Case 2. $k+l=m-1$

$$\begin{aligned}\delta\varphi_m^{k,l} &= \delta((-1)^k(df)^k\cdot\delta\varphi\cdot(d\bar f)^{l-1}\cdot\bar\partial\varphi) \quad \text{by (B5)}\\ &=(-1)^l(df)^k\cdot\delta\varphi\cdot(d\bar f)^l \quad \text{by (A5)}\\ &=(-1)^{l-1}(df)^k\cdot f\cdot(d\bar f)^l+(-1)^l(df)^k\cdot\bar f\cdot(d\bar f)^l \quad \text{by (A1)}\\ &=(-1)^{k+l-1}g_{k+1}^k\cdot d\bar h_l^{l-1}+(-1)^l dg_k^{k-1}\cdot\bar h_{l+1}^l \quad \text{by (B2) and (B3)}\\ &=\bar\partial((-1)^{l-1}g_{k+1}^k\cdot\bar h_l^{l-1})+\partial((-1)^l g_k^{k-1}\cdot\bar h_{l+1}^l)\\ &=\bar\partial\varphi_m^{k,l-1}+\partial\varphi_m^{k-1,l} \quad \text{by (C6)}.\end{aligned}$$

Case 3. $k<m-1<k+l$

$$\begin{aligned}\delta\varphi_m^{k,l} &=(-1)^{m-k-1}(df)^{m-l-1}\cdot\delta\varphi\cdot(d\bar f)^{m-k-1}\wedge\omega^{k+l-m+1} \quad \text{by (B5) and (A5)}\\ &=(-1)^{m-k-1}(df)^{m-l-1}\cdot\delta\varphi\cdot(d\bar f)^{m-k-1}\cdot\partial\bar\partial\varphi\wedge\omega^{k+l-m} \quad \text{by (A6)}\\ &=\bar\partial((-1)^{m-l}(df)^{m-l}\cdot\delta\varphi\cdot(d\bar f)^{m-k-2}\cdot\bar\partial\varphi\wedge\omega^{k+l-m})\\ &\quad+\partial((-1)^{m-l-1}(df)^{m-l-1}\cdot\delta\varphi\cdot(d\bar f)^{m-k-1}\cdot\bar\partial\varphi\wedge\omega^{k+l-m}) \quad \text{by (A4), (A5), (A6)}\\ &=\bar\partial\varphi_m^{k,l-1}+\partial\varphi_m^{k-1,l} \quad \text{by (B5)}.\end{aligned}$$

Case 4. $k=m-1$

$$\begin{aligned}\delta\varphi_m^{m-1,l} &=(df)^{m-l-1}\cdot\delta\varphi\wedge\omega^l \quad \text{by (B6)}\\ &=\bar\partial((-1)^{m-l}(df)^{m-l}\cdot\varphi\wedge\omega^{l-1})+\partial((-1)^{m-l-1}(df)^{m-l-1}\cdot\delta\varphi\cdot\bar\partial\varphi\wedge\omega^{l-1})\\ &=\bar\partial\varphi_m^{m-1,l-1}+\partial\varphi_m^{m-2,l} \quad \text{by (B5) and (B6)}.\end{aligned}$$

Finally, (D10) and (D11) are obvious since $\varphi_m^{m-1,m-1}=\varphi\omega^{m-1}$ and $\eta_m^{m,m}=\omega^m$. Therefore the proof of the relation $\Delta\Phi_m(f,\varphi)=0$ is complete.

4.5.1. Remark. There are several alternative ways of proving $\Delta\Phi_m(f,\varphi)=0$. For example, identities (C) written only for $n=1$ give a relation between $\Phi_m(f,\varphi)$ and $\Phi_{m+1}(f,\varphi)$, and the relation $\Delta\Phi_m(f,\varphi)$ can be proven by induction on m. Otherwise, one can prove directly that $\Delta\Phi_m=\Delta\Phi_n=0$ implies $\Delta(\Phi_m\times\Phi_n)=0$ using (A), (B), and (C) but the calculations would be longer than the above (30 verifications are needed).

5. *Theorem 2*

5.1. Statement of Theorem 2. Let (X,ω) be a Kähler space and $m\geqq 0$ an integer. Then there exist open sets $U_\alpha\subset X$ $(\alpha\in A)$ and $U_{\alpha\beta}^j\subset U_\alpha\cap U_\beta$ $(j\in J_{\alpha\beta})$ depending on X and m alone such that

(i) Any compact m-dimensional complex-analytic subset of X is contained in some U_α.

(ii) Any compact m-dimensional complex-analytic subset of $U_\alpha\cap U_\beta$ is contained in some $U_{\alpha\beta}^j$.

(iii) There exist elements $\chi_\alpha\in A^{m,m}(U_\alpha,\mathbb{R})$ such that

$$\omega^{m+1}|_{U_\alpha}=i\partial\bar\partial\chi_\alpha.$$

(iv) There exist elements $\tau^j_{\alpha\beta} \in A^{m,m}(U^j_{\alpha\beta})$ such that

$$\bar{\partial}\tau^j_{\alpha\beta} = 0 \quad \text{and} \quad (\chi_\alpha - \chi_\beta)|_{U_{\alpha\beta}} = \tau^j_{\alpha\beta} + \bar{\tau}^j_{\alpha\beta}.$$

(v) The $\tau^j_{\alpha\beta}$ are $\bar{\partial}$-closed representatives of elements $\xi^j_{\alpha\beta} \in H^m(U^j_{\alpha\beta}, \Omega^m)$.

5.2. Proof of (i) and (ii). We take an open covering $\mathscr{X}$ of X such that $\underline{X} = (X, \mathscr{X})$ is a Kähler-Čech space with a Kähler-Čech pair (f, φ) as in 4.2.

The U_α are taken as the m-admissible open sets of X and the $U^j_{\alpha\beta}$ as the m-admissible open sets of $U_\alpha \cap U_\beta$. Parts (i) and (ii) of Theorem 2 are restatements of Lemma 3.5.4 of I. By Proposition 1.3.3, each U_α is underlying to some m-admissible $\underline{U}_\alpha \ll \underline{X}$ and each $U^j_{\alpha\beta}$ to some m-admissible $\underline{U}^j_{\alpha\beta} \ll \underline{U}_\alpha \cap \underline{U}_\beta$.

5.3. Proof of (iii). We use the element

$$\Phi_{m+1}(f, \varphi) \in \check{\mathscr{K}}^{m+1}(X, [\mathbb{R}]) = \check{Z}^{2m+2}(\underline{X}; \mathbb{R}, \mathscr{L}^{\cdot}_{m+1})$$

which is Δ-closed in the Čech transform of the complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{L}^0_{m+1} \longrightarrow \ldots \longrightarrow \mathscr{L}^{2m+1}_{m+1} \xrightarrow{\partial\bar{\partial}} \mathscr{L}^{2m+2}_{m+1} \longrightarrow \ldots.$$

Take the restriction [in the sense of (1.1.5)]

$$\Phi_{m+1,\alpha} := \Phi_{m+1}(f, \varphi)|_{\underline{U}_\alpha} \in \check{\mathscr{K}}^{m+1}(\underline{U}_\alpha, [\mathbb{R}]). \tag{5.3.1}$$

Since $\underline{U}_\alpha$ is m-complete, we have

$$H^{2m-k+1}(\underline{U}_\alpha, \mathscr{L}^k_{m+1}) = 0 \quad \text{for} \quad 0 \leqq k \leqq 2m.$$

Indeed, for $k \leqq m$ this is due to the m-completeness of $\underline{U}_\alpha$ and the fact that $\mathscr{L}^k_{m+1} = \Omega^k \oplus A^{k-1} \oplus \bar{\Omega}^k$; for $k > m$, it is due to the fact that $\mathscr{L}^k_{m+1}$ is a fine sheaf.

Corollary 2.3 applies and $\Phi_{m+1,\alpha}$ is Δ-exact if its head is δ-exact, since the canonical morphism

$$\check{H}^{2m+2}(\underline{U}_\alpha; \mathbb{R}, \mathscr{L}^{\cdot}_{m+1}) \to H^{2m+2}(\underline{U}_\alpha, \mathbb{R})$$

is injective. But the head of $\Phi_{m+1,\alpha}$ is $(\delta f)^{m+1}|_{\underline{U}_\alpha}$ whose class in $H^{2m+2}(\underline{U}_\alpha, \mathbb{R})$ is 0 by Lemma 1.3.2, since $\underline{U}_\alpha \ll \underline{X}$ is m-admissible. Therefore

$$\Phi_{m+1,\alpha} = \Delta\Theta_{m+1,\alpha} \tag{5.3.2}$$

for some $\Theta_{m+1,\alpha} \in \mathscr{C}^{2m+1}_{m+1}(\underline{U}_\alpha, [\mathbb{R}])$. In particular, if $\psi_\alpha \in A^{m,m}(U_\alpha)$ is the tail of $\Theta_{m+1,\alpha}$, we have

$$\omega^{m+1}|_{U_\alpha} = \partial\bar{\partial}\psi_\alpha. \tag{5.3.3}$$

It is then sufficient to set

$$\chi_\alpha := \frac{i}{2}(\bar{\psi}_\alpha - \psi_\alpha) \tag{5.3.4}$$

to satisfy condition (iii) of Theorem 2.

5.4. Proof of (iv) and (v). Take a fixed $\underline{U} = \underline{U}^j_{\alpha\beta} \ll \underline{U}_\alpha \cap \underline{U}_\beta$.

There are open inclusions of Čech open sets

$$\begin{array}{ccccc} \underline{U}_\alpha & \xleftarrow{i_\alpha} & \underline{U} & \xrightarrow{i_\beta} & \underline{U}_\beta \\ \downarrow & \swarrow_{j_\alpha} & & {}_{j_\beta}\searrow & \downarrow \\ \underline{X} & & & & \underline{X}\,. \end{array}$$

We may then apply the operator T of (1.1.6) relatively to $j_\alpha, j_\beta : \underline{U} \to \underline{X}$ and set

$$(5.4.1) \qquad \tilde{\Theta}_{m+1} := T\Phi_{m+1}(f, \varphi) \in \mathscr{E}^{2m+1}_{m+1}(\underline{U}, [\mathbb{R}]).$$

This element satisfies the conditions

$$(5.2.1) \qquad \begin{array}{ll} \text{(i)} & \Delta\tilde{\Theta}_{m+1} = j_\beta^*\Phi_{m+1}(f, \varphi) - j_\alpha^*\Phi_{m+1}(f, \varphi) \\ \text{(ii)} & \text{The tail of } \tilde{\Theta}_{m+1} \text{ is } 0\,. \end{array}$$

Indeed, (i) is a consequence of (1.1.7) and (ii) of the fact that T induces 0 on 0-cochains and global sections. Now set

$$(5.4.3) \qquad \Theta_{m+1} := i_\alpha^*(\Theta_{m+1,\alpha}) - i_\beta^*(\Theta_{m+1,\beta}) + \tilde{\Theta}_{m+1} \in \mathscr{E}^{2m+1}_{m+1}(\underline{U}, [\mathbb{R}]).$$

This element satisfies, by (5.3.2) and (5.4.2)

$$(5.4.4) \qquad \begin{array}{ll} \text{(i)} & \Delta\Theta_{m+1} = 0 \\ \text{(ii)} & \text{The tail of } \Theta_{m+1} \text{ is } \quad \psi := (\psi_\alpha - \psi_\beta)|_{\underline{U}}\,. \end{array}$$

We notice that Lemma 2.2(i) does not apply to the canonical morphism

$$\check{H}^{2m+1}(\underline{U}; \mathbb{R}, \mathscr{L}^{\cdot}_{m+1}) \to H^{2m+1}(\underline{U}, \mathbb{R})$$

for among the groups $H^{2m-k}(\underline{U}, \mathscr{L}^k_{m+1})$ there is $H^m(\underline{U}, \mathscr{L}^m_{m+1}) = H^m(\underline{U}, \Omega^m \oplus \bar{\Omega}^m)$ which is not 0 in general. So we apply the operator μ defined in 3.4 to obtain $\mu\Theta_{m+1} \in \mathscr{E}^{2m+1}_m(\underline{U}, [\mathbb{R}])$.

Since μ commutes with D (and δ), $\mu\Theta_{m+1}$ is Δ-closed. This time the canonical morphism

$$\check{H}^{2m+1}(\underline{U}; \mathbb{R}, \mathscr{L}^{\cdot}_m) \to H^{2m+1}(\underline{U}, \mathbb{R})$$

is injective since the groups $H^{2m-k}(\underline{U}, \mathscr{L}^k_m)$ are all 0 for $0 \leq k \leq 2m-1$. Indeed, for $k < m$ this is due to the m-completeness of $\underline{U}$ and, for $k \geq m$, to the fact that $\mathscr{L}^k_m$ is a fine sheaf. So, by Corollary 2.3, $\mu\Theta_{m+1}$ is Δ-exact if its head is δ-exact in $C^{\cdot}(\underline{U}, \mathbb{R})$. But the head of $\mu\Theta_{m+1}$ is equal to the head of Θ_{m+1} which is of the form $c_{m+1}|_{\underline{U}}$ with

$$c_{m+1} \in Z^{2m+1}(\underline{U}_\alpha \cap \underline{U}_\beta, \mathbb{R}).$$

Since $\underline{U} \ll \underline{U}_\alpha \cap \underline{U}_\beta$ is m-admissible, $c_{m+1}|_{\underline{U}}$ is δ-exact (Lemma 1.4.2) and therefore

$$(5.4.5) \qquad \mu\Theta_{m+1} = \Delta Z_m$$

for some $Z_m \in \mathscr{E}^{2m}_m(\underline{U}, \mathbb{R})$.

Now we use the operators β and γ defined in 3.5. Denote by $\mathscr{D}^{\cdot}_m(\underline{U})$ the Čech transform of the $(\bar{\partial} \oplus \partial)$-complex over $\underline{U}$, i.e.

$$(5.4.6) \qquad \mathscr{D}^q_m(\underline{U}) := \check{C}^q(\underline{U}; \Omega^m \oplus \bar{\Omega}^m, \mathscr{G}^{\cdot}_m)$$

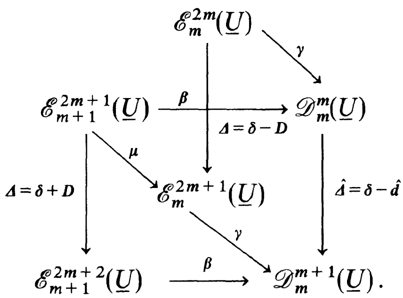
with differential

(5.4.7) $$\hat{\Delta} := \delta + (-1)^{m+q+1}\hat{d} : \mathscr{D}^q_m(\underline{U}) \to \mathscr{D}^{q+1}_m(\underline{U}).$$

Notice that this sign convention differs from (2.1.3).

Diagram (3.5.3) becomes

(5.4.8)



By Lemma 3.5.3 and the sign convention (5.4.7) on $\hat{\Delta}$ we have on $\mathscr{E}^{2m+1}_{m+1}(\underline{U})$

(5.4.9) $$\begin{aligned}\beta\Delta - \hat{\Delta}\beta &= \beta(\delta + D) - (\delta - \hat{d})\beta = (\beta\delta - \delta\beta) + (\beta D + \hat{d}\beta)\\ &= \beta D + \hat{d}\beta = \gamma\mu.\end{aligned}$$

On the other hand, we have on $\mathscr{E}^{2m}_m(\underline{U})$

(5.4.10) $$\gamma\Delta = \hat{\Delta}\gamma.$$

If we apply (5.4.9) to Θ_{m+1} and (5.4.10) to Z_m, we get

$$-\hat{\Delta}\beta\Theta_{m+1} = (\beta\Delta - \hat{\Delta}\beta)\Theta_{m+1} = \gamma\mu\Theta_{m+1} = \gamma\Delta Z_m = \hat{\Delta}\gamma Z_m$$

which means that the element

(5.4.11) $$\Lambda_m := \beta\Theta_{m+1} + \gamma Z_m \in \mathscr{D}^m_m(\underline{U})$$

satisfies

$$\hat{\Delta}\Lambda_m = 0.$$

The tail of Λ_m has the form

$$(\varrho^{m,m}, \sigma^{m,m}) \in A^{m,m}(U) \oplus A^{m,m}(U)$$

with $\bar{\partial}\varrho^{m,m} = \partial\sigma^{m,m} = 0$ (since $\hat{\Delta}\Lambda_m = 0$) and

(5.4.12) $$\varrho^{m,m} + \sigma^{m,m} = \psi$$

by Lemma 3.5.3(iii).

The fact that Λ_m is a $\hat{\Delta}$-cocycle means precisely that $\varrho^{m,m}$ and $\bar{\sigma}^{m,m}$ represent elements of $H^m(\underline{U}, \Omega^m)$. So if we set

(5.4.13) $$\tau^j_{\alpha\beta} := \frac{i}{2}(\bar{\sigma}^{m,m} - \varrho^{m,m})$$

it is clear that conditions (iv) and (v) of Theorem 2 are satisfied.

5.5. *Remark.* (1) We did not use the positivity of ω in the proof of Theorem 2. The result we can actually prove by our method is the following: If U_α and $U^j_{\alpha\beta}$ are the open sets of Theorem 2, then conditions (i) and (ii) remain unchanged. If moreover $\kappa_0, \ldots, \kappa_m$ are arbitrary elements of $\mathscr{K}^1(X)$ and $\omega_q := \partial\bar{\partial}\kappa_q$ for $0 \leqq q \leqq m$, then

(iii) There are elements $\psi_\alpha \in A^{m,m}(U_\alpha)$ such that $(\omega_0 \wedge \ldots \wedge \omega_m)|_{U_\alpha} = \partial\bar{\partial}\psi_\alpha$.

(iv) There are elements $\varrho^j_{\alpha\beta}, \sigma^j_{\alpha\beta} \in A^{m,m}(U^j_{\alpha\beta})$ such that $\bar{\partial}\varrho^j_{\alpha\beta} = \partial\sigma^j_{\alpha\beta} = 0$ and $(\psi_\alpha - \psi_\beta)|_{U^j_{\alpha\beta}} = \varrho^j_{\alpha\beta} + \sigma^j_{\alpha\beta}$.

(v) $\varrho^j_{\alpha\beta}$ and $\bar{\sigma}^j_{\alpha\beta}$ represent cohomology classes of $H^m(U^j_{\alpha\beta}, \Omega^m)$.

(2) The proof we gave was a reasoning on $\mathscr{E}^\cdot_m(\underline{X}, [\mathbb{R}])$. We could have chosen $\mathscr{E}^\cdot_m(\underline{X}, \mathbb{R})$ as well, replacing $\Phi_{m+1}(f, \varphi)$ by

$$\mathrm{Re}(\Phi_{m+1}(f, \varphi)) = \tfrac{1}{2}(\Phi_{m+1}(f, \varphi) + \Phi_{m+1}(f, \varphi)^*)$$

and using Lemma 3.5.3(iv).

IV. The Main Results

1. Stability Theorems

We are now in position to prove that some proper images of Kähler spaces are Kähler.

1.1. Theorem 3. *Let $\pi: X \to X'$ be a geometrically flat morphism of complex spaces with m-dimensional fibers (π is proper surjective and X' reduced by definition). Suppose X is Kähler. Then X' is weakly Kähler.*

If moreover there is a discrete $D' \subset X'$ such that for any $x' \in X' \setminus D'$, either

(i) *X' is weakly normal at x' or*

(ii) *$\pi^{-1}(x')$ admits in X a smoothly embeddable neighborhood*

then X' is Kähler.

Proof. With the notations of Theorem 2, set

$$V'_\alpha := \{x' \in X' | \pi^{-1}(x') \subset U_\alpha\}$$

$$V_\alpha := \pi^{-1}(V'_\alpha)$$

$$V'^j_{\alpha\beta} := \{x' \in X' | \pi^{-1}(x') \subset U^j_{\alpha\beta}\}$$

$$V^j_{\alpha\beta} := \pi^{-1}(V'^j_{\alpha\beta})$$

$$\psi_\alpha := \pi_*(\chi_\alpha|_{V_\alpha})$$

$$g^j_{\alpha\beta} := \pi_*(\tau^j_{\alpha\beta}|_{V^j_{\alpha\beta}}).$$

Since π is surjective, the sets V'_α cover X' and, for fixed α, β, the $V'^j_{\alpha\beta}$ cover $V'_\alpha \cap V'_\beta$. By Proposition 3.4.1 of Chap. I, $\psi_\alpha \in SP^0(V'_\alpha)$, $g^j_{\alpha\beta} \in \mathscr{W}(V'^j_{\alpha\beta})$ and, since $(\psi_\alpha - \psi_\beta)|_{V'^j_{\alpha\beta}} = g^j_{\alpha\beta} + \bar{g}^j_{\alpha\beta}$, $\psi_\alpha - \psi_\beta \in WPH(V'_\alpha \cap V'_\beta, \mathbb{R})$. So X' is weakly Kähler. Now if conditions (i) and (ii) are fulfilled, then $g^j_{\alpha\beta}$ is holomorphic on $V'^j_{\alpha\beta} \setminus D'$ and $\psi_\alpha - \psi_\beta$ pluriharmonic on $V'_\alpha \cap V'_\beta \setminus D'$. If we take a refinement (W'_λ) of (V'_α) such that each point of D' belongs at most to one W'_λ, then it is clear that Theorem 1 applies and X' is Kähler.