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A Probabilistic Proof and Applications of Wiener's Test for the Heat Operator

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1. Introduction

In this paper we study a thinness criterion for the coparabolic operator $\frac{1}{2}\Delta + \partial/\partial t$ in stead of the parabolic (heat) operator $\frac{1}{2}\Delta - \partial/\partial t$. The translation of the results and proofs in one context into those in the other is straightforward. The criterion which is a precise analogue of Wiener's test for the Laplace operator will be stated first in a probabilistic way and then in purely analytic terms.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, $E[\cdot]$ the integration by P, and $W(t)$ $(t \geqq 0)$ a standard N-dimensional Wiener process – the Brownian motion starting at zero standardized by $E[W_i(t)\, W_j(t)] = \delta_{ij} t$ – defined on $(\Omega, \mathscr{F}, P)$. Let A be an analytic set of R^{N+1} and let $h(\xi; A)$, $\xi \in R^{N+1}$ denote the hitting probability of A by the space-time Wiener process starting at $\xi = (s, \mathbf{x})$:

$$h(s, \mathbf{x}; A) = P[(s+t, \mathbf{x} + W(t)) \in A \quad \text{for some} \quad t > 0]$$

where $s \in R$ and $\mathbf{x} \in R^N$. Here and hereafter, abusing the notation, we write simply $f(s, \mathbf{x})$ for $f((s, \mathbf{x}))$ where f is a function of $\xi \in R^{N+1}$. Let us write $x = |\mathbf{x}|$ (the Euclidean length of $\mathbf{x}$), and define

$$p(t, x) = \begin{cases} (2\pi t)^{-N/2} e^{-x^2/2t} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leqq 0 . \end{cases}$$

The function $g(\xi) := p(t, x)$, $\xi = (t, \mathbf{x})$, is a Green (density) function of the space-time Wiener process $(t, W(t))$ which is a density function of the measure $\mu(A) = \int_0^\infty P[(t, W(t)) \in A]\, dt$ relative to the Lebesgue measure of R^{N+1}. By means of the Green function are defined coparabolic balls centered at the origin $O := (0, \mathbf{0})$, which here we parametrize by $a > 0$ as follows

$$B(a) := \{(t, \mathbf{x}) \in R^{N+1} : p(t, x) > (2\pi a)^{-N/2}\},$$

so that $B(a)$ becomes large together with a. For A a subset of R^{N+1} we say that A is *hit i.o.* (infinitely often) as $t \uparrow \infty$ $[t \downarrow 0]$ if there exists an increasing [resp. a decreasing] sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$ [resp. $t_n \to 0$] as $n \to \infty$ and $(t_n, W(t_n)) \in A$ for every n.

Theorem 1. *Let A be an analytic set of $[0, \infty) \times R^N$ and set*

$$A_n = A \cap [B(2^{n+1}) \setminus B(2^n)].$$

i) *$P[A$ is hit i.o. as $t \uparrow \infty] = 0$ or 1 according as $\sum_{n=0}^{\infty} h(O; A_n)$ converges or diverges;*

ii) *$P[A$ is hit i.o. as $t \downarrow 0] = 0$ or 1 according as $\sum_{n=0}^{\infty} h(O; A_{-n})$ converges or diverges.* □

The Green kernel $g(\xi, \eta) := g(\eta - \xi)$ is the fundamental solution for $\frac{1}{2}\Delta + \frac{\partial}{\partial t}$ and this theorem accordingly can be stated in potential theoretic terms. We need the fact that if A is bounded and analytic then there exists a finite Borel measure e_A on R^{N+1} (coparabolic equilibrium measure) such that

$$h(s, \mathbf{x}; A) = \int_{R^{N+1}} p(t-s, |\mathbf{y} - \mathbf{x}|) \, e_A(dt \, d\mathbf{y}). \tag{1.1}$$

The (coparabolic) capacity of A is the total measure of e_A, which we denote by $\mathrm{Cap}(A)$: $\mathrm{Cap}(A) = e_A(R^{N+1})$. It is then clear that for A_n in Theorem 1

$$(2\pi 2^{n+1})^{-N/2} \, \mathrm{Cap}(A_n) \leqq h(O; A_n) \leqq (2\pi 2^n)^{-N/2} \, \mathrm{Cap}(A_n), \tag{1.2}$$

so that the convergence conditions of the series in Theorem 1 are equivalent to those of $\sum_{n=0}^{\infty} 2^{\mp Nn/2} C(A_{\pm n})$.

Let A be an analytic set that has the origin O as its cluster point. A is said to be (coparabolic) *thin* at O if there is a supercoparabolic function u defined on an open neighbourhood of O such that each of its cluster values at O along A is greater than $u(O)$ Doob [2, p. 309]. It is known that A is thin at O if and only if $P[A$ is hit i.o. as $t \downarrow 0] = 0$ [2, p. 656]. Thus the second half of Theorem 1 is paraphrased in purely analytic terms as follows.

Theorem 2. *An analytic set A of R^{N+1} is thin at the origin if and only if $\sum_{n=0}^{\infty} 2^{Nn/2} \, \mathrm{Cap}(A_{-n}) < \infty$.* □

To make a similar rewording of the first half of Theorem 1 we consider the (coparabolic) Martin boundary of the upper half space $t \geqq 0$. Among others there is a minimal Martin boundary point, denoted by ζ_0, corresponding to the limit of $t \to \infty$ and $x/t \to 0$, at which the Martin function is identically one [2, p. 374]. An analytic set A of $[0, \infty) \times R^N$ is (coparabolic) minimal thin at ζ_0 if and only if $P[A$ is hit i.o. as $t \uparrow \infty] = 0$ [2, p. 731]. Therefore Theorem 1 yields

Theorem 2′. *At the Martin boundary point ζ_0 an analytic set A of $[0, \infty] \times R^N$ is minimal thin if and only if $\sum_{n=0}^{\infty} 2^{-Nn/2} \, \mathrm{Cap}(A_n) < \infty$.* □

Theorem 2 has been established by Evans and Gariepy [4]. The method for their proof is potential theoretic, whereas our approach is very probabilistic and has the advantage of being relatively simple not only in its formal presentation but also in the guiding idea which is intuitive and easy to grasp.

If A lies inside the paraboloid $x^2=ct$ for some $c>0$ Theorem 1 is almost as worthless (because dispensable) as its proof is uninvolved, whereas it works very efficiently if A_n are outside such paraboloids with $c=c_n$ tending infinity as $|n|$ becomes large. The latter fact is conceptually because for large $|n|$ the hittings of A_n in the latter case are very rare events which are relatively strongly positively-correlated with one another. In practice, however, the applicability of Theorem 1 is greatly due to the explicit and simple expression of the distribution ν_a of the place at which $(t, W(t))$ hits the coparabolic sphere $S(a):=\partial B(a)$ [the boundary of $B(a)$, $a>0$]. It is shown by Bauer [1] that the parabolic counterpart of ν_a agrees with the measure that Fulks [7] introduced as a measure with which the parabolic function is characterized as having local average property over parabolic spheres [this result implies in particular that $R^{N+1}\setminus B(a)$ is thin at the origin]. In the present context it is convenient to parametrise points of $S(a)$ by the ordinate t and the spherical component θ of the abscissa $\mathbf{x}$ (i.e., $\theta=\mathbf{x}/x$); from ν_a this parametrization induces the measure

$$\hat{\nu}_a(dt\,d\theta)=\tfrac{1}{2}x^N p(t,x)t^{-1}\,dt\,d\theta \qquad (0<t\leqq a, \theta\in\Theta) \tag{1.3}$$

with $\hat{\nu}(\{0\}\times\Theta)=0$ where $x=\sqrt{-Nt\log t/a}$, $\Theta=\{\theta\in R^N:|\theta|=1\}$ and $d\theta$ is a surface element of $(N-1)$-unit-sphere Θ if $N\geqq 2$ and a discrete measure which charges each point of $\Theta=\{-1,1\}$ with unit mass if $N=1$. [To be precise $\hat{\nu}_a$ is the probability measure on $[0,a]\times\Theta$ induced from ν_a by the mapping $(t,\mathbf{x})\to(t,\mathbf{x}/x)$.] Then a little reflection would convince us that Theorem 1 could verify the Kolmogorov's test. In fact not only this is true, but also we can strengthen the harder half of the test as follows.

Theorem 3. *Let $f=f(t,\theta)$ be a positive Borel function of $t\geqq 0$ and $\theta=\mathbf{x}/x$ and G denote the graph of f: $G=\{(t,\mathbf{x}):f(t,\mathbf{x}/x)=x\}$. If*

$$\int_{\substack{0<t<1\\ [t>1]}} t^{-1}\,dt \int_{\Theta} f(t,\theta)^N\, p(t,f(t,\theta))\,d\theta=\infty\,,$$

then $P[G$ is hit i.o. as $t\downarrow 0$ [resp. $t\uparrow\infty$]$]=1$ or, what is the same, G is not thin at the origin [resp. minimal thin at ζ_0]. □

The transformation $\psi:(t,\mathbf{x})\to(1/t,\mathbf{x}/t)$ which transforms $W(t)$ into $\tilde{W}(t):=tW(1/t)$ does not change the probability law of W, i.e.,

$$\tilde{W}(t) \text{ also is a standard Wiener process under } (\Omega,\mathscr{F},P) \tag{1.4}$$

[a *projective invariance* of $W(\cdot)$]. Applying Theorem 1 with $\psi(A)$ and $\tilde{W}(t)$ in place of A and $W(t)$, we accordingly get that $P[A$ is hit i.o. as $t\downarrow 0$ $[t\uparrow\infty]]=0$ or 1 according as $\sum_{n=0}^{\infty[-\infty]} h(O;A'_n)$ converges or diverges where

$$A'_n=\psi(\psi(A)_n)=\{(t,\mathbf{x})\in A:(2\pi 2^{n+1})^{-N/2}<p(1/t,\mathbf{x}/t)\leqq(2\pi 2^n)^{-N/2}\}\,.$$

Note that, A'_n being quite different from A_n, the criterion thus obtained is in appearance not the same as that in Theorem 1.

To make our probabilistic approach self-contained we shall in Appendix give direct probabilistic proofs of (1.1, 1.3). The proof of Theorem 1 will be given in Sect. 2. In Sect. 4 several applications of Theorem 1 will be made to obtain easily computable criteria for the thinness. In particular the Kolmogorov's test and Theorem 3 will be deduced. In Sect. 3 Theorem 1 will be refined in a way as a preparation for the proof of Theorem 3.

2. Proof of Theorem 1

We shall prove only the first half of Theorem 1. The proof for the other half is similar (though not completely parallel). In the rest of the paper n will denote a non-negative integer unless the contrary is explicitly stated. We shall assume that A lies above the ordinate level $t=1$, i.e., $A\subset[1,\infty)\times R^N$. The case $\sum h(O;A_n)<\infty$ is trivially disposed of by the Borel-Cantelli lemma. Since $P[|W(t)|>t$ i.o. as $t\uparrow\infty]=0$ [consider $\tilde{W}(t)=tW(1/t)$], it therefore suffices to show that

$$\text{if } \sum_{n=0}^{\infty} h(O,A_n)=\infty, \quad \text{then} \quad P[A_n \text{ is hit for i.m. } n]=1. \tag{2.1}$$

Here the expression under P means that there exist infinitely many n s.t. $(t, W(t))\in A_n$ for some $t>1$.

The geometry of coparabolic balls $B(a)$ or spheres $S(a):=\partial B(a)$ is fundamental in our proof. Put

$$X_a(t)=\sqrt{-Nt\log t/a}, \qquad 0\leqq t\leqq a.$$

Then $(t,\mathbf{x})\in B(a)$ if and only if $x:=|\mathbf{x}|<X_a(t)$. We see also

$$\frac{d}{dt}X_a(t)=-\frac{N}{2}\,\frac{1+\log t/a}{X_a(t)}. \tag{2.2}$$

Therefore $B(a)$ is an egg-shaped body with the top at $(a,\mathbf{0})$ and its bottom (the center) at the origin $O=(0,\mathbf{0})$, and broadest in the abscissa direction at the ordinate level $t=a/e$. We make another observation of interest: for each $\varepsilon>0$ the mapping $\varphi^\varepsilon:(t,\mathbf{x})\to(\varepsilon t,\sqrt{\varepsilon}\,\mathbf{x})$, which does not change the law of the Wiener process [the *scaling invariance*, i.e., the process $W(\varepsilon t)/\sqrt{\varepsilon}$, $t\geqq 0$ has the same distribution as $W(t)$, $t\geqq 0$], transforms $B(a)$ into $B(\varepsilon a)$, i.e., $\varphi^\varepsilon(B(a))=B(\varepsilon a)$; in particular

$$h(\varepsilon s,\sqrt{\varepsilon}\,\mathbf{x};\varphi^\varepsilon(A_n))=h(s,x;A_n) \quad\text{and}\quad \varphi^{2^{-n}}(A_{n+k})\subset B(2^{k+1})\backslash B(2^k), \tag{2.3}$$

where A_n is supposed to be a subset of $B(2^{n+1})\backslash B(2^n)$. The following inequality immediate from (1.1) and being a trivial extension of (1.2) also plays a key role in our proof: for each $(s,\mathbf{x})\in R^{N+1}$ and each analytic set A of $B(a)\,(a>0)$

$$h(s,\mathbf{x};A)\leqq(2\pi a)^{N/2}\,h(O,A)\sup_{(t,\mathbf{y})\in A} p(t-s,|\mathbf{y}-\mathbf{x}|). \tag{2.4}$$

For later (i.e., beyond the proof of Theorem 1) as well as present needs here is noted that if $s<1/2$, then

$$\Delta x:=X_2(s)-X_1(s)=\frac{sN\log 2}{X_2(s)+X_1(s)}=\frac{\log 2}{2}\cdot\frac{\sqrt{sN}(1+\varepsilon(s))}{\sqrt{-\log s/2}} \tag{2.5}$$

with $0 \leqq \varepsilon(s) \leqq \text{const}(-\log s)^{-1}$: this in particular shows that Δx is much smaller than $X_1(s)$ if s is small.

The next lemma is a result of a simple application of the inequality (2.4).

Lemma 4. *Let $c, s, \delta > 0$, $0 < a < a'$ and A an analytic subset of $D \cap [B(a') \setminus B(a)]$ where $D := \{(t, \mathbf{y}) \in R^{N+1} : t > (1+\delta)s, y > t\}$. Then it follows that for all* $\mathbf{x}$ $h(s, \mathbf{x}; A) \leqq e^x(1 + 1/\delta)a'/a]^{N/2} h(O; A)$. □

Proof. The inequality (2.4) reduces the problem to proving that for $a \leqq b \leqq a'$

$$\sup_{(t,\mathbf{y}) \in D \cap S(b)} p(t-s, |\mathbf{y}-\mathbf{x}|) \leqq e^x \left[1 + \frac{1}{\delta}\right]^{N/2} (2\pi b)^{-N/2}. \tag{2.6}$$

To evaluate the supremum above we express p as follows

$$p(t-s, |\mathbf{y}-\mathbf{x}|) = (2\pi b)^{-N/2} \left(1 - \frac{s}{t}\right)^{-N/2} e^{H(t,\mathbf{y})} \quad \text{for} \quad t > s, (t, \mathbf{y}) \in S(b), \tag{2.7}$$

where $H(t, \mathbf{y}) = \frac{1}{2}(y^2/t - |\mathbf{y}-\mathbf{x}|^2/(t-s))$. Then (2.6) is ready from

$$H(t, \mathbf{y}) < \frac{1}{2t}\{y^2 - |\mathbf{y}-\mathbf{x}|^2\} \leqq \frac{xy}{t} - \frac{x^2}{2t}.$$

The main task for the proof of Theorem 1 is performed by proving

Lemma 5. *There exists two constants $\gamma < 1$ and $C > 0$ such that if $\{A_k\}$ is a sequence of analytic sets the k-th member of which is contained in $B(2^{k+1}) \setminus B(2^k)$, then for $\xi \in \overline{B(1)}$* [the closure of $B(1)$] *and $m = 1, 2, \ldots$*

$$h\left(\xi; \bigcup_{k=1}^{m} A_k\right) \leqq \gamma + C \sum_{k=1}^{m} h(O; A_k). \quad \square \tag{2.8}$$

Proof. Step 1. As the first step we prove that

$$\begin{aligned} &\textit{for each } c > 0 \textit{ there exists constants } \gamma < 1 \textit{ and } C \\ &\textit{depending only on } c \textit{ and } N \textit{ such that } (2.8) \textit{ holds} \\ &\textit{for } \xi \in K_c := \{(t, \mathbf{y}) \in \overline{B(1)} : y^2 \leqq ct\}. \end{aligned} \tag{2.9}$$

[It may be noted that $S(1)$ and the paraboloid $y^2 = ct$ intersect at an exactly one level of ordinate $t = e^{-N/c}$.] Let $F = \{(t, \mathbf{y}) : y > t > 3/2 \text{ or } t \leqq 3/2\} \setminus B(2)$. Then there exists $\gamma < 1$ such that $\sup^* h(\xi; F) \leqq \gamma$, where the $\sup^*$ denotes the supremum taken over all $\xi \in K_c \cap S(1)$. On the other hand by Lemma 4 $\sup^* h\left(\xi; \bigcup_{k=1}^{m} A_k \setminus F\right)$ $\leqq C \sum_{k=1}^{m} h(O; A_k)$. Thus (2.8) holds for $\xi \in K_c \cap S(1)$. As for $\xi \in K_c \setminus S(1)$ we have $\varphi^\varepsilon(\xi) \in K_c \cap S(1)$ with some $\varepsilon > 1$ and in view of the scaling relation (2.3) the same argument as above proves (2.8) with a smaller γ and the same C, proving (2.9).

Step 2. Owing to the strong Markov property of $W(\cdot)$ we have only to show (2.8) for $\xi \in S(1)$. By (2.9) it therefore suffices to prove that there is a (large) constant c such that if $\xi = (s, \mathbf{x})$ satisfies

$$(s, \mathbf{x}) \in S(1) \quad (\text{i.e., } x^2 = -Ns \log s) \quad \text{and} \quad x^2 > cs, \tag{2.10}$$

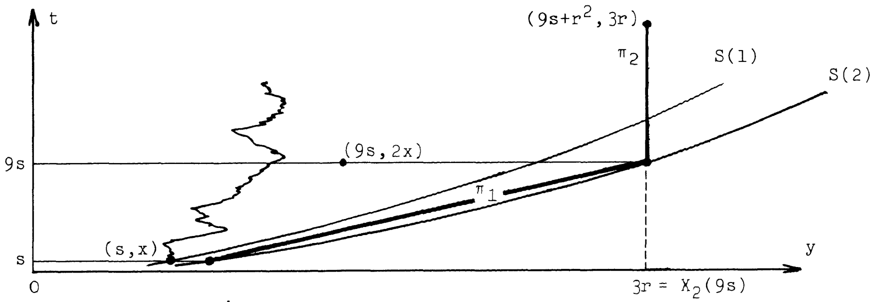

Fig. 1

then (2.8) holds. Let (2.10) hold. Consider two fences in R^{N+1}, say π_1 and π_2, obtained by rotating around the t-axis two line segments lying on the (t, x_1)-plane (x_1 is the first coordinate of R^N-variable $\mathbf{x}$) whose end points have coordinates $(s, X_2(s))$ and $(9s, X_2(9s))$ for π_1; $(9s, X_2(9s))$ and $(9s+[X_2(9s)/3]^2, X_2(9s))$ for π_2, i.e.,

$$\pi_1 := \left\{(t, \mathbf{y}): \frac{y - X_2(s)}{t-s} = \frac{X_2(9s)-X_2(s)}{8s},\ s \leqq t \leqq 9s\right\}$$

$$\pi_2 := \{(t, \mathbf{y}): y = 3r, 9s < t < 9s + r^2\},$$

where $r := X_2(9s)/3$. From (2.9) it follows that $-\log s \geqq \frac{c}{N}$, and then we see for large c

$$X_2(9s) = \sqrt{9}\left\{\frac{-\log 9s/2}{-\log s/2}\right\}^{1/2} \cdot X_2(s) \geqq 3X_2(s), \tag{2.11}$$

which inequality we assume to hold in below. Clearly π_1 lies inside $S(2)$. We can suppose that π_2 also lies inside $S(2)$. In fact, since $X_2(t)$ is increasing up to $t = 2/e$, it suffices to have $9s + X_2(9s)^2/9 \leqq 2/e$, which is clearly possible by taking c large enough. Now setting

$$J_1 = P[\pi_1 \cup \pi_2 \text{ is not hit at all} \mid W(s) = \mathbf{x}]$$

$$J_2 = \sup_{|\mathbf{z}| \leqq 3r} h\left(9s + r^2, \mathbf{z}; \bigcup_{k=1}^{m} A_k\right)$$

and applying the strong Markov property of $W(\cdot)$, we have $h\left(s, \mathbf{x}; \bigcup_{k=1}^{m} A_k\right) \leqq 1 - J_1(1-J_2)$. (Here and hereafter $P[\cdot \mid \mathfrak{A}]$ denotes a conditional law given an event $\mathfrak{A}$.) We may suppose $(9s + r^2, \mathbf{z}) \in K_9$ for $z < 3r$ so as to be able to apply (2.9) with $c = 9$ to obtain $J_2 \leqq \gamma + C \sum_{k=1}^{m} h(O, A_k)$. Therefore

$$h\left(s, \mathbf{x}; \bigcup_{k=1}^{m} A_k\right) \leqq 1 - J_1 + J_1\gamma + J_1 C \sum_{k=1}^{m} h(O, A_k),$$

and if we can show that $J_1 = J_1(s, \mathbf{x}) \geqq p$ with a positive constant p, then (2.8) holds with γ replaced by $\gamma' = 1 - p(1-\gamma) < 1$.

Step 3. Now it suffices to prove that J_1 is bounded below by a positive constant. We set

$$T = s/(\Delta x)^2,\ L = x/\Delta x,\ \kappa = (X_2(9s) - X_2(s)) \cdot \Delta x/8s,$$

where $\Delta x = X_2(s) - X_1(s)$. By applying first Markov property of the Wiener process and then the scaling invariance and the rotation invariance of it, it follows that

$$\begin{aligned} J_1 \geqq P[|W(t)| \leqq L + 1 + \kappa(t-T) \quad &\text{for} \quad T \leqq t \leqq 9T \\ &\text{and} \quad |W(9T)| < 2L|\ |W(T)| = L] \\ \times \inf_{0 \leqq u \leqq 2x} P[\pi_2 \text{ is not hit at all} \mid |W(9s)| = u]&. \end{aligned}$$

Since (2.11) implies $x \leqq r$, the second factor on the right-hand side is greater than or equal to the probability under the infimum with $u = 2r$, which agrees with

$$p_0 := P[|W(t)| \leqq 3 \quad \text{for} \quad 1 \leqq t \leqq 2|\ |W(1)| = 2] > 0.$$

Let us evaluate the first factor, which we name J_3. To this end single out the first event from the two; and also note that, by (2.5, 2.11), $\kappa \geqq (N \log 2)/8$ if c is large enough. Then the conditional probability of it given $|W(T)| = L$ is greater than

$$p_1 := P[|W(t)| \leqq 1 + ((N \log 2)/8)t \quad \text{for} \quad t > 0] > 0,$$

as is clear by first replacing the upside-down frustum $\{(t, y): y \leqq L + 1 + \kappa(t-T),\ T \leqq t \leqq 9T\}$ [which $W(t)$ is to stay within] by another one $\{(t, \mathbf{y}): |\mathbf{y} - W(T)| \leqq 1 + \kappa(t-T),\ T \leqq t \leqq 9T\}$ [which is contained in it] and then applying the translation invariance of the Brownian motion. As for the second event, noticing $T/L^2 = s/x^2 \leqq 1/c$, we apply the scaling invariance of W again to see that if c becomes large the conditional probability of $|W(9T)| \leqq 2L$ given $|W(T)| = L$ approaches 1; hence it can be made greater than $1 - p_1/2 > 0$. Accordingly $1 - J_3 \leqq 1 - p_1 + p_1/2$, i.e., $J_3 \geqq p_1/2$, so that $J_1 \geqq p_1 p_0/2$. The proof of Lemma 5 is complete.

We lastly prepare the following relation

$$\inf_{0 < s \leqq 1} P[|W(t)| < X_2(t) \quad \text{for all} \quad 0 < t < s|\ |W(s)| = X_1(s)] > 0. \tag{2.12}$$

The probability under the infimum is not less than

$$P[|W(t)| < X_2(t) - (X_1(s)/s)t, 0 < t < s \mid W(s) = \mathbf{0}].$$

By conditioning on $W(s/2)$, replacing $X_2(t)$ by a linear function for $s/2 \leqq t < s$, and then scaling $W(\cdot)$ with the constant $\sqrt{s}$ for $0 \leqq t \leqq s/2$ and with $\Delta x = (X_2(s) - X_1(s))$ for $s/2 \leqq t \leqq s$, this probability is bounded below by

$$\begin{aligned} &\int_{R^N} P[W(1/2) \in d\mathbf{y} \mid W(1) = 0] \cdot P[|W(t)| \leqq f(t), 0 < t \leqq 1/2 \mid W(1/2) = \mathbf{y}] \\ &\quad \times P[|W(t)| \leqq g(t), 0 < t < s/2\Delta x^2 \mid W(s/2\Delta x^2) = \sqrt{s}\mathbf{y}/\Delta x] \end{aligned}$$

where $f(t) = [X_2(st) - X_1(s)t]/\sqrt{s}$ and

$$g(t) = \begin{cases} 1 + (\Delta x/s)[X_1(s) - 2(X_2(s) - X_2(s/2))]t & \text{if} \quad s \leqq 1/5 \\ X_2(1/5) - X_1(1/5) & \text{if} \quad s > 1/5. \end{cases}$$

It is easily seen that $f(t) \geqq (1-1/\sqrt{2}) X_2(t)$ for $0 \leqq t \leqq 1/2$, and that if $s \leqq 1/5$, $g(t) \geqq 1+\lambda t$ with some positive constant λ independent of s. It holds that $P[|W(t)| \leqq cX_2(t), 0<t<1/2]>0$ for every $c>0$ (see the first part of Appendix). Either by applying the projective invariance (1.4) to obtain conditioning-free expressions for two probabilities in the integral above or by observing the monotonicity of them in y, we, from these bounds for f and g, conclude (2.12).

Proof of Theorem 1. With Lemma 5 and (2.12) having been prepared the proof of Theorem 1 is rather standard. Let A and A_n be as in Theorem 1. We can assume $\sum_{k=1}^{\infty} h(O; A_{2k}) = \infty$ and $\lim_{k\to\infty} h(O; A_{2k}) = 0$, which will entail no loss od generality. Let $\mathfrak{A}_k$ denote the event $\{$there exists $t>0$ such that $(t, W(t)) \in A_k$ and $(s, W(s)) \notin B(2^{k+2})$ for all $0<s<t\}$. By the Blumentahl's $0-1$ law we have only to show that $P[\mathfrak{A}_{2k}$ occurs for i.m. $k]>0$. By considering the time-reversed motion $W(2^{2k+1}-t)$ and applying its strong Markov property it is seen that $P(\mathfrak{A}_k) \geqq qh(O; A_k)$ where q denotes the infimum in (2.12). Therefore for every $\delta>0$ and every sufficiently large integer L there exists $M>L$ such that

$$\delta \leqq \sum_{n=L}^{M} P[\mathfrak{A}_{2n}] \leqq \sum_{n=L}^{M} h(O; A_{2n}) \leqq 2\delta/q. \tag{2.13}$$

Then in view of (2.3) Lemma 5 shows that there exist constants $\delta>0$ and $\gamma'<1$ such that (2.13) implies $h\left(s, \mathbf{x}; \bigcup_{j=n+1}^{M} A_{2j}\right) \leqq \gamma'$ for all $(s, \mathbf{x}) \in \bar{A}_{2n}$ and $L \leqq n < M$. Since under $\mathfrak{A}_{2n}$ the event $\mathfrak{A}_{2j}$, $j>n$, can occur only if A_{2j} is hit after the first hitting of A_{2n}, by the strong Markov property this inequality yields $P\left[\bigcup_{j=n+1}^{M} \mathfrak{A}_{2j} \middle| \mathfrak{A}_{2n}\right] \leqq \gamma'$, or, what is the same,

$$P\left[\mathfrak{A}_{2n} \setminus \bigcup_{j=n+1}^{M} \mathfrak{A}_{2j}\right] \geqq (1-\gamma') P[\mathfrak{A}_{2n}].$$

The events, for $n=L, \ldots, M$, measured on the left-hand side are mutually disjoint and the union of them coincides with the union of $\mathfrak{A}_{2n}$. Thus, by the first inequality of (2.13), $P\left[\bigcup_{n=L}^{M} \mathfrak{A}_{2n}\right] \geqq (1-\gamma')\delta$, which shows that $\lim_{L\to\infty} P\left[\bigcup_{n\geqq L} \mathfrak{A}_{2n}\right] = P[\mathfrak{A}_{2n}$ occur for i.m. $n]>0$ as required.

A Remark to the Proof of Theorem 1. If A is contained in a parabolic body $x^2<ct$, the assertion of Theorem 1 may directly proved by a simple method. Indeed one sees, by (2.7), that if $k \geqq 2$ and A_{n+k} lies within such a paraboloid, then

$$\sup_{(s,\mathbf{x}) \in B(a_{n+k})} \sup_{(t,\mathbf{y}) \in A_{n+k}} p(t-s, |\mathbf{y}-\mathbf{x}|) \leqq \text{const}\, e^{c/2} (a_{n+k})^{-N/2},$$

where $a_n = 2^n$, and accordingly can follow both Lamperti [9] and Ito and McKean [8]. If one take a_n, in stead of 2^n, such that a_{n+1}/a_n tends to infinity rapidly enough, this method can be applied to prove corresponding results (without the above constraint on A), but they are of course not so useful as Theorem 1.

3. A Refinement of Theorem 1

In this section will be given a refinement of Theorem 1 where each shell $B(2^{n+1})\backslash B(2^n)$ is sliced with abscissa hyperplanes to form an infinite sequence of annuli into which the set A whose thinness is in question is to be partitioned. Given $a>0$, set $X_a(t)=(-Nt\log t/a)^{1/2}$ as before and let X_a^{-1} be the inverse function of X_a restricted to $0<t\leqq a/e$. Define a sequence $\{t_k\}_{k=-1}^{\infty}$ by $t_{-1}=1$, $t_0=1/e$ for the first two entries and then inductively by

$$x_k=X_1(t_k),\ t_{k+1}=X_2^{-1}(x_k) \quad \text{for} \quad k\geqq 0 \tag{3.1}$$

and set

$$D_k=\{(t,\mathbf{x})\in B(2)\backslash B(1): t_{k+1}<t\leqq t_k\}, \qquad k=-1,0,1,\ldots.$$

These D_k's together then constitute a partition of $B(2)\backslash B(1)$. Recalling the mapping $\varphi^{\varepsilon}:(t,\mathbf{x})\to(\varepsilon t,\sqrt{\varepsilon}\,\mathbf{x})$ together with the fact stated in (2.3) we finally define

$$D_k^{(n)}=\varphi^{2^n}(D_k).$$

Thus we have a partition $\{D_k^{(n)}\}_{k,n}$ of $(0,\infty)\times R^N$ which is much finer than $\{B(2^{n+1})\backslash B(2^n)\}$. The next proposition therefore is stronger than Theorem 1 (and Theorem 2).

Proposition 6. *An analytic set A of R^{N+1} is minimal thin at ζ_0 [thin at the origin] if and only if*

$$\sum_{\substack{n\geqq 0\\ [n\leqq 0]}} 2^{-Nn/2}\sum_{k=-1}^{\infty}\operatorname{Cap}(A\cap D_k^{(n)})<\infty,$$

or equivalently, $\sum_{n,k} h(O;A\cap D_k^{(n)})<\infty$. □

Before proceeding to the proof of Proposition 6 we point out the following relation

$$\Delta x_k := x_k-x_{k-1}\sim\tfrac{1}{2}\sqrt{N\log 2}\,\sqrt{t_k-t_{k+1}} \quad \text{as} \quad k\to\infty \tag{3.2}$$

("$\sim$" means that the ratio of two quantities which hold it between approaches 1), which we shall use both in proving and in applying Proposition 6. The verification of (3.2) is ready from (2.5) if one observes

$$t_k-t_{k+1}=t_k\left(1-\frac{\log t_k}{\log t_{k+1/2}}\right)\sim(\log 2)\frac{-t_k}{\log t_k} \quad \text{as} \quad k\to\infty. \tag{3.3}$$

Proof of Proposition 6. We consider only the criterion of thinness at infinity. We have merely to show that if $\sum_{k,n} h(O;A\cap D_k^{(n)})=\infty$, then $P[A$ is hit i.o. as $t\uparrow\infty]=1$, for the converse is ready from the Borel-Cantelli lemma. In view of Theorem 1 this implication follows from

Lemma 7. *Set $A_n=A\cap[B(2^{n+1})\backslash B(2^n)]$ and $A_{n,k}=A\cap D_k^{(n)}$. If $h(O;A_n)\to 0$ as $n\to\infty$, then there exists a constant C which may depend on A but not on n such that* $\sum_{k=-1}^{\infty} h(O;A_{n,k})\leqq C\cdot h(O;A_n)$ *for all n.* □

Proof. By arguing as in the proof of Theorem 1 the problem is reduced to showing

$$1-h\left(s,\mathbf{x};\bigcup_{j=-1}^{k-1} A_{n,j}\right)>q \quad \text{if} \quad (s,\mathbf{x})\in A_{n,k+1} \tag{3.4}$$

for all $k\geqq 0$ and for all sufficiently large n where q is a positive constant which may depend on A but not on k nor on n. By the strong Markov property of $W(\cdot)$ the left-hand side of (3.4) is greater than

$$P[|W(2^n t_k)|<r\,|\,W(s)=\mathbf{x}]\left\{1-\sup_{y\leqq r} h\left(2^n t_k,\mathbf{y};\bigcup_{j=-1}^{k-1} A_{n,j}\right)\right\}$$

where $r=X_{2^{n-1}}(2^n t_k)$. That the first factor is bounded below by a positive constant follows from (3.2) and the relation

$$x-r\leqq 2^{n/2}x_k-r=2^{n/2}(X_1(t_k)-X_{1/2}(t_k))\sim \text{const}\, 2^{n/2}(x_{k-1}-x_k)$$

as $k\to\infty$ uniformly in n; for n large enough the same is true for the second factor as an application of Lemma 5 (with $m=1$) verifies. Accordingly we get (3.4). This completes the proof of Lemma 7 and hence that of Proposition 6.

4. Applications

1. As a simplest application of Theorem 1 we prove the Kolmogorov's test which was stated in Lévy's book [10, p. 266] without proof and has been proved in various ways by Petrovskii [12], Erdös [3], Feller [5], and Motoo [11] (among these only Petrovskii's proof is purely analytic; the others' proofs are probabilistic but still quite different from one another). Let $f(t)$ be a non-negative Borel function of $t\geqq 0$ and set $F=\{(t,\mathbf{x})\in R^{N+1}\colon t\geqq 0,\ |\mathbf{x}|\geqq f(t)\}$. Then the Kolmogorov's test reads as follows:

Assume $f(t)/\sqrt{t}\uparrow\infty$ as $t\downarrow 0$ [as $t\uparrow\infty$]. Then the set F is thin at the origin [resp. minimal thin at the Martin boundary point ζ_0] if and only if

$$\int_{\substack{0<t<1\\ [t>1]}} f(t)^N p(t,f(t))\frac{dt}{t}<\infty\,. \tag{4.1}$$

Proof. The proof is carried out only in the case where the thinness at ζ_0 is concerned. The other case is reduced to it by using the projective invariance (1.4) (the same comment may be applied throughout this section, though the direct modification of proof also may be possible in some places). First we prove the sufficiency of (4.1). For this part we do not apply Theorem 1, but make use of the explicit expression (1.3) of the hitting distribution ν_a. The assumed monotonicity of $f(t)/\sqrt{t}$ implies that the boundary ∂F crosses $S(a)$ at exactly one ordinate level $t=t(a)$ (if a is sufficiently large). [This is seen either by comparing $f(t)$ with the function $c\sqrt{t}$ or by noting that for it to be true it is sufficient that the lower derivative of $f(t)^2/t$ on the right is greater than $-1/t$.] Let $t_k=2^{2k/N}$ and a_k the value of a for which ∂F crosses $S(a)$ at the level $t=t_k$; if there are many such a's take the infimum of them. Let B_k denote the part of $S(a_k)$ below the level $t=t_{k+1}$. Let τ_a be the first hitting time to $S(a)$ by $(t, W(t))$. Then from (1.3, 2.2) it follows that for t/a small and $0 \leqq s < t$

$$P[s<\tau_a \leqq t] = \int_\Theta d\theta \int_s^t (2\pi a)^{-N/2} X_a(u)^{N+1} \frac{-1}{Nu(1+\log u/a)} dX_a(u)$$
$$= \left(\int_\Theta d\theta/N\right)(2\pi a)^{-N/2}(X_a(t)^N - X_a(s)^N)(1+o(1)), \tag{4.2}$$

where $o(1)\to 0$ as $t/a \to 0$. From $\lim f(t)/\sqrt{t}=\infty$, it follows that $t_k/a_k \to 0$ and $X_{a(k)}(t_{k+1})/X_{a(k)}(t_k) \to 2^{1/N}$ as $k\to\infty$ where $a(k)=a_k$. Now keeping these in mind apply (4.2) and then, noticing $X_{a(k)}(t) \geqq f(t)$ for $t<t_k$, use (1.3) to see that for large k

$$h(O;B_k) = P[\tau_{a(k)} \leqq t_{k+1}] \leqq 4P[t_{k-1} < \tau_{a(k)} \leqq t_k](1+o(1))$$
$$\leqq \text{const} \int_{t_{k-1}}^{t_k} f(t)^N p(t, f(t)) t^{-1} dt.$$

Let F_k be the part of F between two levels $t=t_k$ and $t=t_{k+1}$. Then $h(O;F_k) \leqq h(O;B_k)$ since $R^{N+1}\setminus B(a)$ is thin at O. Accordingly (4.1) implies $\sum h(O;F_k)<\infty$ and hence $P[F$ is hit i.o. as $t\uparrow\infty]=0$ or, equivalently, F is minimal thin at ζ_0.

The proof of the converse part is similar, but this time t_k is defined as the ordinate at which level F intersects $S(2^k)$. Let A_k and A'_k be the parts of $S(2^k)$ and $S(2^{k+1})$, respectively, between two levels $t=t_k$ and $t=t_{k+1}$. Note that A'_k is included in $F'_k := F \cap (B(2^{k+1})\setminus B(2^k))$, whereas A_k lies outside F. By (1.3) and by writing $a(k)=2^k$ we as above see

$$h(O;A'_k \geqq \nu_{a(k+1)}(A'_k) = C_N \int_{t_k}^{t_{k+1}} X_{a(k+1)}(t)^N (2\pi a(k+1))^{-N/2} t^{-1} dt$$
$$\geqq 2^{-1-N/2} \nu_{a(k)}(A_k)$$
$$\geqq 2^{-2-N/2} C_N \int_{t_k}^{t_{k+1}} f(t)^N p(t, f(t)) t^{-1} dt$$

for k large enough where C_N is a positive constant. Thus the divergence of the definite integral in (4.1) implies $\sum h(O;F_k)=\infty$; hence by Theorem 1 $P[F$ is hit i.o. as $t\uparrow\infty]=1$. The proof is complete.

2. For the necessity of the condition (4.1) in the Kolmogorov's test a slight modification of its proof dispenses the monotonicity assumption on $f(t)/\sqrt{t}$. In the next theorem not only this assumption is removed, but also the conclusion is strengthened. (F is replaced by the graph of f).

Theorem 8. *Let $f=f(t,\theta)$ be a positive Borel function of $t\geqq 0$ and of $\theta=\mathbf{x}/x$ and G denote the graph of f: $G:=\{(t,\mathbf{x}): f(t,\mathbf{x}/x)=x\}$. Let $\psi(x)$ be a bounded positive function of $x>0$ which is non-increasing and slowly varying as $x\downarrow 0$ and satisfies $\int_0^1 \psi(x)x^{-1}dx<\infty$; if $N=2$ it is further assumed that $\psi(x)(-\log x)$ is bounded. Put*

$\Psi_1(x)=1_{(0,1)}(x)$ and $\Psi_N(x)=x^{N-2}\psi(x)1_{(0,1)}(x)$ for $N\geqq 2$ (1_Δ denotes the indicator function of a set Δ). For G to be not thin at the origin [minimal thin at ζ_0] it is sufficient that

$$\underset{\substack{0<t<1\\ [t>1]}}{\int}\frac{dt}{t}\int_\Theta \{\Psi_N(f(t,\theta)/\sqrt{t})+f(t,\theta)^N p(t,f(t,\theta))\}\,d\theta=\infty . \quad \square$$

Remark. A positive function $\psi(x)$ is slowly varying as $x\downarrow 0$ if $\psi(\kappa x)/\psi(x)\to 1$ as $x\downarrow 0$ for every $\kappa>0$. For every $\delta>0$ the function $\min\{1,|\log x|^{-1-\delta}\}$ can serve as ψ in Theorem 8. Of the integrand parenthesized in the inner integral above the first term becomes dominant to the second as $f(t,\theta)/\sqrt{t}$ approaches zero. By comparison Ψ_N obviously can be deleted from the integrand to obtain Theorem 3 of Introduction.

For the proof we prepare two lemmas. Given a positive continuous function $h(t)$ of $t\in[\alpha,\beta]$, let us consider a "projection" π_h which is defined by

$$\pi_h(t,\mathbf{x})=(t,h(t)\mathbf{x}/x), \qquad (t,\mathbf{x})\in[\alpha,\beta]\times R^N .$$

Lemma 9. *Let h and π_h be as above. Let G be a graph of a Borel function f as in Theorem 8 and A a Borel subset of G. If $\alpha\leqq t\leqq\beta$ and $x\geqq h(t)$ for all $(t,\mathbf{x})\in A$, then* $\mathrm{Cap}(\pi_h A)\leqq e^{\gamma/2}\,\mathrm{Cap}(A)$ *where*

$$\gamma := \sup_{\alpha\leqq s<t\leqq\beta}\frac{(h(t)-h(s))^2}{t-s}. \quad \square$$

Proof. We shall make use of the formula

$$\mathrm{Cap}(A)=\sup\{\mu(A)\colon \mu \text{ is supported by } A \text{ and } g\mu\leqq 1\}, \tag{4.3}$$

where, for μ a finite Borel measure on R^{N+1}, $g\mu(\xi)=\int g(\xi,\eta)\,\mu(d\eta)=\int p(t-s,|\mathbf{y}-\mathbf{x}|)\,\mu(dt\,d\mathbf{y})$ ($\xi=(s,\mathbf{x})$) (cf. Watson [13]; also Doob [2, p. 243]). For a measure μ supported by A, let μ' denote a measure on $A'=\pi_h A$ induced by π_h from μ. In view of (4.3) it suffices to show that if $g\mu'\leqq 1$, then $g\mu\leqq e^{\gamma/2}$, because the mapping: $\mu\to\mu'$ is reversible. Let us write $\mathbf{x}^t=h(t)\mathbf{x}/x$ etc. We shall actually prove that for $\alpha\leqq s\leqq\beta$

$$g\mu(s,\mathbf{x})\leqq e^{\gamma/2}g\mu'(s,\mathbf{x}^s) \quad \text{if} \quad x\geqq h(s)$$

which is enough for our end by virtue of the maximum principle. Since

$$g\mu'(s,\mathbf{x}^s)=\int p(t-s,|\mathbf{y}^t-\mathbf{x}^s|)\,\mu(dt\,d\mathbf{y}),$$

we have only to show that for $\alpha\leqq s<t\leqq\beta$

$$|\mathbf{y}^t-\mathbf{x}^s|^2\leqq|\mathbf{y}-\mathbf{x}|^2+(h(t)-h(s))^2 \quad \text{if} \quad x\geqq h(s) \quad \text{and} \quad y\geqq h(t). \tag{4.4}$$

If $\mathbf{x}^s\cdot\mathbf{y}^t\leqq\min\{|\mathbf{x}^s|^2,|\mathbf{y}^t|^2\}$, then is true this inequality even with the second term on the right side discarded. If $\mathbf{x}^s\cdot\mathbf{y}^t\geqq|\mathbf{y}^t|^2$, then

$$|\mathbf{y}^t-\mathbf{x}^s|^2\leqq\min_{b>0}|\mathbf{x}^s-b\mathbf{y}|^2+(h(t)-h(s))^2$$

which clearly yields (4.4). The remaining case, where we have $\mathbf{x}^s\cdot\mathbf{y}^t\geqq|\mathbf{x}^s|^2$, can be treated by interchanging the roles of $\mathbf{x}^s$ and $\mathbf{y}^t$ in the above. Thus Lemma 9 has been proved.

Lemma 10. *Let G and $\Psi_N(x)$ be as in Theorem 8. Then there is a constant C which may depend on Ψ_N but not on G such that if A is a Borel subset of $G \cap \{(t, \mathbf{x}): 0 \leqq t \leqq 1, x \leqq 1\}$, then*

$$\operatorname{Cap}(A) \geqq C \int_{(t,\mathbf{y}) \in A} \Psi_N(f(t,\theta))\, dt\, d\theta \tag{4.5}$$

where $\mathbf{y}$ *stands for* $f(t,\theta)\theta$. □

Remark. When $N=1$ (4.5) is surpassed by

$$\operatorname{Cap}(A) \geqq \sqrt{2\pi}\, |\pi_0 A| \left(\sup_{0 \leqq s \leqq 1} \int_{\pi_0 A \cap (s,1]} dt/\sqrt{t-s} \right)^{-1}, \tag{4.6}$$

where $\pi_0 A = \{t: (t, \mathbf{y}) \in A$ for some $\mathbf{y}\}$ and $|\cdot|$ denotes the Lebesgue measure on R. This inequality may afford a better sufficient condition for thinness than that of Theorem 8.

Proof of Lemma 10. Let μ be the measure on A whose value of A' a Borel subset of A is given by the integral on the right-hand side of (4.5) but with A' in place of A. It suffices to prove that $g\mu(t, \mathbf{x}) \leqq C^{-1}$ for $0 \leqq s \leqq 1$, $x \leqq 1$. Let $x \leqq 1$ and $0 \leqq s \leqq 1$. Noting that if $|\mathbf{y}-\mathbf{x}| > x/2$ then $|\mathbf{y}-\mathbf{x}| > y/3$, we see

$$\begin{aligned} g\mu(s,\mathbf{x}) \leqq & \int_{\substack{|\mathbf{y}-\mathbf{x}|<x/2 \\ (t,\mathbf{y}) \in A}} p(t-s, |\mathbf{y}-\mathbf{x}|)\, \Psi_N(f(t,\theta))\, dt\, d\theta \\ & + \int_s^1 dt \int_\Theta p(t-s, f(t,\theta)/3)\, \Psi_N(f(t,\theta))\, d\theta . \end{aligned} \tag{4.7}$$

The case $N=1$ is readily disposed of and omitted. Let $N \geqq 2$. To get an upper bound of the first integral in (4.7) apply the projection π_h with $h \equiv x/2$ and make a comparison as in Lemma 9. Since $x^{N-1} d\theta$ equals a constant multiple of a surface element on $(N-1)$-dimensional sphere of radius $x/2$, we then see that it is bounded above by

$$C_1 \frac{\psi(x)}{x} \int_0^1 dt \int_0^x p(t,u) u^{N-2} du = C_1' \frac{\psi(x)}{x} \int_0^x du \int_u^\infty e^{-y^2/2} y^{N-3} dy,$$

which is dominated by a constant multiple of $\psi(x)$ if $N \geqq 3$ and that of $\psi(x)\,|\log x|$ if $N=2$. As for the second integral in (4.7) we have only to use the inequality

$$p(u, y/3) y^{N-2} \psi(y) \leqq C_2 \psi(\sqrt{u})/u$$

to bound it above by $C_2' \int_0^1 \psi(\sqrt{u}) u^{-1} du = 2C_2' \int_0^1 \psi(u) u^{-1} du$. By arguing in each of the two cases $y < \sqrt{u}$ and $y \geqq \sqrt{u}$ the inequality above is ready from the monotonicity of ψ and the fact that the slowly varying function is always of the form $\psi(x) = a(x) \exp\left(\int_x^1 \varepsilon(u) u^{-1} du \right)$ where $a(x) \to 1$ and $\varepsilon(x) \to 0$ as $x \to 0$ (cf. Feller [6, p. 274]). Consequently $g\mu$ is bounded. Lemma 10 has been proved.

Proof of Theorem 8. Proposition 6 in the previous section will be of use in this proof. Let $G_{n,k} = G \cap D_{n,k}$. Then applying Lemma 9 with $h(t) = X_{2^n}(t)$,

$2^n t_{k+1} \leqq t \leqq 2^n t_k$ we see that if $k \geqq 0$

$$\sqrt{2}^N h(O; G_{n,k}) \geqq (2\pi 2^n)^{-N/2} \operatorname{Cap}(G_{n,k}) \geqq (2\pi 2^n)^{-N/2} e^{\gamma/2} \operatorname{Cap}(\pi_h G_{n,k}),$$

where

$$\gamma := \sup \left\{ \frac{(X_1(t) - X_1(s))^2}{t-s} : t_{k+1} \leqq t - s \leqq t_k, \ k = 0, 1, 2, \ldots \right\}.$$

From (3.2, 3.3, 2.2) it follows that $\gamma < \infty$. Since $x^N p(t,x)$ is decreasing with respect to x in the region $x^2 > Nt$ $(t > 0)$ which contains $D_{n,k}$ if $k \geqq 0$, and since the right-most member of the inequalities above equals $e^{\gamma/2} h(O, \pi_h G_{n,k}) = e^{\gamma/2} v_{2^n}(\pi_h G_{n,k})$, an application of (1.3) deduces that if $k \geqq 0$

$$h(O; G_{n,k}) \geqq 2^{-N/2} e^{\gamma/2} \int_{(t,\mathbf{y}) \in G_{n,k}} f(t,\theta)^N p(t, f(t,\theta)) \frac{1}{t} \, dt d\theta,$$

where $\mathbf{y} = f(t,\theta)\theta$. Finally Lemma 10 together with the scaling relation (2.3) proves

$$h(O; G_{n,-1}) \geqq C \int_{(t,\mathbf{y}) \in G_{n,-1}} \Psi_N(f(t,\theta)/\sqrt{t}) t^{-1} \, dt d\theta.$$

The assertion of Theorem 8 follows from Proposition 6 and the last two inequalities.

3. The criterion given by Theorem 8 may be fairly accurate if $f(t,\theta)/\sqrt{t}$ is large, but not if small. To see the latter consider the tube $T_{s,x} = \{(t,\mathbf{y}) : 0 < t < s, |\mathbf{y}| = x\}$. Replacing $\Psi_N(x)$ in (4.5) by x^{N-2} or $|\log x|^{-1}$ according as $N \geqq 3$ or $N = 2$, we make a computation similar to that in the proof of Lemma 10 to see that as $x \downarrow 0$

$$\operatorname{Cap}(T_{1,x}) \asymp x^{N-2} \quad \textit{or} \quad |\log x|^{-1} \ \textit{according as} \ N \geqq 3 \ \textit{or} \ = 2, \tag{4.8}$$

where "$\asymp$" means that the ratio of two quantities which hold this symbol between is bounded away from zero and infinity. (For getting the upper bound it may be noted that $g\mu_1 \leqq g\mu_2$ implies $\mu_1(R^{N+1}) \leqq \mu_2(R^{N+1})$.) By (4.8) together with the scaling relation

$$\operatorname{Cap}(\varphi^\varepsilon A) = \varepsilon^{N/2} \operatorname{Cap}(A).$$

Theorem 1 yields the Dvoretsky-Erdös test [Spitzer's test] which may read as follows: if $f(t,\theta) = f(t)$ a function of t only and $h(t) := f(t)/\sqrt{t}$ decreases to zero as $t \uparrow \infty$, then G is not thin at ζ_0 if and only if $\int^{\infty} h(t)^{N-2} t^{-1} dt = \infty$ in the case $N \geqq 3$ $\left[\text{resp. } \int^{+\infty} |\log h(t)|^{-1} t^{-1} dt = \infty \text{ in the case } N = 2\right]$, which condition is somewhat weaker than the divergence condition given in Theorem 8.

Complementary to (4.8) we have $\operatorname{Cap}(T_{\Delta t,x}) \asymp \sqrt{\Delta t}\, x^{N-1}$ for $x^2 > \Delta t > 0$, $N \geqq 1$. It follows from (2.5) and (3.2) that $x_k \sqrt{t_k - t_{k+1}}/t_k \sim$ const.; then from (1.2) that for $t \geqq \Delta t > 0$

$$\begin{aligned} J(t, \Delta t, x) &:= P[|W(s)| = x \text{ for some } s \in (t, t + \Delta t)] \\ &\asymp \sqrt{\Delta t}\, x^{N-1} p(t,x) \text{ as long as } \frac{\sqrt{\Delta t}}{x}, \frac{x}{t}\sqrt{\Delta t} \leqq C, \end{aligned} \tag{4.9}$$

where C is any constant, which the constants involved in "$\asymp$" may depend on (if $N=1$, then $\sqrt{\Delta t}/x \leqq C$ may be deleted, so that $x=0$ is allowed). The relation in (4.9), which may be rewritten $(x\sqrt{\Delta t}/t)J(t, \Delta t, x) \asymp x^N p(t,x)\Delta t/t$, shows that the ratio of the probability $P[W(s) \in S(a)$ for some $s \in [t, t+\Delta t)$ but not for any $s \in (0,t)]$ to the probability $J(t, \Delta t, X_a(t))$ stays off zero or does not according as $X_a(t)\sqrt{\Delta t/t}$ does or does not. This observation suggests that the divergence condition in Theorem 8 fails to be critical if the heights of $G \cap D_{n,k}$ are much smaller than those of $D_{n,k}$. As an example let $f(t) = \sqrt{2ct \log\log t}$ if $2^n \leqq t < 2^n(1+n^{-\gamma})$, $n=1,2,\ldots$ and $f(t)=\infty$ otherwise. Then $P[|W(t)| = f(t)$ for some $t \in [2^n, 2^{n+1})] \asymp J(2^n, 2^n n^{-\gamma}, f(2^n)) \asymp n^{-\gamma/2-c}\sqrt{\log n}^{N-1}$, (as $n \to \infty$), (1) where for the first relation we employed Lemma 9. This shows that $P[|W(t)| = f(t)$ i.o. as $t \uparrow \infty] = 1$ if and only if $0 \leqq \gamma/2 \leqq 1-c$, whereas a sufficient condition provided by Theorem 8 is $0 \leqq \gamma \leqq 1-c$.

4. Let $q(r), r \geqq 0$ be a non-negative non-decreasing function. If $N=3$, it is supposed that $q(0+)=0$ and $\lim_{r\to 0} r|\log q(r)|/q(r) = 0$. Let θ_0 be a fixed unit vector of R^N. Put

$$f(t,\theta) = \sqrt{2t \log(|\log t| \vee 1)}\,(1+q(|\theta-\theta_0|)), \qquad t>0,$$

and $F = \{(t, \mathbf{x}) : t>0, x \geqq f(t, \mathbf{x}/x)\}$. Then *for F to be thin at the origin it is necessary and sufficient that*

$$q(0)>0 \qquad \textit{if} \quad N=1 \quad \textit{or} \quad 2$$

$$\int_{0+} \frac{q(u)^{-3/2}}{-\log u}\, dq(u) < \infty \qquad \textit{if} \quad N=3, \tag{4.10}$$

$$\int_{0+} u q(u)^{-2} dq(u) < \infty \qquad \textit{if} \quad N=4, \tag{4.11}$$

$$\int_{0+} u^{N-1} q(u)^{-N/2-2} dq(u) < \infty \qquad \textit{if} \quad N \geqq 5. \tag{4.12}$$

(The point $u=0$ is not contained in the range of the integration for these integrals.) The same criteria are applied to the minimalthinness at ζ_0.

Before starting the proof it is remarked that for the minimalthinness at ζ_0 is necessary the convergence of the series

$$\sum_{n=1}^{\infty} 2^{-nN/2} \int_{0+} X_{2^n}(t) |\{\mathbf{x} : (t, \mathbf{x}) \in F \cap S(2^n)\}| \frac{dt}{t}, \tag{4.13}$$

($|\cdot|$ is the $(N-1)$-dimensional surface area) which follows from Theorem 1 simply by taking into account only the frontal part of $F_n := F \cap (B(2^{n+1}) \setminus B(2^n))$ in computation of $h(O, F_n)$. It turns out that this necessary condition is also a sufficient one if either $N=4$ and $q(r)=r^\gamma$ or $N \geqq 5$, but far from that if $N \leqq 3$.

We treat only the thinness at infinity. The case $N=1$ or 2 are easy and omitted. The case $N=3$ or 4 is somewhat delicate. We begin with the case $N \geqq 5$. Let $a(n) = 2^n$. Let T_n and $k_n(t)$, $e<t<T_n$, be determined by $\sqrt{2T_n \log\log T_n} = X_{a(n)}(T_n)$ and by $X_{a(n)}(t) = \sqrt{2t \log\log t}(1+k_n(t))$, respectively. Let $L_n = \log\log T_n$ and

$T_n' = T_n(\log T_n)^{-1/N}$. Then by a simple computation

$$\frac{N}{5L_n}\log\frac{T_n}{t} \leqq k_n(t) \leqq \frac{N}{3L_n}\log\frac{T_n}{t} \quad \text{for} \quad T_n' \leqq t \leqq T_n, \tag{4.14}$$

for n large enough. Fixing n let $a=a(n)$, $T=T_{n+1}$, $T'=T_n'$, $L=L_n$, $k^0(t)=k_n(t)$ and $T^0=T_n$. Then

$$h(O;F_n)=\frac{1}{2}\int_{T'}^{T} X_a^N p(t,X_a)\frac{dt}{t}\int h(t,X_a\theta;F_n)d\theta+\delta_n, \tag{4.15}$$

where $X_a=X_a(t)$ and $\delta_n \leqq \hat{v}_a((0,T']\times\Theta)$. It follows that $\Sigma\delta_n<\infty$, which may be seen directly or by noting that for $f(t)=\sqrt{3t\log\log t}$ the integral in the Kolmogorov test converges and $f(T_n)/X_{a(n)}(T_n')\to 1$ as $n\to\infty$. Put $A_{n,t}=\{\theta:|\theta-\theta_0|>q^{-1}(k_n(t))\}$ for $t\leqq T^0$ and $A_{n,t}=\theta$ for $t>T^0$, where $q^{-1}(u)=\sup\{r\geqq 0: q(r)\leqq u\}$ with the convention $\sup\emptyset=0$. We devide the inner integral appearing in (4.15) into the integral on $A_{n,t}$ and that on $\Theta\setminus A_{n,t}$, which we denote by $I_n(t)$ and $II_n(t)$, respectively. The contribution from $II_n(t)$ to the double integral in (4.15), which corresponds to the n-th summand of (4.13), is relatively easy to estimate. Since $h(t,X_a\theta;F_n)=1$ for $\theta\notin A_{n,t}$, it is bounded above and below by constant multiples of

$$a^{-N/2}\int_{T'}^{T^0} X_a(t)^N[q^{-1}(k^0(t))]^{N-1}\frac{dt}{t}. \tag{4.16}$$

Applying (4.14) $X_a(t)\leqq\sqrt{-Nt\log T'/a}=\sqrt{3tL}\,(t>T')$ and $X_a(t)\geqq\sqrt{2tL}\,(t<L)$ and changing the variable according to $u=N(\log T^0/t)/3L$ we bound (4.16) from above and below by constant multiples of

$$\frac{1}{N}L^{N/2+1}(T^0/a)^{N/2}\int_0^{1/3}[q^{-1}(u)]^{N-1}e^{-3Lu/2}du.$$

Since $T_{n+1}-T_n\asymp T_n$, the convergence of the sum of the last quantity over n is equivalent to that of the integral of it by $d(\log T^0)=dT^0/T^0$; and since $(T^0/a)^{N/2}d(\log T^0)\sim d(\log\log T^0)$, the latter integral equals $1/N$ times

$$\int_1^{\infty} L^{N/2+1}dL\int_0^{1/3}[q^{-1}(u)]^{N-1}e^{-3Lu/2}du=\text{const}\int_0^{1/3}[q^{-1}(u)]^{N-1}u^{-N/2-2}du.$$

This proves especially the necessity of (4.12).

To get an upper bound of $I_n(t)$ we further put $R=X_{a(n)}(T_{n+1})$, $r=r_n(t)=X_{a(n)}(t)$, $k(t)=k_{n+1}(t)$ and $b=a(n+1)$. Let $W_1^x(\cdot)$ and $Y^x(\cdot)$ be respectively the 1-dimensional Brownian motion and the $(N-1)$-dimensional Bessel process both starting at x which are mutually independent and defined on the same probability space as so far used. Then, by writing $A=A_{n,t}$,

$$I_n(t)\leqq\int_A P\left[W_1^r(s)>r+\frac{R-r}{T-t}s,\ Y^{l(r\theta)}(s)\leqq X_b(t+s)q^{-1}(k(t))\right.$$
$$\left.\text{for some}\quad 0<s<T-t\right]d\theta,$$

where $l(r\theta)$ is the distance of the point $r\theta$ from the line $u\theta_0$, $u\in R$. Since $(R-r)/(T-t)\geqq(dr/dt)|_{t=T}\geqq L/3R$ if n is large enough, the integral on the right-hand side is at most

$$\int_A P\left[W_1^0(s)>\frac{Lr}{3R}s,\ Y^{l(\theta)}(s)<\frac{1}{r}X_b(t+r^2s)\,q^{-1}(k(t))\quad\text{for some}\quad 0<s<\frac{T-t}{r^2}\right]d\theta.$$

Since the law of the last leaving time of $W_1^0(t)$ from the line ct, $t\geqq 0$ is given by $c\sqrt{t}\,e^{-tc^2/2}t^{-1}\,dt$ and since $X_b(t+r^2s)/r\leqq 2\sqrt{T/t}$, if we write $c=Lr/3R$ and $y=2\sqrt{T/t}\,q^{-1}(k(t))$ and denote by τ_y^r the first hitting time to y by $Y^r(\cdot)$, the last integral is dominated by

$$\text{const}\int_{q^{-1}(k^0(t))}^{1} r^{N-2}\,dr\int_0^\infty\frac{c\sqrt{s}}{s}e^{-sc^2/2}P[\tau_y^r<s]\,ds.$$

Consequently, applying the scaling relation: $P[\tau_y^r<s]=P[\tau_{cy}^{cr}<c^2s]$, we obtain

$$I_n(t)\leqq\text{const}\,c^{-N+1}\int_{cq^{-1}(k^0(t))}^{c} r^{N-2}\,dr\int_0^\infty\frac{1}{\sqrt{s}}e^{-s}P[\tau_{cy}^r<2s]\,ds.\tag{4.17}$$

By dominating $P[\tau_{cy}^r<s]$ by the hitting time distribution of the 1-dimensional Brownian motion when $r\geqq 2cy$, the right-hand side of (4.17) is at most a constant multiple of

$$c^{-N+1}\int_{2cy}^\infty r^{N-2}\,dr\int_0^\infty\frac{r-cy}{\sqrt{u^3}}e^{-(r-cy)^2/2u}\,du\int_u^\infty\frac{1}{\sqrt{s}}e^{-s/2}\,ds$$
$$+c^{-N+1}\int_{cq^{-1}(k^0(t))}^{2cy} r^{N-2}(cy/r)^{N-3}\,dr,$$

which, after a bit of calculation, we can bound by $\text{const}(c^{-N+1}+y^{N-1})$ $=\text{const}(\sqrt{T/t})^{N-1}\{L^{-N+1}+[q^{-1}(k(t))]^{N-1}\}$. Then, as before,

$$\sum_n\int_{T_n'}^{T_{n+1}} r_n^N p(t,r_n)\,I_n(t)\frac{dt}{t}\leqq\text{const}\left\{\int_0^1[q^{-1}(u)]^{N-1}u^{-N/2-2}\,du+C_N\right\},$$

where $C_N:=\int_1^\infty L^{-N/2+1}\,dL$ is finite if $N\geqq 5$. This shows the sufficiency of (4.12).

For $N=3$ or 4 the convergence in (4.10) or (4.11) implies that of the sum of the quantities in (4.16) over n, so that it suffices to evaluate the sum corresponding to $I_n(t)$. We make use of the formula

$$\varphi_\varepsilon^r(\lambda):=\int_0^\infty e^{-\lambda t}P[\tau_\varepsilon^r<t]\,dt=\frac{1}{\lambda}\frac{K_\nu(\sqrt{2\lambda}r)/r^{-\nu}}{K_\nu(\sqrt{2\lambda}\varepsilon)/\varepsilon^{-\nu}},\qquad \nu=\frac{N-3}{2}.$$

(K_ν is a modified Bessel function in the standard notation), together with $K_{1/2}(z)$ $=\sqrt{\pi}/\sqrt{8z}+O(z^{3/2})$ $(z\downarrow 0)$ and $K_0(z)=-\log z+\text{const}+o(z)$ $(z\downarrow 0)$. By the Tchebychev inequality $P[\tau_\varepsilon^r < s] \leqq e^s \varphi_\varepsilon^r(1)$. Applying this for $0<s<1$ and the scaling relation mentioned before, we have

$$\int_0^\infty \frac{1}{\sqrt{s}} e^{-s} P[\tau_\varepsilon^r < 2s]\, ds \leqq 3\varphi_{\varepsilon/\sqrt{2}}^{r/\sqrt{2}}(1).$$

If $N=4$, by (4.17) and by $K_\nu(z)=O(e^{-z}/\sqrt{z})$ as $z\to\infty$,

$$I_n(t) \leqq c^{-N-1}(cy)^{N-3} \operatorname{const} \int_0^\infty r^{N-2}(e^{-r} \wedge r^{-N+3})\, dr = \operatorname{const} c^{-2} y.$$

Repeating the same argument as made through several lines from (4.16) down to the end of the paragraph which contains it we would obtain a corresponding upper bound which proves the sufficiency of (4.11). The lower bound can be computed in a similar way (disregard the part of F_n above the level $t=T_n$). If $N=3$, $I_n(t) \leqq \operatorname{const} c^{-2}/|\log cb|$ whenever $cb<1$. This time we put $T_n' = T_n/L_n^4$, which still entails that in (4.15) $\sum \delta_n < \infty$. Then, by proceeding as before, the convergence of $\sum_n \int_{T'}^{T} X_a^N p(t, X_a) I_n(t) t^{-1}\, dt$ follows from that of

$$\int^{+\infty} L^{1/2}\, dL \int_0^{4(\log L)/L} e^{-Lu/2} |\log L q^{-1}(u)|^{-1}\, du.$$

If the condition $q^{-1}(u)=o(u/|\log u|)$ is employed, after changing the order and variable of integration one sees that the latter convergence in turn follows from (4.10). Thus the sufficiency of (4.10) has been proved. The necessity is similar and easier as in the other cases.

5. Appendix

This appendix consists of four parts. In the third part we shall give a probabilistic proof of the result by Bauer [1] stated in Introduction. Our proof is based on the fact that $R^{N+1}\setminus B(a)$ is thin at the origin and on a result of Fulks [7], which will be proved in the first and the second part, respectively. The last part is devoted to showing the existence of the equilibrium measure e_A in (1.1).

1. For $c>0$ put $f(t)=\sqrt{-ct\log t}$ $(0<t<1)$ and $F=\{(t,\mathbf{x}): x \geqq f(t), 0<t<1\}$. We here prove that F is thin at O for every c, which in particular implies the thinness of $R^{N+1}\setminus B(a)$ at O. This claim of course follows from Kolmogorov's test, but in our deduction of it we made use of it granted that ν_a defined via (1.3) is the hitting distribution to $S(a)$, which fact we are going to prove. Since the probability of the event $\mathfrak{A}_k := \{|W(e^{-k+1})| \geqq f(e^{-k})\}$ is not less than 1/2 times the probability of the event {the part of F between $t=e^{-k}$ and $t=e^{-k+1}$ is ever hit} for $k>1$, the asserted thinness follows from

$$\sum_k P[\mathfrak{A}_k] = \sum_k P[W(e) \geqq e^{k/2} f(e^{-k})] \leqq \operatorname{const} \sum_k \sqrt{k}^{N-2} e^{-(c/2e)k} < \infty.$$

2. For reader's convenience the Fulks derivation of an explicit form of v_a is briefly given in a way relevant to the present context. Let $\mathcal{L}$ and $\mathcal{L}^*$ be respectively the parabolic and coparabolic operators: $\mathcal{L}=\frac{1}{2}\Delta-(\partial/\partial t)$, $\mathcal{L}^*=\frac{1}{2}\Delta+(\partial/\partial t)$. Let $D_\varepsilon=\{(t,\mathbf{y})\in B(a):t>\varepsilon\}$, $\varepsilon>0$. Given a function v *which is defined and satisfies* $\mathcal{L}^*v=0$ *in a neighbourhood of* $B(a)$, we apply the divergence theorem to integrate the expression $u\mathcal{L}^*v-v\mathcal{L}u=\frac{1}{2}\nabla\cdot(u\nabla v-v\nabla u)+\partial(uv)/\partial t$ over D_ε with $u=g$ [recall $g(t,\mathbf{x})=p(t,x)$]. Since $\mathcal{L}g=0$ for $t>0$ and $g=(\sqrt{2\pi a})^{-N}$ on $S(a)$, this after rearrangement yields

$$\int_{x<X_a(\varepsilon)} gv\,d\mathbf{x}=-\frac{1}{2}\int_{S_\varepsilon} v\nabla g\cdot n_1\,d\sigma+\frac{1}{\sqrt{2\pi a}^N}\int_{S_\varepsilon}\left(\frac{1}{2}\nabla v\cdot n_1+vn_2\right)d\sigma$$

where $S_\varepsilon=S(a)\cap\partial D_\varepsilon$, n_1 and n_2 are respectively the abscissa and the ordinate components of the outward unit normal vector to S_ε, and $d\sigma$ is a surface element of S_ε. Taking $u\equiv 1$ in place of g, which results in the same equality as above but with 1 replacing g, one sees that the second term on the right-hand side of this relation vanishes as $\varepsilon\downarrow 0$. The left-hand side equals $E[v(W(\varepsilon));|W(\varepsilon)|<X_a(\varepsilon)]$, which converges to $v(O)$ since $P[|W(\varepsilon)|\geqq X_a(\varepsilon)]=P[|W(1)|\geqq\sqrt{-N\log\varepsilon/a}]\to 0$. Consequently

$$v(O)=-\tfrac{1}{2}\int_{S(a)} v\nabla g\cdot n_1\,d\sigma\,. \tag{A.1}$$

But, by introducing a variable $\theta=\mathbf{x}/x$, we have $-\nabla g\cdot n_1=(xg/t)|n_1|$ and $|n_1|\,d\sigma=x^{N-1}\,d\theta\,dt$ at $(t,\mathbf{x})\in S(a)$, so that the relation (A.1) can be written

$$v(O)=\int_{(0,a]\times\Theta} v(t,X_a(t)\theta)\,\hat{v}_a(dt\,d\theta)\,, \tag{A.2}$$

or, what is the same, $v(O)=v_a(v):=\int_{S(a)} v\,dv_a$, where

$$\hat{v}_a(dt\,d\theta)=\frac{1}{2}(2\pi a)^{-N/2}\,X_a(t)^N\frac{dt}{t}\,d\theta\,, \tag{A.3}$$

and v_a is a measure on $S(a)$ induced from $\hat{v}_a$ by the mapping $(t,\theta)\to(t,X_a(t)\theta)\in S(a)$.

3. Here we shall prove that v_a *defined above via* (A.3) *agrees with the hitting distribution to* $S(a)$ *for* $(t, W(t))$. For $\xi=(s,\mathbf{x})$ let P_ξ denote the probability law of $(t+s, W(t)+\mathbf{x})$ the space-time Wiener process starting at ξ induced on the space of continuous functions $C([0,\infty),R^{N+1})$ equipped with the cylindrical Borel field and β_t [or sometimes $\beta(t)$] a generic element of this space. The expectation by P_ξ is denoted by E_ξ. Let T be the first hitting time to $S(a)$ by β_t: $T=\inf\{t>0:\beta_t\in S(a)\}$. Given f a continuous function on $S(a)$, we put

$$u(\xi)=E_\xi[f(\beta_T)]\,,$$

where by convention $f(\beta_T)=0$ if $T=\infty$. Our aim is to show that $u(O)=v_a(f)$. Let us claim that for a sequence ξ_n in $B(a)$ which converges to the origin it holds that

$$\limsup_{\delta\downarrow 0}\,\limsup_{n\to\infty}P_{\xi_n}[T<\delta]=0\quad\text{implies}\quad u(O)=\lim_{n\to\infty}u(\xi_n)\,. \tag{A.4}$$

Indeed if $\xi \in \overline{B(a)}$, the Markov property shows

$$E_\xi[f(\beta_T)] = E_\xi[E_{\beta_\delta}[f(\beta_T)]; T > \delta] + E_\xi[f(\beta_T); T < \delta],$$

from which it follows that

$$|E_\xi[f(\beta_T)] - E_\xi[u(\beta_\delta)]| \leqq 2\|f\|_\infty P_\xi[T < \delta].$$

Applying this inequality with $\xi = 0$ and $\xi = \xi_n$ and rewriting $E_\xi[u(\beta_\delta)]$ as $E_O[u(\beta_\delta + \xi)]$, we see that the premiss of (A.4) implies

$$\limsup_{n\to\infty} |u(\xi_n) - u(O)| \leqq \limsup_{\delta\to 0} \limsup_{n\to\infty} |E_O[u(\beta_\delta + \xi_n)] - E_O[u(\beta_\delta)]|,$$

where the thinness of $S(a)$ at O also is applied. Since $S(a)$ is not thin at every point of it except the origin and $(a, \mathbf{0})$, $u(\xi)$ is continuous off these two points. Therefore we get the conclusion of (A.4).

As being well established in the theory of Markov processes $\mathscr{L}^* u = 0$ off $S(a)$. Let $\xi_\varepsilon = (\varepsilon, \mathbf{0})$ for $\varepsilon > 0$. Since

$$(\partial/\partial a)\{X_a(a-t)\}^2/N = -\log(1 - t/a) - t/a > 0, \qquad 0 < t < a,$$

if $b < a - \varepsilon$ the shifted ball $B(b) + \xi_\varepsilon$ is in the interior of $B(a)$, so that by the Fulks result (A.2) $u(\xi_\varepsilon) = \nu_b(u(\cdot + \xi_\varepsilon))$. Let T' be the first hitting time to $S(a/2)$. Then $P_{(\varepsilon, \mathbf{0})}[T < \delta] \leqq P_O[T' < \delta]$ for $\varepsilon < a/2$. Since $R^{N+1} \setminus B(a/2)$ is thin at the origin, the right-hand side of this inequality vanishes as $\delta \to 0$; hence by (A.4) $u(\xi_\varepsilon)$ converges to $u(O)$. On the other hand since $u(\xi)$ is continuous at every point except the top and bottom points of $S(a)$ and coincides with f on $S(a)$ except for these two and since ν_a has no point mass, by the dominated convergence theorem $\lim_{\varepsilon\to 0} \nu_b(u(\cdot + \xi_\varepsilon)) = \nu_b(u)$ and owing to the explicit form of ν_b provided by (A.3) $\lim_{b\uparrow a} \nu_b(u) = \nu_a(u)$. Consequently $u(O) = \nu_a(f)$ as required.

4. In this part we shall continue to use notations of previous three parts but does not make use of results obtained in them in any essential way. Let us consider the "dual" space-time process $(-t+s, W(t)+\mathbf{x})$, $t \geqq 0$, starting at $\xi = (s, \mathbf{x})$, and mark * on the corresponding objects. Thus P^*_ξ is the law of the dual process, $S^*(a) = -S(a)$ and $g^*(\xi, \eta) = g(\eta, \xi)\,(= g(\xi - \eta))$, etc. ($\beta_t$ is not marked because it is merely a sample path; and not also the first hitting times defined by means of it.) *Given a bounded analytic set A of R^{N+1}, we take $\xi_0 \in R^{N+1}$ and $a > 0$ such that* $\bar{A}$ (the closure of A) $\subset B^* := B^*(a) + \xi_0$. *Let T_A and T_{S^*} be the first passage times of β_t to A and to $S^* := \partial B^*$, respectively, and put*

$$\mu^*(\cdot) = (2\pi a)^{N/2} P^*_{\xi_0}[\beta(T_{S^*}) \in \cdot],$$

and

$$e_A(\cdot) = \int P^*_\eta[\beta(T_A) \in \cdot]\, \mu^*(d\eta). \tag{A.5}$$

Then e_A is the equilibrium measure of A, i.e. it satisfies (1.1).

Let $t > s$, $\xi = (s, \mathbf{x})$ and $\eta = (t, \mathbf{y})$. First we prove that for a.a. $\mathbf{y}$

$$E_\xi[g(\beta(T_A), \eta)] = P[(u, W^{\mathbf{x};\mathbf{y}}_{t-s}(u)) \in A \quad \text{for some} \quad 0 < u < t - s]\, g(\xi, \eta) \tag{A.6}$$

where $W_t^{\mathbf{x},\mathbf{y}}(u) := W(u) + \{(t-u)\mathbf{x} + u(\mathbf{y} - W(t))\}/t$, $0 \leqq u \leqq t$, a Brownian bridge starting at $\mathbf{x}$ and ending at $\mathbf{y}$. For the proof of (A.6) we can assume $\xi = O$. Let τ_A be the first hitting time to A by $(t, W(t))$. Then, by recalling that $g(\xi, \eta) = p(t-s, |\mathbf{y}-\mathbf{x}|)$ is a transition density of $W(\cdot)$, for a bounded continuous function $\varphi(\mathbf{y})$ we see

$$\begin{aligned}\int_{R^N} E_O[g(\beta_{T_A}, (t, \mathbf{y}))]\, \varphi(\mathbf{y})\, d\mathbf{y} &= E[E[\varphi(W(t-u)+\mathbf{x})]_{u=\tau_A, \mathbf{x}=W(\tau_A)}; \tau_A < t] \\ &= E[\varphi(W(t)); \tau_A < t] \\ &= \int_{R^N} P[\tau_A < t \,|\, W(t) = \mathbf{y}]\; p(t, \mathbf{y})\; \varphi(\mathbf{y})\, d\mathbf{y}\end{aligned}$$

(where the strong Markov property is applied for the second equality), proving (A.6). The process $W_t^{\mathbf{x},\mathbf{y}}(t-u)$, $0 \leqq u \leqq t$ is also a Brownian bridge but starting at $\mathbf{y}$ and ending at $\mathbf{x}$. Keeping this in mind compare the right-hand side of (A.6) with that of its dual version, then you can see that for a.a. (ξ, η)

$$E_\xi[g(\beta(T_A), \eta)] = E_\eta^*[g^*(\beta(T_A), \xi)]\,. \tag{A.7}$$

Since the both sides of this equality are continuous in (ξ, η) off $\bar{A} \times \bar{A}$, it holds outside $\bar{A} \times \bar{A}$. Let us prove that (A.7) is true for all ξ and η. To this end we can assume that $A \subset \{(t', \mathbf{x}') \in R^{N+1} : s < t' < t\}$. Let $\xi \notin \bar{A}$, $\eta \in \bar{A}$ and $A_n = \{(t', \mathbf{x}) \in A : t' < t - 1/n\}$. Then $T_{A_n} \downarrow T_A$ a.s. P_ξ, and $P_\xi[T_{A_n} = T_A \,|\, T_{A_n} < \infty] = 1$. The latter relation implies $E_\xi[g(\beta(T_{A_n}), \xi)] \leqq E_\xi[g(\beta(T_A), \eta)]$; and application of Fatou's lemma then shows that the expectation on the left-hand side of this inequality converges to that on the right-hand side as $n \to \infty$. As for the convergence of $E_\eta^*[g(\beta(T_{A_n}), \xi)]$ we can simply apply the bounded convergence theorem. Consequently we have (A.7). Repeating the same argument we see that (A.7) is true for $\xi, \eta \in \bar{A}$; hence, by symmetry, for all ξ and η.

We now integrate both sides of (A.7) by $\mu^*(d\eta)$. For the moment let $\xi_0 = O$. Then $\mu^*(\sqrt{2\pi a})^N v_a^*$ and if $\xi \notin \overline{B^*(a)}$ it holds that $v_a^* g^*(\xi) = g^*(\xi)$, since then $\mathscr{L}^*\{g^*(\cdot, \xi)\} = 0$ in a neighbourhood of $\overline{B^*(a)}$. This shows that $\mu^* g^* = g\mu^* = 1$ on $S^* \setminus \{\xi_0\}$ and hence on B^*. Now the integral of the left-hand side equals $E_\xi[g\mu^*(\beta(T_A))] = P_\xi[T_A < \infty] = h(\xi, A)$, while by the very definition of e_A that of the right-hand side is ge_A. Thus our claim is proved.

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