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5.5. *Remark.* (1) We did not use the positivity of  $\omega$  in the proof of Theorem 2. The result we can actually prove by our method is the following: If  $U_\alpha$  and  $U_{\alpha\beta}^j$  are the open sets of Theorem 2, then conditions (i) and (ii) remain unchanged. If moreover  $\kappa_0, \dots, \kappa_m$  are arbitrary elements of  $\mathcal{K}^{-1}(X)$  and  $\omega_q := \partial\bar{\partial}\kappa_q$  for  $0 \leq q \leq m$ , then

(iii) There are elements  $\psi_\alpha \in A^{m,m}(U_\alpha)$  such that  $(\omega_0 \wedge \dots \wedge \omega_m)|_{U_\alpha} = \partial\bar{\partial}\psi_\alpha$ .

(iv) There are elements  $\varrho_{\alpha\beta}^j, \sigma_{\alpha\beta}^j \in A^{m,m}(U_{\alpha\beta}^j)$  such that  $\bar{\partial}\varrho_{\alpha\beta}^j = \partial\sigma_{\alpha\beta}^j = 0$  and  $(\psi_\alpha - \psi_\beta)|_{U_{\alpha\beta}^j} = \varrho_{\alpha\beta}^j + \sigma_{\alpha\beta}^j$ .

(v)  $\varrho_{\alpha\beta}^j$  and  $\bar{\sigma}_{\alpha\beta}^j$  represent cohomology classes of  $H^m(U_{\alpha\beta}^j, \Omega^m)$ .

(2) The proof we gave was a reasoning on  $\mathcal{E}_m^*(\underline{X}, [\mathbb{R}])$ . We could have chosen  $\mathcal{E}_m^*(\underline{X}, \mathbb{R})$  as well, replacing  $\Phi_{m+1}(f, \varphi)$  by

$$\operatorname{Re}(\Phi_{m+1}(f, \varphi)) = \frac{1}{2}(\Phi_{m+1}(f, \varphi) + \Phi_{m+1}(f, \varphi)^*)$$

and using Lemma 3.5.3(iv).

## IV. The Main Results

### 1. Stability Theorems

We are now in position to prove that some proper images of Kähler spaces are Kähler.

**1.1. Theorem 3.** *Let  $\pi: X \rightarrow X'$  be a geometrically flat morphism of complex spaces with  $m$ -dimensional fibers ( $\pi$  is proper surjective and  $X'$  reduced by definition). Suppose  $X$  is Kähler. Then  $X'$  is weakly Kähler.*

*If moreover there is a discrete  $D' \subset X'$  such that for any  $x' \in X' \setminus D'$ , either*

(i)  $X'$  is weakly normal at  $x'$  or

(ii)  $\pi^{-1}(x')$  admits in  $X$  a smoothly embeddable neighborhood

*then  $X'$  is Kähler.*

*Proof.* With the notations of Theorem 2, set

$$V'_\alpha := \{x' \in X' \mid \pi^{-1}(x') \subset U_\alpha\}$$

$$V_\alpha := \pi^{-1}(V'_\alpha)$$

$$V_{\alpha\beta}^j := \{x' \in X' \mid \pi^{-1}(x') \subset U_{\alpha\beta}^j\}$$

$$V_{\alpha\beta}^j := \pi^{-1}(V_{\alpha\beta}^j)$$

$$\psi_\alpha := \pi_* (\chi_\alpha|_{V_\alpha})$$

$$g_{\alpha\beta}^j := \pi_* (\tau_{\alpha\beta}^j|_{V_{\alpha\beta}^j}).$$

Since  $\pi$  is surjective, the sets  $V'_\alpha$  cover  $X'$  and, for fixed  $\alpha, \beta$ , the  $V_{\alpha\beta}^j$  cover  $V'_\alpha \cap V'_\beta$ . By Proposition 3.4.1 of Chap. I,  $\psi_\alpha \in SP^0(V'_\alpha)$ ,  $g_{\alpha\beta}^j \in \mathcal{W}(V_{\alpha\beta}^j)$  and, since  $(\psi_\alpha - \psi_\beta)|_{V_{\alpha\beta}^j} = g_{\alpha\beta}^j + \bar{g}_{\alpha\beta}^j$ ,  $\psi_\alpha - \psi_\beta \in WPH(V'_\alpha \cap V'_\beta, \mathbb{R})$ . So  $X'$  is weakly Kähler. Now if conditions (i) and (ii) are fulfilled, then  $g_{\alpha\beta}^j$  is holomorphic on  $V_{\alpha\beta}^j \setminus D'$  and  $\psi_\alpha - \psi_\beta$  pluriharmonic on  $V'_\alpha \cap V'_\beta \setminus D'$ . If we take a refinement  $(W'_\lambda)$  of  $(V'_\alpha)$  such that each point of  $D'$  belongs at most to one  $W'_\lambda$ , then it is clear that Theorem 1 applies and  $X'$  is Kähler.

**1.2. Corollary.** *Let  $\pi: X \rightarrow X'$  be a proper open surjective morphism. Suppose  $X$  is Kähler and  $X'$  normal. Then  $X'$  is Kähler.*

Many other consequences may be formulated. For example

**1.3. Corollary.** *Let  $\pi: X \rightarrow X'$  be a flat projective morphism. Suppose  $X$  is Kähler and  $X'$  reduced. Then  $X'$  is Kähler.*

*Proof.* The fibers of a projective morphism have smoothly embeddable neighborhoods by construction of  $\mathbb{P}(\mathcal{F})$  for a coherent sheaf  $\mathcal{F}$ .

**1.4. Remark.** Conditions (i) and (ii) of Theorem 3 are actually unnecessary. See note 3.6 of Chap. I.

## 2. The Space of Cycles of a Kähler Space

We use the notations of Chap. I, 3.

**2.1. Theorem 4.** *Let  $X$  be a Kähler space and  $m \geq 0$  an integer. Then the Barlet space  $\mathbf{B}_m(X)$  of  $m$ -cycles of  $X$  is weakly Kähler. Moreover, the open subset  $\mathbf{B}_m(X)^{(0)}$  of  $\mathbf{B}_m(X)$  is Kähler.*

*Proof.* By an argument similar to the above, set

$$W_\alpha := \{c \in \mathbf{B}_m(X) \mid |c| \subset U_\alpha\}$$

$$W_{\alpha\beta}^j := \{c \in \mathbf{B}_m(X) \mid |c| \subset U_{\alpha\beta}^j\}$$

$$\Phi_\alpha := F_{\chi_\alpha}, \quad G_{\alpha\beta}^j := F_{\tau_{\alpha\beta}^j},$$

Then  $\Phi_\alpha \in SP^0(W_\alpha)$ ,  $G_{\alpha\beta}^j$  is weakly holomorphic on  $W_{\alpha\beta}^j$  and holomorphic on  $W_{\alpha\beta}^j \cap \mathbf{B}_m(X)^{(0)}$ ,  $(\Phi_\alpha - \Phi_\beta)|_{W_{\alpha\beta}^j} = G_{\alpha\beta}^j + \bar{G}_{\alpha\beta}^j$  and the result follows.

**2.2. Corollary.** *Let  $X$  be a Kähler space. Then the weak normalization of  $\mathbf{B}_m(X)$  is Kähler.*

*Proof.* By a well-known result [5, 12, 18] every connected component of  $\mathbf{B}_m(X)$  is compact and, by Theorem 4 above, weakly Kähler. The result follows from Proposition 4.2.4 of Chap. II.

## 3. Fujiki's Class $\mathcal{C}$

**3.1. Definition** (Fujiki [12]). A reduced compact complex space  $X$  is said to belong to class  $\mathcal{C}$  if it is a holomorphic image of a compact Kähler space.

By Hironaka's resolution of singularities it is sufficient to take holomorphic images of compact Kähler manifolds.

Let us define for the moment the class  $\mathcal{C}^*$  of reduced compact spaces bimeromorphically equivalent to compact Kähler manifolds, i.e. admitting compact Kähler modifications.

It is then true that  $\mathcal{C}$  is stable under holomorphic images and subspaces; but it seems difficult to prove, for example, that a reduced subspace of a space in  $\mathcal{C}^*$  is in  $\mathcal{C}^*$ . Of course,  $\mathcal{C}^* \subset \mathcal{C}$ .