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Case 2. $k+l=m-1$

$$\begin{aligned}
\delta\varphi_m^{k,l} &= \delta((-1)^k(df)^k \cdot \delta\varphi \cdot (d\bar{f})^{l-1} \cdot \bar{\delta}\varphi) \quad \text{by (B5)} \\
&= (-1)^l(df)^k \cdot \delta\varphi \cdot (d\bar{f})^l \quad \text{by (A5)} \\
&= (-1)^{l-1}(df)^k \cdot f \cdot (d\bar{f})^l + (-1)^l(df)^k \cdot \bar{f} \cdot (d\bar{f})^l \quad \text{by (A1)} \\
&= (-1)^{k+l-1}g_{k+1}^k \cdot d\bar{h}_l^{l-1} + (-1)^l dg_k^{k-1} \cdot \bar{h}_{l+1}^l \quad \text{by (B2) and (B3)} \\
&= \bar{\delta}((-1)^{l-1}g_{k+1}^k \cdot \bar{h}_l^{l-1}) + \partial((-1)^l g_k^{k-1} \cdot \bar{h}_{l+1}^l) \\
&= \bar{\delta}\varphi_m^{k,l-1} + \partial\varphi_m^{k-1,l} \quad \text{by (C6)}.
\end{aligned}$$

Case 3. $k < m-1 < k+l$

$$\begin{aligned}
\delta\varphi_m^{k,l} &= (-1)^{m-k-1}(df)^{m-l-1} \cdot \delta\varphi \cdot (d\bar{f})^{m-k-1} \wedge \omega^{k+l-m+1} \quad \text{by (B5) and (A5)} \\
&= (-1)^{m-k-1}(df)^{m-l-1} \cdot \delta\varphi \cdot (d\bar{f})^{m-k-1} \cdot \partial\bar{\delta}\varphi \wedge \omega^{k+l-m} \quad \text{by (A6)} \\
&= \bar{\delta}((-1)^{m-l}(df)^{m-l} \cdot \delta\varphi \cdot (d\bar{f})^{m-k-2} \cdot \bar{\delta}\varphi \wedge \omega^{k+l-m}) \\
&\quad + \partial((-1)^{m-l-1}(df)^{m-l-1} \cdot \delta\varphi \cdot (d\bar{f})^{m-k-1} \cdot \bar{\delta}\varphi \wedge \omega^{k+l-m}) \quad \text{by (A4), (A5), (A6)} \\
&= \bar{\delta}\varphi_m^{k,l-1} + \partial\varphi_m^{k-1,l} \quad \text{by (B5)}.
\end{aligned}$$

Case 4. $k=m-1$

$$\begin{aligned}
\delta\varphi_m^{m-1,l} &= (df)^{m-l-1} \cdot \delta\varphi \wedge \omega^l \quad \text{by (B6)} \\
&= \bar{\delta}((-1)^{m-l}(df)^{m-l} \cdot \varphi \wedge \omega^{l-1}) + \partial((-1)^{m-l-1}(df)^{m-l-1} \cdot \delta\varphi \cdot \bar{\delta}\varphi \wedge \omega^{l-1}) \\
&= \bar{\delta}\varphi_m^{m-1,l-1} + \partial\varphi_m^{m-2,l} \quad \text{by (B5) and (B6)}.
\end{aligned}$$

Finally, (D10) and (D11) are obvious since $\varphi_m^{m-1,m-1} = \varphi\omega^{m-1}$ and $\eta_m^{m,m} = \omega^m$. Therefore the proof of the relation $\Delta\Phi_m(f, \varphi) = 0$ is complete.

4.5.1. Remark. There are several alternative ways of proving $\Delta\Phi_m(f, \varphi) = 0$. For example, identities (C) written only for $n=1$ give a relation between $\Phi_m(f, \varphi)$ and $\Phi_{m+1}(f, \varphi)$, and the relation $\Delta\Phi_m(f, \varphi)$ can be proven by induction on m . Otherwise, one can prove directly that $\Delta\Phi_m = \Delta\Phi_n = 0$ implies $\Delta(\Phi_m \times \Phi_n) = 0$ using (A), (B), and (C) but the calculations would be longer than the above (30 verifications are needed).

5. Theorem 2

5.1. Statement of Theorem 2. Let (X, ω) be a Kähler space and $m \geq 0$ an integer. Then there exist open sets $U_\alpha \subset X$ ($\alpha \in A$) and $U_{\alpha\beta}^j \subset U_\alpha \cap U_\beta$ ($j \in J_{\alpha\beta}$) depending on X and m alone such that

(i) Any compact m -dimensional complex-analytic subset of X is contained in some U_α .

(ii) Any compact m -dimensional complex-analytic subset of $U_\alpha \cap U_\beta$ is contained in some $U_{\alpha\beta}^j$.

(iii) There exist elements $\chi_\alpha \in A^{m,m}(U_\alpha, \mathbb{R})$ such that

$$\omega^{m+1}|_{U_\alpha} = i\partial\bar{\delta}\chi_\alpha.$$

(iv) There exist elements $\tau_{\alpha\beta}^j \in A^{m,m}(U_{\alpha\beta}^j)$ such that

$$\bar{\partial}\tau_{\alpha\beta}^j = 0 \quad \text{and} \quad (\chi_\alpha - \chi_\beta)|_{U_{\alpha\beta}} = \tau_{\alpha\beta}^j + \bar{\tau}_{\alpha\beta}^j.$$

(v) The $\tau_{\alpha\beta}^j$ are $\bar{\partial}$ -closed representatives of elements $\xi_{\alpha\beta}^j \in H^m(U_{\alpha\beta}^j, \Omega^m)$.

5.2. *Proof of (i) and (ii).* We take an open covering \mathcal{X} of X such that $\underline{X} = (X, \mathcal{X})$ is a Kähler-Čech space with a Kähler-Čech pair (f, φ) as in 4.2.

The U_α are taken as the m -admissible open sets of X and the $U_{\alpha\beta}^j$ as the m -admissible open sets of $U_\alpha \cap U_\beta$. Parts (i) and (ii) of Theorem 2 are restatements of Lemma 3.5.4 of I. By Proposition 1.3.3, each U_α is underlyingly to some m -admissible $\underline{U}_\alpha \ll \underline{X}$ and each $U_{\alpha\beta}^j$ to some m -admissible $\underline{U}_{\alpha\beta}^j \ll \underline{U}_\alpha \cap \underline{U}_\beta$.

5.3. *Proof of (iii).* We use the element

$$\Phi_{m+1}(f, \varphi) \in \check{\mathcal{X}}^{m+1}(X, [\mathbb{R}]) = \check{Z}^{2m+2}(\underline{X}; \mathbb{R}, \mathcal{L}_{m+1}^{\cdot})$$

which is Δ -closed in the Čech transform of the complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{L}_{m+1}^0 \longrightarrow \dots \longrightarrow \mathcal{L}_{m+1}^{2m+1} \xrightarrow{\bar{\partial}} \mathcal{L}_{m+1}^{2m+2} \longrightarrow \dots$$

Take the restriction [in the sense of (1.1.5)]

$$(5.3.1) \quad \Phi_{m+1,\alpha} := \Phi_{m+1}(f, \varphi)|_{\underline{U}_\alpha} \in \check{\mathcal{X}}^{m+1}(\underline{U}_\alpha, [\mathbb{R}]).$$

Since \underline{U}_α is m -complete, we have

$$H^{2m-k+1}(\underline{U}_\alpha, \mathcal{L}_{m+1}^k) = 0 \quad \text{for} \quad 0 \leq k \leq 2m.$$

Indeed, for $k \leq m$ this is due to the m -completeness of \underline{U}_α and the fact that $\mathcal{L}_{m+1}^k = \Omega^k \oplus A^{k-1} \oplus \bar{\Omega}^k$; for $k > m$, it is due to the fact that \mathcal{L}_{m+1}^k is a fine sheaf.

Corollary 2.3 applies and $\Phi_{m+1,\alpha}$ is Δ -exact if its head is δ -exact, since the canonical morphism

$$\check{H}^{2m+2}(\underline{U}_\alpha; \mathbb{R}, \mathcal{L}_{m+1}^{\cdot}) \rightarrow H^{2m+2}(\underline{U}_\alpha, \mathbb{R})$$

is injective. But the head of $\Phi_{m+1,\alpha}$ is $(\delta f)^{m+1}|_{\underline{U}_\alpha}$ whose class in $H^{2m+2}(\underline{U}_\alpha, \mathbb{R})$ is 0 by Lemma 1.3.2, since $\underline{U}_\alpha \ll \underline{X}$ is m -admissible. Therefore

$$(5.3.2) \quad \Phi_{m+1,\alpha} = \Delta \Theta_{m+1,\alpha}$$

for some $\Theta_{m+1,\alpha} \in \mathcal{E}_{m+1}^{2m+1}(\underline{U}_\alpha, [\mathbb{R}])$. In particular, if $\psi_\alpha \in A^{m,m}(U_\alpha)$ is the tail of $\Theta_{m+1,\alpha}$, we have

$$(5.3.3) \quad \omega^{m+1}|_{U_\alpha} = \bar{\partial}\bar{\partial}\psi_\alpha.$$

It is then sufficient to set

$$(5.3.4) \quad \chi_\alpha := \frac{i}{2}(\bar{\psi}_\alpha - \psi_\alpha)$$

to satisfy condition (iii) of Theorem 2.

5.4. *Proof of (iv) and (v).* Take a fixed $\underline{U} = \underline{U}_{\alpha\beta}^j \ll \underline{U}_\alpha \cap \underline{U}_\beta$.

There are open inclusions of Čech open sets

$$\begin{array}{ccccc}
 \underline{U}_\alpha & \xleftarrow{i_\alpha} & \underline{U} & \xrightarrow{i_\beta} & \underline{U}_\beta \\
 \downarrow & & \swarrow j_\alpha & \searrow j_\beta & \downarrow \\
 \underline{X} & & & & \underline{X}
 \end{array}$$

We may then apply the operator T of (1.1.6) relatively to $j_\alpha, j_\beta: \underline{U} \rightarrow \underline{X}$ and set

$$(5.4.1) \quad \tilde{\Theta}_{m+1} := T\Phi_{m+1}(f, \varphi) \in \mathcal{E}_{m+1}^{2m+1}(\underline{U}, [\mathbb{R}]).$$

This element satisfies the conditions

$$(5.2.1) \quad \begin{aligned} \text{(i)} \quad & \Delta \tilde{\Theta}_{m+1} = j_\beta^* \Phi_{m+1}(f, \varphi) - j_\alpha^* \Phi_{m+1}(f, \varphi) \\ \text{(ii)} \quad & \text{The tail of } \tilde{\Theta}_{m+1} \text{ is } 0. \end{aligned}$$

Indeed, (i) is a consequence of (1.1.7) and (ii) of the fact that T induces 0 on 0-cochains and global sections. Now set

$$(5.4.3) \quad \Theta_{m+1} := j_\alpha^*(\Theta_{m+1, \alpha}) - j_\beta^*(\Theta_{m+1, \beta}) + \tilde{\Theta}_{m+1} \in \mathcal{E}_{m+1}^{2m+1}(\underline{U}, [\mathbb{R}]).$$

This element satisfies, by (5.3.2) and (5.4.2)

$$(5.4.4) \quad \begin{aligned} \text{(i)} \quad & \Delta \Theta_{m+1} = 0 \\ \text{(ii)} \quad & \text{The tail of } \Theta_{m+1} \text{ is } \psi := (\psi_\alpha - \psi_\beta)|_{\underline{U}}. \end{aligned}$$

We notice that Lemma 2.2(i) does not apply to the canonical morphism

$$\check{H}^{2m+1}(\underline{U}; \mathbb{R}, \mathcal{L}_{m+1}^\bullet) \rightarrow H^{2m+1}(\underline{U}, \mathbb{R})$$

for among the groups $H^{2m-k}(\underline{U}, \mathcal{L}_{m+1}^k)$ there is $H^m(\underline{U}, \mathcal{L}_{m+1}^m) = H^m(\underline{U}, \Omega^m \oplus \bar{\Omega}^m)$ which is not 0 in general. So we apply the operator μ defined in 3.4 to obtain $\mu\Theta_{m+1} \in \mathcal{E}_m^{2m+1}(\underline{U}, [\mathbb{R}])$.

Since μ commutes with D (and δ), $\mu\Theta_{m+1}$ is Δ -closed. This time the canonical morphism

$$\check{H}^{2m+1}(\underline{U}; \mathbb{R}, \mathcal{L}_m^\bullet) \rightarrow H^{2m+1}(\underline{U}, \mathbb{R})$$

is injective since the groups $H^{2m-k}(\underline{U}, \mathcal{L}_m^k)$ are all 0 for $0 \leq k \leq 2m-1$. Indeed, for $k < m$ this is due to the m -completeness of \underline{U} and, for $k \geq m$, to the fact that \mathcal{L}_m^k is a fine sheaf. So, by Corollary 2.3, $\mu\Theta_{m+1}$ is Δ -exact if its head is δ -exact in $C^*(\underline{U}, \mathbb{R})$. But the head of $\mu\Theta_{m+1}$ is equal to the head of Θ_{m+1} which is of the form $c_{m+1}|_{\underline{U}}$ with

$$c_{m+1} \in Z^{2m+1}(\underline{U}_\alpha \cap \underline{U}_\beta, \mathbb{R}).$$

Since $\underline{U} \ll \underline{U}_\alpha \cap \underline{U}_\beta$ is m -admissible, $c_{m+1}|_{\underline{U}}$ is δ -exact (Lemma 1.4.2) and therefore

$$(5.4.5) \quad \mu\Theta_{m+1} = \Delta Z_m$$

for some $Z_m \in \mathcal{E}_m^{2m}(\underline{U}, \mathbb{R})$.

Now we use the operators β and γ defined in 3.5. Denote by $\mathcal{D}_m^q(\underline{U})$ the Čech transform of the $(\bar{\partial} \oplus \partial)$ -complex over \underline{U} , i.e.

$$(5.4.6) \quad \mathcal{D}_m^q(\underline{U}) := \check{C}^q(\underline{U}; \Omega^m \oplus \bar{\Omega}^m, \mathcal{G}_m^\bullet)$$

with differential

$$(5.4.7) \quad \hat{\Delta} := \delta + (-1)^{m+q+1} \hat{d} : \mathcal{D}_m^q(\underline{U}) \rightarrow \mathcal{D}_m^{q+1}(\underline{U}).$$

Notice that this sign convention differs from (2.1.3).

Diagram (3.5.3) becomes

$$(5.4.8) \quad \begin{array}{ccccc} & & \mathcal{E}_m^{2m}(\underline{U}) & & \\ & & \downarrow \gamma & & \\ & & \mathcal{D}_m^m(\underline{U}) & & \\ \mathcal{E}_{m+1}^{2m+1}(\underline{U}) & \xrightarrow{\beta} & & \xrightarrow{\Delta = \delta - D} & \mathcal{D}_m^m(\underline{U}) \\ \downarrow \mu & & \downarrow \Delta = \delta - D & & \downarrow \hat{\Delta} = \delta - \hat{d} \\ \mathcal{E}_m^{2m+1}(\underline{U}) & & & & \mathcal{D}_m^{m+1}(\underline{U}) \\ \downarrow \Delta = \delta + D & & \downarrow \gamma & & \downarrow \beta \\ \mathcal{E}_{m+1}^{2m+2}(\underline{U}) & & \mathcal{D}_m^{m+1}(\underline{U}) & & \end{array}$$

By Lemma 3.5.3 and the sign convention (5.4.7) on $\hat{\Delta}$ we have on $\mathcal{E}_{m+1}^{2m+1}(\underline{U})$

$$(5.4.9) \quad \begin{aligned} \beta \Delta - \hat{\Delta} \beta &= \beta(\delta + D) - (\delta - \hat{d})\beta = (\beta\delta - \delta\beta) + (\beta D + \hat{d}\beta) \\ &= \beta D + \hat{d}\beta = \gamma\mu. \end{aligned}$$

On the other hand, we have on $\mathcal{E}_m^{2m}(\underline{U})$

$$(5.4.10) \quad \gamma \Delta = \hat{\Delta} \gamma.$$

If we apply (5.4.9) to Θ_{m+1} and (5.4.10) to Z_m , we get

$$-\hat{\Delta} \beta \Theta_{m+1} = (\beta \Delta - \hat{\Delta} \beta) \Theta_{m+1} = \gamma \mu \Theta_{m+1} = \gamma \Delta Z_m = \hat{\Delta} \gamma Z_m$$

which means that the element

$$(5.4.11) \quad A_m := \beta \Theta_{m+1} + \gamma Z_m \in \mathcal{D}_m^m(\underline{U})$$

satisfies

$$\hat{\Delta} A_m = 0.$$

The tail of A_m has the form

$$(\varrho^{m,m}, \sigma^{m,m}) \in A^{m,m}(\underline{U}) \oplus A^{m,m}(\underline{U})$$

with $\bar{\partial} \varrho^{m,m} = \partial \sigma^{m,m} = 0$ (since $\hat{\Delta} A_m = 0$) and

$$(5.4.12) \quad \varrho^{m,m} + \sigma^{m,m} = \psi$$

by Lemma 3.5.3(iii).

The fact that A_m is a $\hat{\Delta}$ -cocycle means precisely that $\varrho^{m,m}$ and $\bar{\sigma}^{m,m}$ represent elements of $H^m(\underline{U}, \Omega^m)$. So if we set

$$(5.4.13) \quad \tau_{\alpha\beta}^i := \frac{i}{2} (\bar{\sigma}^{m,m} - \varrho^{m,m})$$

it is clear that conditions (iv) and (v) of Theorem 2 are satisfied.

5.5. *Remark.* (1) We did not use the positivity of ω in the proof of Theorem 2. The result we can actually prove by our method is the following: If U_α and $U_{\alpha\beta}^j$ are the open sets of Theorem 2, then conditions (i) and (ii) remain unchanged. If moreover $\kappa_0, \dots, \kappa_m$ are arbitrary elements of $\mathcal{K}^{-1}(X)$ and $\omega_q := \partial\bar{\partial}\kappa_q$ for $0 \leq q \leq m$, then

(iii) There are elements $\psi_\alpha \in A^{m,m}(U_\alpha)$ such that $(\omega_0 \wedge \dots \wedge \omega_m)|_{U_\alpha} = \partial\bar{\partial}\psi_\alpha$.

(iv) There are elements $\varrho_{\alpha\beta}^j, \sigma_{\alpha\beta}^j \in A^{m,m}(U_{\alpha\beta}^j)$ such that $\bar{\partial}\varrho_{\alpha\beta}^j = \partial\sigma_{\alpha\beta}^j = 0$ and $(\psi_\alpha - \psi_\beta)|_{U_{\alpha\beta}^j} = \varrho_{\alpha\beta}^j + \sigma_{\alpha\beta}^j$.

(v) $\varrho_{\alpha\beta}^j$ and $\bar{\sigma}_{\alpha\beta}^j$ represent cohomology classes of $H^m(U_{\alpha\beta}^j, \Omega^m)$.

(2) The proof we gave was a reasoning on $\mathcal{E}_m^*(\underline{X}, [\mathbb{R}])$. We could have chosen $\mathcal{E}_m^*(\underline{X}, \mathbb{R})$ as well, replacing $\Phi_{m+1}(f, \varphi)$ by

$$\operatorname{Re}(\Phi_{m+1}(f, \varphi)) = \frac{1}{2}(\Phi_{m+1}(f, \varphi) + \Phi_{m+1}(f, \varphi)^*)$$

and using Lemma 3.5.3(iv).

IV. The Main Results

1. Stability Theorems

We are now in position to prove that some proper images of Kähler spaces are Kähler.

1.1. Theorem 3. *Let $\pi: X \rightarrow X'$ be a geometrically flat morphism of complex spaces with m -dimensional fibers (π is proper surjective and X' reduced by definition). Suppose X is Kähler. Then X' is weakly Kähler.*

If moreover there is a discrete $D' \subset X'$ such that for any $x' \in X' \setminus D'$, either

(i) X' is weakly normal at x' or

(ii) $\pi^{-1}(x')$ admits in X a smoothly embeddable neighborhood

then X' is Kähler.

Proof. With the notations of Theorem 2, set

$$V'_\alpha := \{x' \in X' \mid \pi^{-1}(x') \subset U_\alpha\}$$

$$V_\alpha := \pi^{-1}(V'_\alpha)$$

$$V_{\alpha\beta}^j := \{x' \in X' \mid \pi^{-1}(x') \subset U_{\alpha\beta}^j\}$$

$$V_{\alpha\beta}^j := \pi^{-1}(V_{\alpha\beta}^j)$$

$$\psi_\alpha := \pi_* (\chi_\alpha|_{V_\alpha})$$

$$g_{\alpha\beta}^j := \pi_* (\tau_{\alpha\beta}^j|_{V_{\alpha\beta}^j}).$$

Since π is surjective, the sets V'_α cover X' and, for fixed α, β , the $V_{\alpha\beta}^j$ cover $V'_\alpha \cap V'_\beta$. By Proposition 3.4.1 of Chap. I, $\psi_\alpha \in SP^0(V'_\alpha)$, $g_{\alpha\beta}^j \in \mathcal{W}^j(V_{\alpha\beta}^j)$ and, since $(\psi_\alpha - \psi_\beta)|_{V_{\alpha\beta}^j} = g_{\alpha\beta}^j + \bar{g}_{\alpha\beta}^j$, $\psi_\alpha - \psi_\beta \in WPH(V'_\alpha \cap V'_\beta, \mathbb{R})$. So X' is weakly Kähler. Now if conditions (i) and (ii) are fulfilled, then $g_{\alpha\beta}^j$ is holomorphic on $V_{\alpha\beta}^j \setminus D'$ and $\psi_\alpha - \psi_\beta$ pluriharmonic on $V'_\alpha \cap V'_\beta \setminus D'$. If we take a refinement (W'_λ) of (V'_α) such that each point of D' belongs at most to one W'_λ , then it is clear that Theorem 1 applies and X' is Kähler.