5.5. *Remark.* (1) We did not use the positivity of $\omega$ in the proof of Theorem 2. The result we can actually prove by our method is the following: If $U_\alpha$ and $U^j_{\alpha\beta}$ are the open sets of Theorem 2, then conditions (i) and (ii) remain unchanged. If moreover $\kappa_0, \ldots, \kappa_m$ are arbitrary elements of $\mathcal{K}^1(X)$ and $\omega_q := \partial\bar{\partial}\kappa_q$ for $0 \leqq q \leqq m$, then

(iii) There are elements $\psi_\alpha \in A^{m,m}(U_\alpha)$ such that $(\omega_0 \wedge \ldots \wedge \omega_m)|_{U_\alpha} = \partial\bar{\partial}\psi_\alpha$.

(iv) There are elements $\varrho^j_{\alpha\beta}, \sigma^j_{\alpha\beta} \in A^{m,m}(U^j_{\alpha\beta})$ such that $\bar{\partial}\varrho^j_{\alpha\beta} = \partial\sigma^j_{\alpha\beta} = 0$ and $(\psi_\alpha - \psi_\beta)|_{U^j_{\alpha\beta}} = \varrho^j_{\alpha\beta} + \sigma^j_{\alpha\beta}$.

(v) $\varrho^j_{\alpha\beta}$ and $\bar{\sigma}^j_{\alpha\beta}$ represent cohomology classes of $H^m(U^j_{\alpha\beta}, \Omega^m)$.

(2) The proof we gave was a reasoning on $\mathscr{E}^{\cdot}_m(\underline{X}, [\mathbb{R}])$. We could have chosen $\mathscr{E}^{\cdot}_m(\underline{X}, \mathbb{R})$ as well, replacing $\Phi_{m+1}(f, \varphi)$ by

$$\mathrm{Re}(\Phi_{m+1}(f, \varphi)) = \tfrac{1}{2}(\Phi_{m+1}(f, \varphi) + \Phi_{m+1}(f, \varphi)^*)$$

and using Lemma 3.5.3(iv).

## IV. The Main Results

### *1. Stability Theorems*

We are now in position to prove that some proper images of Kähler spaces are Kähler.

**1.1. Theorem 3.** *Let $\pi: X \to X'$ be a geometrically flat morphism of complex spaces with m-dimensional fibers ($\pi$ is proper surjective and $X'$ reduced by definition). Suppose $X$ is Kähler. Then $X'$ is weakly Kähler.*

*If moreover there is a discrete $D' \subset X'$ such that for any $x' \in X' \backslash D'$, either*

(i) *$X'$ is weakly normal at $x'$ or*

(ii) *$\pi^{-1}(x')$ admits in $X$ a smoothly embeddable neighborhood*

*then $X'$ is Kähler.*

*Proof.* With the notations of Theorem 2, set

$$\begin{aligned}
V'_\alpha &:= \{x' \in X' | \pi^{-1}(x') \subset U_\alpha\}\\
V_\alpha &:= \pi^{-1}(V'_\alpha)\\
V'^j_{\alpha\beta} &:= \{x' \in X' | \pi^{-1}(x') \subset U^j_{\alpha\beta}\}\\
V^j_{\alpha\beta} &:= \pi^{-1}(V'^j_{\alpha\beta})\\
\psi_\alpha &:= \pi_*(\chi_\alpha|_{V_\alpha})\\
g^j_{\alpha\beta} &:= \pi_*(\tau^j_{\alpha\beta}|_{V^j_{\alpha\beta}}).
\end{aligned}$$

Since $\pi$ is surjective, the sets $V'_\alpha$ cover $X'$ and, for fixed $\alpha, \beta$, the $V'^j_{\alpha\beta}$ cover $V'_\alpha \cap V'_\beta$. By Proposition 3.4.1 of Chap. I, $\psi_\alpha \in SP^0(V'_\alpha)$, $g^j_{\alpha\beta} \in \mathscr{W}(V'^j_{\alpha\beta})$ and, since $(\psi_\alpha - \psi_\beta)|_{V'^j_{\alpha\beta}} = g^j_{\alpha\beta} + \bar{g}^j_{\alpha\beta}$, $\psi_\alpha - \psi_\beta \in WPH(V'_\alpha \cap V'_\beta, \mathbb{R})$. So $X'$ is weakly Kähler. Now if conditions (i) and (ii) are fulfilled, then $g^j_{\alpha\beta}$ is holomorphic on $V'^j_{\alpha\beta} \backslash D'$ and $\psi_\alpha - \psi_\beta$ pluriharmonic on $V'_\alpha \cap V'_\beta \backslash D'$. If we take a refinement $(W'_\lambda)$ of $(V'_\alpha)$ such that each point of $D'$ belongs at most to one $W'_\lambda$, then it is clear that Theorem 1 applies and $X'$ is Kähler.

**1.2. Corollary.** *Let $\pi: X \to X'$ be a proper open surjective morphism. Suppose $X$ is Kähler and $X'$ normal. Then $X'$ is Kähler.*

Many other consequences may be formulated. For example

**1.3. Corollary.** *Let $\pi: X \to X'$ be a flat projective morphism. Suppose $X$ is Kähler and $X'$ reduced. Then $X'$ is Kähler.*

*Proof.* The fibers of a projective morphism have smoothly embeddable neighborhoods by construction of $\mathbb{P}(\mathscr{F})$ for a coherent sheaf $\mathscr{F}$.

*1.4. Remark.* Conditions (i) and (ii) of Theorem 3 are actually unnecessary. See note 3.6 of Chap. I.

## *2. The Space of Cycles of a Kähler Space*

We use the notations of Chap. I, 3.

**2.1. Theorem 4.** *Let $X$ be a Kähler space and $m \geqq 0$ an integer. Then the Barlet space $\mathbf{B}_m(X)$ of $m$-cycles of $X$ is weakly Kähler. Moreover, the open subset $\mathbf{B}_m(X)^{(0)}$ of $\mathbf{B}_m(X)$ is Kähler.*

*Proof.* By an argument similar to the above, set

$$W_\alpha := \{c \in \mathbf{B}_m(X) \mid |c| \subset U_\alpha\}$$

$$W^j_{\alpha\beta} := \{c \in \mathbf{B}_m(X) \mid |c| \subset U^j_{\alpha\beta}\}$$

$$\Phi_\alpha := F_{\chi_\alpha}, \qquad G^j_{\alpha\beta} := F_{\tau^j_{\alpha\beta}},$$

Then $\Phi_\alpha \in SP^0(W_\alpha)$, $G^j_{\alpha\beta}$ is weakly holomorphic on $W^j_{\alpha\beta}$ and holomorphic on $W^j_{\alpha\beta} \cap \mathbf{B}_m(X)^{(0)}$, $(\Phi_\alpha - \Phi_\beta)|_{W^j_{\alpha\beta}} = G^j_{\alpha\beta} + \bar{G}^j_{\alpha\beta}$ and the result follows.

**2.2. Corollary.** *Let $X$ be a Kähler space. Then the weak normalization of $\mathbf{B}_m(X)$ is Kähler.*

*Proof.* By a well-known result [5, 12, 18] every connected component of $\mathbf{B}_m(X)$ is compact and, by Theorem 4 above, weakly Kähler. The result follows from Proposition 4.2.4 of Chap. II.

## *3. Fujiki's Class $\mathscr{C}$*

**3.1. Definition** (Fujiki [12]). A reduced compact complex space $X$ is said to belong to class $\mathscr{C}$ if it is a holomorphic image of a compact Kähler space.

By Hironaka's resolution of singularities it is sufficient to take holomorphic images of compact Kähler manifolds.

Let us define for the moment the class $\mathscr{C}^*$ of reduced compact spaces bimeromorphically equivalent to compact Kähler manifolds, i.e. admitting compact Kähler modifications.

It is then true that $\mathscr{C}$ is stable under holomorphic images and subspaces; but it seems difficult to prove, for example, that a reduced subspace of a space in $\mathscr{C}^*$ is in $\mathscr{C}^*$. Of course, $\mathscr{C}^* \subset \mathscr{C}$.