infty(X_\lambda)$$

such that

$$\varphi_\lambda - \varphi_\mu = f_{\lambda\mu} + \bar{f}_{\lambda\mu}, \qquad f_{\lambda\mu} \in \mathcal{O}(X_\lambda \cap X_\mu).$$

Proof. By definition, there is an open covering (U_α) together with $\psi_\alpha \in SP^\infty(U_\alpha)$ such that $\psi_\alpha - \psi_\beta \in PH(U_\alpha \cap U_\beta, \mathbb{R})$. This means that $U_\alpha \cap U_\beta$ admits an open covering $(U^j_{\alpha\beta})_{j \in J_{\alpha\beta}}$ such that

$$(\psi_\alpha - \psi_\beta)|_{U^j_{\alpha\beta}} = g^j_{\alpha\beta} + \bar{g}^j_{\alpha\beta}, \qquad g^j_{\alpha\beta} \in \mathcal{O}(U^j_{\alpha\beta}).$$

Apply the Covering Lemma above to obtain an open covering (X_λ) of X with $X_\lambda \subset U_{\alpha(\lambda)}$ and $X_\lambda \cap X_\mu \subset U^{j(\lambda,\mu)}_{\alpha(\lambda)\alpha(\mu)}$. Then if we set

$$\varphi_\lambda := \psi_{\alpha(\lambda)}|_{X_\lambda}$$

$$f_{\lambda\mu} := g^{j(\lambda,\mu)}_{\alpha(\lambda)\alpha(\mu)}|_{X_\lambda \cap X_\mu}$$

these elements satisfy the required conditions.

4.2. Kähler-Čech Pairs. It will be convenient to multiply the above elements φ_λ and $f_{\lambda\mu}$ by $i=\sqrt{-1}$ to obtain

$$\text{(4.2.1)}\qquad \begin{aligned} &\text{(i)} && \varphi_\lambda-\varphi_\mu=f_{\lambda\mu}-\bar{f}_{\lambda\mu}\\ &\text{(ii)} && -i\varphi_\lambda\in SP^\infty(X_\lambda)\\ &\text{(iii)} && \partial\bar\partial\varphi_\lambda=\omega|_{X_\lambda}. \end{aligned}$$

So the Kähler metric of X is

$$\kappa=\{(X_\lambda,-i\varphi_\lambda)\}.$$

A pair (f,φ) with $f\in C^1(\underline{X},\Omega^0)$ and $\varphi\in C^0(\underline{X},A^0)$ satisfying (4.2.1) will be called a *Kähler-Čech pair* and $\underline{X}=(X,\mathscr{X})$ will be called a *Kähler-Čech space.* Since $(\delta\varphi)_{\lambda\mu}=\varphi_\mu-\varphi_\lambda$, we have the identities

$$\text{(A)}\qquad \begin{aligned} &(1) && \delta\varphi=\bar f-f\\ &(2) && \delta f=\delta\bar f\\ &(3) && d\delta f=0\\ &(4) && \partial\delta\varphi=-df\\ &(5) && \bar\partial\delta\varphi=d\bar f\\ &(6) && \partial\bar\partial\varphi=\varepsilon(\omega)\\ &(7) && d\omega=0. \end{aligned}$$

Identity (A2) shows that $\delta f\in Z^2(\underline{X},\mathbb{R})$. The diagram

$$\text{(4.2.2)}\qquad \begin{tikzcd} & & -i\delta\varphi \arrow[dd, "-i\partial"] & i\varphi \arrow[l, "\delta"'] \arrow[dd, "-i\partial"'] \arrow[dr, "i\partial\bar\partial"] & & \\ \delta f & f \arrow[l, "\delta"'] \arrow[ur, "-2\,\mathrm{Im}"] \arrow[dr, "d"'] & & & \partial\bar\partial\varphi & \omega \arrow[l, "\varepsilon"'] \\ & & df & \partial\varphi \arrow[l, "\delta"] \arrow[ur, shift left, "\bar\partial"'] \arrow[ur, shift right, "d"] & & \end{tikzcd}$$

shows that $-2\,\mathrm{Im}f=-i\delta\varphi$ represents the Kähler class $\hat{c}_1(\kappa)$ of (X,κ) in $H^1(X,PH_{X,\mathbb{R}})$, δf represents $c_1(\kappa)\in H^2(X,\mathbb{R})$ and df represents $\tilde{c}_1(\kappa)\in H^1(X,\Omega^1_X)$. Moreover (4.2.2) confirms that ω is a d-closed representative of $c_1(\kappa)$ and a $\bar\partial$-closed representative of $\tilde{c}_1(\kappa)$, i.e. that diagram (1.2.4) of Chap. II is indeed commutative.

In terms of the $\partial\bar\partial$-complex $\mathscr{L}^{\cdot}_{1,\mathbb{R}}$ given by

$$\text{(4.2.3)}\qquad \begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "\binom{1}{1}"] & \mathscr{L}^0_{1,\mathbb{R}} \arrow[r, "D"] \arrow[d, equal] & \mathscr{L}^1_{1,\mathbb{R}} \arrow[r, "D"] \arrow[d, equal] & \mathscr{L}^2_{1,\mathbb{R}} \arrow[r, "D"] \arrow[d, equal] & \mathscr{L}^3_{1,\mathbb{R}} \arrow[d, equal] \\ & & (\Omega^0\oplus\bar\Omega^0)_{\mathbb{R}} \arrow[r, "(-1\ \ 1)"] & (A^{0,0})_{i\mathbb{R}} \arrow[r, "\partial\bar\partial"] & A^{1,1}_{\mathbb{R}} \arrow[r, "d"] & (A^{1,2}\oplus A^{2,1})_{\mathbb{R}} \end{tikzcd}$$

[where $(\cdot)_{\mathbb{R}}$ denotes self-conjugate elements and $(\cdot)_{i\mathbb{R}}$ anti-self-conjugate elements] we constructed an element

$$\text{(4.2.4)} \qquad \Phi_1(f,\varphi):=\begin{array}{|c|c|c} \hline \delta f & f & \\ \hline \bar{f} & \varphi & \\ \hline & & \multicolumn{1}{|c|}{\omega} \\ \cline{3-3} \end{array} \in \mathscr{E}_1^2(\underline{X},\mathbb{R})$$

(with the notations of 3.6) and relations (A) mean precisely that $\Delta\Phi_1(f,\varphi)=0$.

4.3. Generalization to Higher Powers. We now construct the announced element

$$\Phi_m(f,\varphi)\in \check{Z}^{2m}(X;\mathbb{R},\mathscr{L}^{\cdot}_m)$$

whose head is $(\delta f)^m$ and tail ω^m, and whose existence is the key step in the proof of Theorem 2. Actually, if we set

$$\text{(i)} \quad \mathscr{K}^m(\underline{X}):=\check{Z}^{2m}(\underline{X};\mathbb{C},\mathscr{L}^{\cdot}_m), \qquad \mathscr{K}(\underline{X}):=\bigoplus_{m\geq 0}\mathscr{K}^m(\underline{X})$$

$$\text{(4.3.1)(ii)} \quad \mathscr{K}^m(\underline{X},[\mathbb{R}]):=\check{Z}^{2m}(\underline{X};\mathbb{R},\mathscr{L}^{\cdot}_m), \qquad \mathscr{K}(\underline{X},[\mathbb{R}]:=\bigoplus_{m\geq 0}\mathscr{K}^m(\underline{X},[\mathbb{R}])$$

$$\text{(iii)} \quad \mathscr{K}^m(\underline{X},\mathbb{R}):=\check{Z}^{2m}(\underline{X};\mathbb{R},\mathscr{L}^{\cdot}_{m,\mathbb{R}}), \qquad \mathscr{K}(\underline{X},\mathbb{R}):=\bigoplus_{m\geq 0}\mathscr{K}^m(\underline{X},\mathbb{R})$$

then there is an associative product law on $\mathscr{K}(\underline{X})$ with respect to which it is a graded $\mathbb{C}$-algebra admitting $\mathscr{K}(\underline{X},[\mathbb{R}])$ as a $\mathbb{R}$-subalgebra, but not $\mathscr{K}(\underline{X},\mathbb{R})$. Then $\Phi_m(f,\varphi)$ is simply the m-th power of $\Phi_1(f,\varphi)$ in $\mathscr{K}(\underline{X},\mathbb{R})$.

$\Phi_m(f,\varphi)$ is defined by

$$\text{(4.3.2)} \quad \Phi_m(f,\varphi):=\begin{array}{|c|cccc} \hline a_m & g_m^0 & \cdots & g_m^{m-1} & \multicolumn{1}{|c}{} \\ \hline \bar{h}_m^0 & \varphi_m^{0,0} & \cdots & \varphi_m^{m-1,0} & \multicolumn{1}{|c}{} \\ \vdots & \vdots & & \vdots & \multicolumn{1}{|c}{} \\ \bar{h}_m^{m-1} & \varphi_m^{0,m-1} & \cdots & \varphi_m^{m-1,m-1} & \multicolumn{1}{|c}{} \\ \hline & & & & \multicolumn{1}{|c|}{\eta_m^{m,m}} \\ \cline{5-5} \end{array} \in \mathscr{K}^m(\underline{X},[\mathbb{R}]),$$

where $a_m\in C^{2m}(\underline{X},\mathbb{R})$, $g_m^k\in C^{2m-k-1}(\underline{X},\Omega^k)$,

$$\bar{h}_m^l\in C^{2m-l-1}(\underline{X},\bar{\Omega}^l), \qquad \varphi_m^{k,l}\in C^{2m-k-l-2}(\underline{X},A^{k,l}), \qquad \eta_m^{m,m}\in H^0(X,A^{m,m})$$

are given by the relations (B) below. Recall that $\delta f=\delta\bar{f}$ by (A2). We use the cup-product of Čech cochains as defined in 1.2.

(B)

(1) $a_m=(\delta f)^m$

(2) $g_m^k=(-1)^k(df)^k\cdot f\cdot(\delta f)^{m-k-1}$

(3) $\bar{h}_m^l=(\delta f)^{m-l-1}\cdot\bar{f}\cdot(d\bar{f})^l$

(4) $\varphi_m^{k,l}=(-1)^{k+l}(df)^k\cdot f\cdot(\delta f)^{m-k-l-2}\cdot\bar{f}\cdot(d\bar{f})^l$ for $k+l<m-1$

(5) $\varphi_m^{k,l}=(-1)^{m-l-1}(df)^{m-l-1}\cdot\delta\varphi\cdot(d\bar{f})^{m-k-2}\cdot\bar{\partial}\varphi\wedge\omega^{k+l-m+1}$
for $k<m-1\leq k+l$

(6) $\varphi_m^{m-1,l}=(-1)^{m-l-1}(df)^{m-l-1}\cdot\varphi\wedge\omega^l$

(7) $\eta_m^{m,m}=\omega^m$.

Domains of validity of formulae (B)

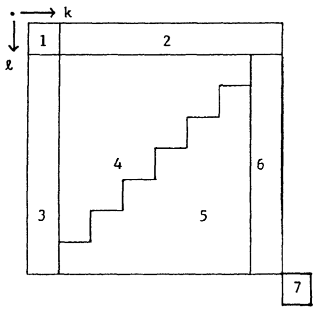

Fig. 1

Before proving that $\Delta\Phi_m(f,\varphi)=0$ we mention

4.4. Relation Between $\Phi_m(f,\varphi)$, $\Phi_n(f,\varphi)$, and $\Phi_{m+n}(f,\varphi)$. A formal consequence of identities (A) and (B) is the following:

(1) $a_{m+n}=a_m\cdot a_n$

(2) $g_{m+n}^k=g_m^k\cdot a_n$ for $0\leqq k<m$

(3) $\qquad=(-1)^m dg_m^{m-1}\cdot g_n^{k-m}$ for $m\leqq k<m+n$

(4) $\bar{h}_{m+n}^l=a_m\cdot\bar{h}_n^l$ for $0\leqq l<n$

(5) $\qquad=\bar{h}_m^{l-n}\cdot d\bar{h}_n^{n-1}$ for $n\leqq l<m+n$

(C) (6) $\varphi_{m+n}^{k,l}=(-1)^l g_m^k\cdot\bar{h}_n^l$ for $0\leqq k<m,\quad 0\leqq l<n$

(7) $\qquad=(-1)^m dg_m^{m-1}\cdot\varphi_n^{k-m,l}$ for $m\leqq k<m+n,\quad 0\leqq l<n$

(8) $\qquad=(-1)^n\varphi_m^{k,l-n}\cdot d\bar{h}_n^{n-1}$ for $n\leqq l<m+n-k-1$

(9) $\qquad=(-1)^{m-1}\delta\varphi_m^{m+n-l-1,l-n}\cdot\bar{\partial}\varphi_n^{k+l-m-n+1,n-1}$
for $l\geqq n,\quad m+n-1\leqq k+l<m+2n-1$

(10) $\qquad=\varphi_m^{k-n,l-n}\wedge\eta_n^{n,n}$ for $k+l\geqq m+2n-1$

(11) $\eta_{m+n}^{m+n,m+n}=\eta_m^{m,m}\wedge\eta_n^{n,n}$.

Domains of validity of formulae (C)

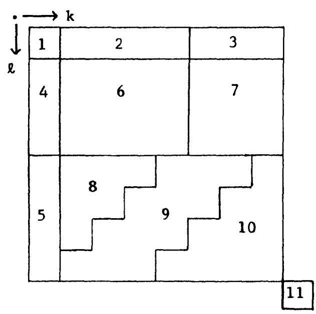

Fig. 2

Actually, the identities (C) define the announced product law

$$\tilde{\mathscr{K}}^m(\underline{X}) \times \tilde{\mathscr{K}}^n(\underline{X}) \to \tilde{\mathscr{K}}^{m+n}(\underline{X}).$$

It is true (the proof be omitted) that the law in question is associative (it will be denoted by the symbol $\times$) and, if Φ and Ψ are Δ-closed, $\Phi \times \Psi$ is also Δ-closed. However it is not compatible with the involution defined in 3.3 and this is the reason for which we work in $\tilde{\mathscr{K}}^m(\underline{X}, [\mathbb{R}])$ instead of $\tilde{\mathscr{K}}^m(\underline{X}, \mathbb{R})$.

Identities (C) will be used to prove

4.5. The Relation $\Delta\Phi_m(f,\varphi)=0$. In order to prove that the element $\Phi_m(f,\varphi)$ defined in (4.3.3) is Δ-closed, we must prove according to (3.6.3) the relations

$$\text{(D)}\quad \begin{array}{ll}
(1) & \delta a_m = 0 \\
(2) & \delta g_m^0 = a_m \\
(3) & \delta g_m^k = d g_m^{k-1} \quad \text{for} \quad 1 \leqq k < m \\
(4) & \delta \bar{h}_m^0 = a_m \\
(5) & \delta \bar{h}_m^l = d \bar{h}_m^{l-1} \quad \text{for} \quad 1 \leqq l < m \\
(6) & \delta \varphi_m^{0,0} = -g_m^0 + \bar{h}_m^0 \\
(7) & \delta \varphi_m^{k,0} = (-1)^{k-1} g_m^k + \partial \varphi_m^{k-1,l} \quad \text{for} \quad 1 \leqq k < m \\
(8) & \delta \varphi_m^{0,l} = \bar{\partial} \varphi_m^{0,l-1} + (-1)^l \bar{h}_m^l \quad \text{for} \quad 1 \leqq l < m \\
(9) & \delta \varphi_m^{k,l} = \bar{\partial} \varphi_m^{k,l-1} + \partial \varphi_m^{k-1,l} \quad \text{for} \quad 1 \leqq k, l < m \\
(10) & \varepsilon(\eta_m^{m,m}) = \partial\bar{\partial} \varphi_m^{m-1,m-1} \\
(11) & d\eta_m^{m,m} = 0 .
\end{array}$$

Proof of (D1). It is obvious.

Proof of (D2). $\delta g_m^0 = \delta(f \cdot (\delta f)^{m-1}) = (\delta f)^m = a_m$.

Proof of (D3).

$$\begin{aligned}
\delta g_m^k &= \delta((-1)^k (df)^k \cdot f \cdot (\delta f)^{m-k-1}) = (df)^k \cdot (\delta f)^{m-k} \\
&= d((-1)^{k-1}(df)^{k-1} \cdot f \cdot (\delta f)^{m-k}) = d g_m^{k-1} .
\end{aligned}$$

Proof of (D4).

$$\begin{aligned}
\delta \bar{h}_m^0 &= \delta((\delta f)^{m-1} \cdot \bar{f}) = (\delta f)^m \quad \text{by (A2)} \\
&= a_m .
\end{aligned}$$

Proof of (D5).

$$\begin{aligned}
\delta \bar{h}_m^l &= \delta((\delta f)^{m-l-1} \cdot \bar{f} \cdot (d\bar{f})^l) = (\delta f)^{m-l} \cdot (d\bar{f})^l \quad \text{by (A2)} \\
&= d((\delta f)^{m-l} \cdot \bar{f} \cdot (d\bar{f})^{l-1}) = d\bar{h}_m^{l-1} .
\end{aligned}$$

Proof of (D6).

$$\delta\varphi_m^{0,0}=\delta(f\cdot(\delta f)^{m-2}\cdot\bar{f})=(\delta f)^{m-1}\cdot\bar{f}-f\cdot(\delta f)^{m-1}=\bar{h}_m^0-g_m^0\,.$$

Proof of (D7). *Case 1.* $k<m-1$

$$\begin{aligned}\delta\varphi_m^{k,0}&=\delta((-1)^k dg_k^{k-1}\cdot\varphi_{m-k}^{0,0})\quad\text{by (C7)}\\&=dg_k^{k-1}\cdot\delta\varphi_{m-k}^{0,0}=dg_k^{k-1}\cdot(-g_{m-k}^0+\bar{h}_{m-k}^0)\quad\text{by (D6)}\\&=-dg_k^{k-1}\cdot g_{m-k}^0+\partial(g_k^{k-1}\cdot\bar{h}_{m-k}^0)\\&=(-1)^{k-1}g_m^k+\partial\varphi_m^{k-1,0}\quad\text{by (C3) and (C6)}.\end{aligned}$$

Case 2. $k=m-1$

$$\begin{aligned}\delta\varphi_m^{m-1,0}&=\delta((-1)^{m-1}(df)^{m-1}\cdot\varphi)=(df)^{m-1}\cdot\delta\varphi=(df)^{m-1}\cdot(-f+\bar{f})\quad\text{by (A1)}\\&=-(df)^{m-1}\cdot f+\partial((-1)^{m-2}(df)^{m-2}\cdot f\cdot\bar{f})\\&=(-1)^m g_m^{m-1}+\partial\varphi_m^{m-2,0}\quad\text{by (B2) and (B5)}.\end{aligned}$$

Proof of (D8). *Case 1.* $l<m-1$

$$\begin{aligned}\delta\varphi_m^{m-1,0}&=\delta((-1)^{m-1}(df)^{m-1}\cdot\varphi)=(df)^{m-1}\cdot\delta\varphi=(df)^{m-1}\cdot(-f+\bar{f})\quad\text{by (A1)}\\&=-(df)^{m-1}\cdot f+\partial((-1)^{m-2}(df)^{m-2}\cdot f\cdot\bar{f})\\&=(-1)^m g_m^{m-1}+\partial\varphi_m^{m-2,0}\quad\text{by (B2) and (B5)}.\end{aligned}$$

Proof of (D8). *Case 1.* $l<m-1$

$$\begin{aligned}\delta\varphi_m^{0,l}&=\delta((-1)^l\varphi_{m-l}^{0,0}\cdot d\bar{h}^{l-1})\quad\text{by (C8)}\\&=(-1)^l\delta\varphi_{m-l}^{0,0}\cdot d\bar{h}_l^{l-1}=(-1)^l(-g_{m-l}^0+\bar{h}_{m-l}^0)\cdot d\bar{h}_l^{l-1}\quad\text{by (D6)}\\&=\bar{\partial}((-1)^{l-1}g_{m-l}^0\cdot\bar{h}^{l-1})+(-1)^l\bar{h}_{m-l}^0\cdot d\bar{h}_l^{l-1}\\&=\bar{\partial}\varphi_m^{0,l-1}+(-1)^l\bar{h}_m^l\quad\text{by (C6) and (C5)}.\end{aligned}$$

Case 2. $l=m-1$

$$\begin{aligned}\delta\varphi_m^{0,m-1}&=\delta(\delta\varphi\cdot(d\bar{f})^{m-2}\cdot\bar{\partial}\varphi)=(-1)^{m-1}\delta\varphi\cdot(d\bar{f})^{m-1}\quad\text{by (A5)}\\&=(-1)^m(f-\bar{f})\cdot(d\bar{f})^{m-1}\quad\text{by (A1)}\\&=\bar{\partial}((-1)^{m-2}f\cdot\bar{f}\cdot(d\bar{f})^{m-2})+(-1)^{m-1}\bar{f}\cdot(d\bar{f})^{m-2}\\&=\bar{\partial}\varphi_m^{0,m-2}+(-1)^{m-1}\bar{h}_m^{m-1}\quad\text{by (B4) and (B5)}.\end{aligned}$$

Proof of (D9). *Case1.* $k+l<m-1$.

We can write $m=r+s$ with $r>k$ and $s>l$. Then

$$\begin{aligned}\delta\varphi_m^{k,l}&=\delta\varphi_{r+s}^{k,l}=\delta((-1)^l g_r^k\cdot\bar{h}_s^l)\quad\text{by (C6)}\\&=(-1)^l\delta g_r^k\cdot\bar{h}_s^l+(-1)^{k+l-1}g_r^k\cdot\delta\bar{h}_s^l\\&=(-1)^l dg_r^{k-1}\cdot\bar{h}_s^l+(-1)^{k+l-1}g_r^k\cdot d\bar{h}_s^{l-1}\quad\text{by (D3) and (D5)}\\&=\partial((-1)^l g_r^{k-1}\cdot\bar{h}_s^l)+\bar{\partial}((-1)^{l-1}g_r^k\cdot\bar{h}_s^{l-1})\\&=\partial\varphi_m^{k-1,l}+\bar{\partial}\varphi_m^{k,l-1}\quad\text{by (C6)}.\end{aligned}$$

Case 2. $k+l=m-1$

$$\begin{aligned}\delta\varphi_m^{k,l}&=\delta((-1)^k(df)^k\cdot\delta\varphi\cdot(d\bar f)^{l-1}\cdot\bar\partial\varphi) \quad \text{by (B5)}\\&=(-1)^l(df)^k\cdot\delta\varphi\cdot(d\bar f)^l \quad \text{by (A5)}\\&=(-1)^{l-1}(df)^k\cdot f\cdot(d\bar f)^l+(-1)^l(df)^k\cdot\bar f\cdot(d\bar f)^l \quad \text{by (A1)}\\&=(-1)^{k+l-1}g_{k+1}^k\cdot d\bar h_l^{l-1}+(-1)^l dg_k^{k-1}\cdot\bar h_{l+1}^l \quad \text{by (B2) and (B3)}\\&=\bar\partial((-1)^{l-1}g_{k+1}^k\cdot\bar h_l^{l-1})+\partial((-1)^l g_k^{k-1}\cdot\bar h_{l+1}^l)\\&=\bar\partial\varphi_m^{k,l-1}+\partial\varphi_m^{k-1,l} \quad \text{by (C6)}.\end{aligned}$$

Case 3. $k<m-1<k+l$

$$\begin{aligned}\delta\varphi_m^{k,l}&=(-1)^{m-k-1}(df)^{m-l-1}\cdot\delta\varphi\cdot(d\bar f)^{m-k-1}\wedge\omega^{k+l-m+1} \quad \text{by (B5) and (A5)}\\&=(-1)^{m-k-1}(df)^{m-l-1}\cdot\delta\varphi\cdot(d\bar f)^{m-k-1}\cdot\partial\bar\partial\varphi\wedge\omega^{k+l-m} \quad \text{by (A6)}\\&=\bar\partial((-1)^{m-l}(df)^{m-l}\cdot\delta\varphi\cdot(d\bar f)^{m-k-2}\cdot\bar\partial\varphi\wedge\omega^{k+l-m})\\&\quad+\partial((-1)^{m-l-1}(df)^{m-l-1}\cdot\delta\varphi\cdot(d\bar f)^{m-k-1}\cdot\bar\partial\varphi\wedge\omega^{k+l-m}) \quad \text{by (A4), (A5), (A6)}\\&=\bar\partial\varphi_m^{k,l-1}+\partial\varphi_m^{k-1,l} \quad \text{by (B5)}.\end{aligned}$$

Case 4. $k=m-1$

$$\begin{aligned}\delta\varphi_m^{m-1,l}&=(df)^{m-l-1}\cdot\delta\varphi\wedge\omega^l \quad \text{by (B6)}\\&=\bar\partial((-1)^{m-l}(df)^{m-l}\cdot\varphi\wedge\omega^{l-1})+\partial((-1)^{m-l-1}(df)^{m-l-1}\cdot\delta\varphi\cdot\bar\partial\varphi\wedge\omega^{l-1})\\&=\bar\partial\varphi_m^{m-1,l-1}+\partial\varphi_m^{m-2,l} \quad \text{by (B5) and (B6)}.\end{aligned}$$

Finally, (D10) and (D11) are obvious since $\varphi_m^{m-1,m-1}=\varphi\omega^{m-1}$ and $\eta_m^{m,m}=\omega^m$. Therefore the proof of the relation $\Delta\Phi_m(f,\varphi)=0$ is complete.

4.5.1. Remark. There are several alternative ways of proving $\Delta\Phi_m(f,\varphi)=0$. For example, identities (C) written only for $n=1$ give a relation between $\Phi_m(f,\varphi)$ and $\Phi_{m+1}(f,\varphi)$, and the relation $\Delta\Phi_m(f,\varphi)$ can be proven by induction on m. Otherwise, one can prove directly that $\Delta\Phi_m=\Delta\Phi_n=0$ implies $\Delta(\Phi_m\times\Phi_n)=0$ using (A), (B), and (C) but the calculations would be longer than the above (30 verifications are needed).

5. *Theorem 2*

5.1. Statement of Theorem 2. Let (X,ω) be a Kähler space and $m\geqq 0$ an integer. Then there exist open sets $U_\alpha\subset X$ $(\alpha\in A)$ and $U_{\alpha\beta}^j\subset U_\alpha\cap U_\beta$ $(j\in J_{\alpha\beta})$ depending on X and m alone such that

(i) Any compact m-dimensional complex-analytic subset of X is contained in some U_α.

(ii) Any compact m-dimensional complex-analytic subset of $U_\alpha\cap U_\beta$ is contained in some $U_{\alpha\beta}^j$.

(iii) There exist elements $\chi_\alpha\in A^{m,m}(U_\alpha,\mathbb{R})$ such that

$$\omega^{m+1}|_{U_\alpha}=i\partial\bar\partial\chi_\alpha\,.$$