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The Open Mapping and Closed Range Theorems

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0. Introduction

This paper originated from a study of Browder's version [3] of Banach's closed range theorem for (not necessarily continuous) linear operators between locally convex spaces. It seems that the most reasonable way to proceed is to perform a simultaneous development of the closed range theorem and the open mapping theorem. Insofar as the open mapping theorem is concerned, our results largely parallel those of Köthe [5, 7], but our proofs are less technical and (see Theorem 11) we can dispense with the local convexity of the spaces concerned. Furthermore (see Remark 19) our analysis throws light on the result obtained in [7] that, in the locally convex case, every open map is weakly singular. Insofar as the closed range theorem is concerned, our results represent improvements of the results of Browder [3] and Baker [2]. See the discussion in Sect. 8.

Our presentation leans heavily on the properties of seminorms. We use the characterization of barreled spaces in terms of lower semicontinuous seminorms (see Sect. 1) and our other main tools are the concepts of *Mackey seminorm* (see Sect. 1), the quotient of a seminorm by a linear map (see Sect. 2) and *adequate map* (see Definition 7 and Remark 8). These tools obviate our having to deal with the more abstract concept of a quotient topological vector space.

We introduce two definitions of the adjoint of a linear map. The "small adjoint," introduced in Sect. 3, enables us to give the most succinct statements for the results in Sects. 3–5. The "large adjoint," introduced in Sect. 6, corresponds to the definition usually made.

In Sect. 7 we specialize to the case when local convexity is assumed for the range space (and sometimes also for the domain space).

A number of the results in this paper assume that the range of the operator under consideration is metrizable. We discuss weakenings of this condition in Sect. 8.

In order to keep our treatment short and simple, we have not touched on of the many other interesting concepts dealt with by some authors (e.g., Köthe [5, 7], Baker [2], Mennicken and Sagraloff [8]) such as nearly open operators, weakly open operators, etc.

Other authors who have contributed to the literature include Dieudonné and Schwartz [4], Mochizuki [9], Pták [10], Schaefer [12], and Treves [13, 14].

1. Preliminaries

If E is a real topological vector space, we write E^* for the algebraic dual of E and E' for the topological dual of E. If P is a seminorm on E, we write

$$E_P^* := \{a : a \in E^*, a \leqq P \text{ on } E\}$$

and

$$E'_P := \{a : a \in E', a \leqq P \text{ on } E\}.$$

From the Banach-Alaoglu theorem, E_P^* is $w(E^*, E)$-compact.

We say that E is *barreled* if every lsc (lower semicontinuous) seminorm on E is continuous (see [6, 21.2(2)–(3), p. 257]). (We do not assume that E is necessarily Hausdorff or locally convex.) The authors are grateful to the referee for pointing out that if E is ultrabarreled (in the sense of [11]) then E is barreled and, from [11, Sect. 7, p. 256] the converse is not true even if E is locally convex.

Lemma 1. *Let E be barreled and Q be any seminorm on E. Then E'_Q is $w(E', E)$-compact.*

Proof. Define $R : E \to \mathbb{R}$ by $R := \sup\{a : a \in E'_Q\}$. R is a lsc seminorm on E and $R \leqq Q$. Since E is barreled, R is continuous. The result follows since $E'_Q = E_R^*$.

We shall say that P is a *Mackey seminorm on E* if $E_P^* \subset E'$. [If E is Hausdorff and locally convex, then P is Mackey $\Leftrightarrow$ P is continuous with respect to the Mackey topology $\tau(E, E')$. However, our definition does *not* require that E be either Hausdorff or locally convex.]

Lemma 2. *Any Mackey seminorm is lsc.*

Proof. This follows since, from the Hahn-Banach theorem, any seminorm is the supremum of the linear functionals that it dominates.

Lemma 3. *Let E be pseudometrizable. Then any Mackey seminorm on E is continuous. (This is the counterpart of the result that any metrizable locally convex space has the Mackey topology.)*

Proof. Suppose, on the contrary, that P is a discontinuous Mackey seminorm on E. Then $\exists \{x_n\}_{n \geqq 1} \subset E$ such that $x_n \to 0$ and, $\forall n \geqq 1$, $P(x_n) \geqq 1$. Let $\{U_n\}_{n \geqq 1}$ be a decreasing neighborhood base at 0. Then $\exists$ a subsequence $\{y_n\}_{n \geqq 1}$ of $\{x_n\}_{n \geqq 1}$ such that, $\forall n \geqq 1$, $y_n \in 3^{-n} U_n$. Thus, writing $z_n := 2.3^n y_n$, $z_n \to 0$ and $P(z_n) \geqq 2.3^n$.

We define $n_1 := 1$ and choose $b_1, n_2, b_2, \ldots$ inductively as follows: (using the Hahn-Banach theorem) $b_k \in E_P^* \subset E'$ such that

$$\langle z_{n_k}, b_k \rangle = P(z_{n_k}) \geqq 2.3^k$$

and (since $z_n \to 0$) $n_k > n_{k-1}$ such that

$$\left| \left\langle z_{n_k}, \sum_{j=1}^{k-1} 3^{-j} b_j \right\rangle \right| < \frac{1}{2}.$$

Write $b := 2\sum_{j=1}^{\infty} 3^{-j} b_j \in E_P^* \subset E'$. Then, $\forall k \geqq 1$,

$$\langle z_{n_k}, b\rangle \geqq 2.3^{-k}\langle z_{n_k}, b_k\rangle + \sum_{j=k+1}^{\infty} 2.3^{-j}\langle z_{n_k}, b_j\rangle - 2\cdot\frac{1}{2}$$

$$\geqq \left(2.3^{-k} - \sum_{j=k+1}^{\infty} 2.3^{-j}\right) P(z_{n_k}) - 1 \geqq 1,$$

which is a contradiction since $z_{n_k} \to 0$.

2. Semi-Open Linear Maps

We shall suppose for the rest of this paper that E and F are real Hausdorff topological vector spaces and $T: E \to F$ is linear (but not necessarily continuous). We do not assume that E or F is locally convex until Sect. 7.

If P is a seminorm on E we define the seminorm P/T on $T(E)$ by

$$P/T(y) := \inf P(T^{-1}y)\,(y \in T(E)).$$

Lemma 4. *Let* $b \in T(E)^*$. *Then* $b \in T(E)^*_{P/T} \Leftrightarrow b \circ T \in E_P^*$.

Proof. $(\Rightarrow)$ $\forall x \in E$, $\langle x, b\circ T\rangle = \langle Tx, b\rangle \leqq P/T(Tx) = \inf P(T^{-1}Tx) \leqq P(x)$.

$(\Leftarrow)$ $\forall y \in T(E)$ and $x \in T^{-1}y$, $\langle y, b\rangle = \langle Tx, b\rangle = \langle x, b\circ T\rangle \leqq P(x)$. Taking the infimum over x, $\langle y, b\rangle \leqq P/T(y)$.

Definition 5. We shall say that T is *semi-open* if P/T is a continuous seminorm on $T(E)$ whenever P is a continuous seminorm on E.

Lemma 6. (6.1) $\Rightarrow$ (6.2) $\Rightarrow$ (6.3) $\Rightarrow$ (6.4). *If* $T(E)$ *is barreled then* (6.1)–(6.4) *are equivalent. If* $T(E)$ *is metrizable then* (6.1)–(6.3) *are equivalent.*

(6.1) *T is semi-open.*

(6.2) *If $b \in T(E)^*$ and $b \circ T \in E'$ then $b \in T(E)'$.*

(6.3) *If P is a continuous seminorm on E then P/T is Mackey on $T(E)$.*

(6.4) *If P is a continuous seminorm on E then P/T is lsc on $T(E)$.*

Proof. ((6.1) $\Rightarrow$ (6.2)). Let $b \in T(E)^*$ and $b \circ T \in E'$. Since T is semi-open, setting $P := |\langle\cdot, b\circ T\rangle|$, $P/T = |\langle\cdot, b\rangle|$ is continuous on $T(E)$, hence $b \in T(E)'$. Thus (6.2) is true.

((6.2) $\Rightarrow$ (6.3)). Let $b \in T(E)^*_{P/T}$. From Lemma 4, $b \circ T \in E_P^* \subset E'$. From (6.2), $b \in T(E)'$. Hence P/T is Mackey.

It is immediate from Lemma 2 that (6.3) $\Rightarrow$ (6.4).

If $T(E)$ is barreled then it is immediate that (6.4) $\Rightarrow$ (6.1). If $T(E)$ is metrizable then it is immediate from Lemma 3 that (6.3) $\Rightarrow$ (6.1).

3. Connections with the Range of the Small Adjoint

We define $\Delta := \{\delta : \delta \in \overline{T(E)}', \delta \circ T \in E'\} \subset \overline{T(E)}'$ and $T^\tau : \Delta \to E'$ by

$$T^\tau\delta := \delta \circ T\ (\delta \in \Delta).$$

[It is understood in this that $\overline{T(E)}$ has the topology induced from the topology of F.] We define $G := \{(x, Tx) : x \in E\} \subset E \times F$.

Definition 7. We shall say that T is *adequate* if

$$x \in E, a \in E', \langle x, T^{\tau}(\Delta)\rangle = \{0\} \quad \text{and} \quad \langle T^{-1}(0), a\rangle = \{0\} \Rightarrow \langle x, a\rangle = 0.$$

Remark 8. T is clearly adequate if Δ separates the points of $T(E)$ or $T^{-1}(0)$ is dense in E – in particular if F is locally convex and T is continuous. The condition (8.1) below also ensures that T is adequate.

$$(8.1) \qquad (x, 0) \in G^{00} \Rightarrow x \in T^{-1}(0)^{00},$$

where ^{0} stands for the operation of polarity (orthogonality). To see this, suppose that $\langle x, T^{\tau}(\Delta)\rangle = \{0\}$ and $\langle T^{-1}(0), a\rangle = \{0\}$. If $(c, b) \in G^0 \subset E' \times F'$ then $b \circ T = -c \in E'$ hence $\delta := b|_{\overline{T(E)}} \in \Delta$ and $c = -\delta \circ T = -T^{\tau}\delta$. Thus

$$\langle (x, 0), (c, b)\rangle = \langle x, c\rangle = -\langle x, T^{\tau}\delta\rangle = 0,$$

that is to say, $(x, 0) \in G^{00}$. From (8.1), $x \in T^{-1}(0)^{00}$ hence $\langle x, a\rangle = 0$.

If F is locally convex then the above argument can be reversed and gives that (8.1) $\Leftrightarrow$ T is adequate. If E and F are both locally convex then, from the above and the bipolar theorem, T is adequate $\Leftrightarrow$ T is weakly singular (see Remark 19). The authors are grateful to the referee for pointing out this last fact.

Lemma 9. *We consider the condition:*

$$(9.1) \qquad T^{\tau}(\Delta) \ \textit{is} \ w(E', E)\textit{-closed}.$$

Then (6.2) $\Rightarrow$ (9.1). *If T is adequate then* (6.2) $\Leftrightarrow$ (9.1).

Proof. ((6.2) $\Rightarrow$ (9.1)). Let $a \in E'$ and suppose that $\exists$ a net $\delta_\alpha \in \Delta$ such that $T^{\tau}\delta_\alpha \to a$ in $w(E', E)$: we shall prove that $a \in T^{\tau}(\Delta)$. Let $x \in T^{-1}(0)$. Then

$$\langle x, a\rangle = \lim_\alpha \langle x, T^{\tau}\delta_\alpha\rangle = \lim_\alpha \langle Tx, \delta_\alpha\rangle = \lim_\alpha \langle 0, \delta_\alpha\rangle = 0.$$

Thus we have proved that if $x \in T^{-1}(0)$ then $\langle x, a\rangle = 0$. It follows algebraically that $\exists b \in T(E)^*$ such that $b \circ T = a$. From (6.2), $b \in T(E)'$ and, from standard extension arguments, $\exists \delta \in \overline{T(E)}'$ such that $\delta|_{T(E)} = b$, hence $\delta \circ T = a$. Then $\delta \in \Delta$ and $T^{\tau}\delta = a$, from which $a \in T^{\tau}(\Delta)$, as required.

((9.1) $\Rightarrow$ (6.2)) Let $b \in T(E)^*$ and $b \circ T \in E'$. Suppose that $x \in E$ and $\langle x, T^{\tau}(\Delta)\rangle = \{0\}$. Since T is adequate and $\langle T^{-1}(0), b \circ T\rangle = \{0\}$, $\langle x, b \circ T\rangle = 0$. Thus we have proved that

$$x \in E \text{ and } \langle x, T^{\tau}(\Delta)\rangle = \{0\} \Rightarrow \langle x, b \circ T\rangle = 0.$$

From (9.1) and the separation theorem, $b \circ T \in T^{\tau}(\Delta)$, hence $\exists \delta \in \Delta$ such that $T^{\tau}\delta = b \circ T$. Then $b = \delta|_{T(E)} \in T(E)'$. Thus (6.2) is true, as required.

Theorem 10. *Let T be adequate. If $T(E)$ is barreled then* (6.1)–(6.4) *and* (9.1) *are equivalent. If $T(E)$ is metrizable then* (6.1)–(6.3) *and* (9.1) *are equivalent.*

Proof. This is immediate from Lemmas 6 and 9.

4. The Semi-Open Mapping Theorem

We recall that E is a *Pták space* if a subspace L of E' is $w(E',E)$-closed whenever, $\forall$ continuous seminorms P on E, $L \cap E_P^*$ is $w(E',E)$-compact. (We do not assume that E is necessarily locally convex. See Remarks 15 for further comments on Pták spaces.)

Theorem 11 (Semi-Open Mapping Theorem). *If E is a Pták space, T is adequate and $T(E)$ is barreled then T is semi-open and $T^\tau(\Delta)$ is $w(E',E)$-closed.*

Proof. Let P be a continuous seminorm on E. From standard extension arguments and Lemma 4,

$$T^\tau(\Delta) \cap E_P^* = \{b \circ T : b \in T(E)', b \circ T \leqq P\} = \{b \circ T : b \in T(E)'_{P/T}\};$$

from Lemma 1 and the $w(T(E)', T(E)) - w(E^*, E)$ continuity of the map $b \to b \circ T$, this set is $w(E^*,E)$-compact. Since E is a Pták space, $T^\tau(\Delta)$ is $w(E',E)$-closed. The result now follows from Theorem 10.

5. Connections with the Range of T

If $z \in \overline{T(E)}$ we define $\Delta_z := \{\delta : \delta \in \Delta, \langle z, \delta \rangle = 0\}$.

Lemma 12. *If T is semi-open and $z \in \overline{T(E)}$ then, for all continuous seminorms P on E, $T^\tau(\Delta_z) \cap E_P^*$ is $w(E',E)$-compact.*

Proof. Let P be a continuous seminorm on E. Since T is semi-open P/T is (uniformly) continuous on $T(E)$. From standard extension arguments, $\exists$ a continuous seminorm R on $\overline{T(E)}$ such that $R|_{T(E)} = P/T$. We now prove that

$$T^\tau(\Delta_z) \cap E_P^* = \{\delta \circ T : \delta \in \overline{T(E)}_R^*, \langle z, \delta \rangle = 0\}. \tag{12.1}$$

Let $a \in T^\tau(\Delta_z) \cap E_P^*$. Then $\exists \delta \in \Delta$ such that $\langle z, \delta \rangle = 0$ and $T^\tau \delta = a$. From Lemma 4 ($\Leftarrow$), $\delta|_{T(E)} \in T(E)_{P/T}^*$. Since both δ and R are continuous on $\overline{T(E)}$, $\delta \in \overline{T(E)}_R^*$. This establishes ($\subset$) in (12.1). Conversely, let $\delta \in \overline{T(E)}_R^*$ and $\langle z, \delta \rangle = 0$. Since $\delta|_{T(E)} \in T(E)_{P/T}^*$, from Lemma 4 ($\Rightarrow$), $\delta \circ T \in E_P^* \subset E'$, from which $\delta \in \Delta$, hence $\delta \in \Delta_z$. This establishes ($\supset$) in (12.1). The result now follows from (12.1), the Banach-Alaoglu theorem and the $w(\overline{T(E)}^*, \overline{T(E)}) - w(E^*, E)$ continuity of the map $\delta \to \delta \circ T$.

Lemma 13. *If Δ separates the points of $\overline{T(E)}$ and, $\forall z \in \overline{T(E)}$, $T^\tau(\Delta_z)$ is $w(E',E)$-closed then $T(E)$ is closed in F.*

Proof. Let $z \in \overline{T(E)} \setminus T(E)$. Clearly $z \neq 0$. Since Δ separates the points of $\overline{T(E)}$, $\exists \delta_0 \in \Delta$ such that $\langle z, \delta_0 \rangle = 1$. If now $\delta \in \Delta_z$ then $\langle z, \delta \rangle \neq \langle z, \delta_0 \rangle$ hence [since $z \in \overline{T(E)}$ and both δ_0 and δ are continuous on $\overline{T(E)}$] $\delta|_{T(E)} \neq \delta_0|_{T(E)}$, from which $T^\tau \delta \neq T^\tau \delta_0$. Thus $T^\tau \delta_0 \notin T^\tau(\Delta_z)$. By hypothesis and the separation theorem, $\exists x \in E$ such that $\langle x, T^\tau \delta_0 \rangle = 1$ and

$$\delta \in \Delta_z \Rightarrow \langle x, T^\tau \delta \rangle = 0. \tag{13.1}$$

Let $\delta \in \Delta$. Since $\langle z, \delta \rangle \delta_0 - \delta \in \Delta_z$, from (13.1) and the fact that $\langle x, T^\tau \delta_0 \rangle = 1$,

$$0 = \langle x, T^\tau(\langle z, \delta \rangle \delta_0 - \delta) \rangle = \langle z - Tx, \delta \rangle.$$

Thus we have proved that

$$\forall \delta \in \Delta, \langle z - Tx, \delta \rangle = 0.$$

Since Δ separates the points of $\overline{T(E)}$, $z = Tx \in T(E)$. This contradiction of the assumption that $z \notin T(E)$ establishes that $\overline{T(E)} \setminus T(E) = \emptyset$, hence $T(E)$ is closed, as required.

Theorem 14 (First Half of the Closed Range Theorem). *Let E be a Pták space and Δ separate the points of $\overline{T(E)}$. Then*

$$T \textit{ is semi-open} \Rightarrow T(E) \textit{ is closed in } F.$$

If, further, $T(E)$ is barreled or metrizable then

$$T^{\tau}(\Delta) \textit{ is } w(E', E)\textit{-closed} \Rightarrow T \textit{ is semi-open}.$$

Proof. This is immediate from Theorem 10 and Lemmas 12 and 13.

Remarks 15. The argument of this section is related to Grothendieck's completeness theorem, but the argument of Theorem 14 does really establish that $T(E)$ is *closed*. We do not know whether, under the conditions of Theorem 14, the conclusion can be strengthened to "$T(E)$ is complete." We observe that it is only the topology of $T(E)$ (as a subspace of F) that is at issue in Theorem 14. We do not otherwise have to be concerned with the topology of F. Theorem 14 complements [7, 37.5 (4), p. 104], which requires that E be locally convex.

The authors are grateful to the referee for the following observation. Let $E = l^{1/2}$, $F = l^1$, and $T: E \to F$ be the canonical inclusion. Then $T(E)$ is metrizable and (being dense) is not closed in F. Further, $\Delta = F' = l^{\infty}$ and $T^{\tau}(\Delta) = E'$ which is $w(E', E)$-closed in E'. Thus, from Theorem 14, E is *not* a Pták space. This example shows that a complete metrizable topological vector space is not necessarily a Pták space. Combining this with [1, 10 (7), p. 54] we see that a B-complete space (in the sense of [1]) is not necessarily a Pták space. On the other hand, if $E' = \{0\}$ then E is a Pták space; consequently, it is easy to give an example of a Pták space that is not complete. Combining this with [1, 10 (9), p. 56] we see that a Pták space (in the sense of [1]) is not necessarily B-complete.

6. The Range of the Large Adjoint

Remark 16. The map T^{τ} with domain Δ can be thought of as an adjoint map for T. This definition is obviously the appropriate one for the analysis of Sects. 3–5. It is, however, more usual to define the domain of the adjoint by

$$D := \{d : d \in F', d \circ T \in E'\} \subset F'$$

and the adjoint $T^t : D \to E'$ by

$$T^t d := d \circ T \quad (d \in D).$$

It is this distinction that prompted us to use the phrases "small adjoint" and "large adjoint" in the section headings for Sects. 3 and this section. If the following condition is satisfied:

(16.1) If $\delta \in \overline{T(E)}'$ and $\delta \circ T \in E'$ then $\exists d \in F'$ such that $d|_{\overline{T(E)}} = \delta$

then $T^t(D) = T^t(\Delta)$. In this case, (9.1) is equivalent to:

(16.2) $$T^t(D) \text{ is } w(E', E)\text{-closed}.$$

For the analysis of the next section, it is sufficient to observe that (16.1) is satisfied if F is locally convex.

7. The Locally Convex Case

In this section we suppose that F is locally convex. We shall also suppose for much of this section that E is locally convex.

Lemma 17. *We consider the condition*

(17.1) $$T \textit{ is an open map from } E \textit{ onto } T(E).$$

Then (17.1) $\Rightarrow$ (6.1). *If* E *is locally convex then* (17.1) $\Leftrightarrow$ (6.1).

Proof. This follows from the identity

$$\{y : y \in T(E), P/T(y) < 1\} = T\{x : x \in E, P(x) < 1\},$$

which is immediate from the definition of P/T.

Theorem 18 (On the Equivalence of Conditions). *Let E be locally convex and T be adequate. If $T(E)$ is either barreled or metrizable then* (6.2), (6.3), (16.2), *and* (17.1) *are equivalent.*

Proof. This is immediate from Theorem 10 and Lemma 17.

Remark 19 (On Weak Singularity). T is said to be *weakly singular* (see [7, 36.1 (7), p. 81]) if

$$(x, 0) \in \bar{G} \Rightarrow x \in \overline{T^{-1}(0)}.$$

We now show that if (6.4) is satisfied and E is locally convex then T is weakly singular. Suppose that $(x, 0) \in \bar{G}$. Then $\exists$ a net x_α in E such that $x_\alpha \to x$ and $Tx_\alpha \to 0$. Let P be a continuous seminorm on E. Then, eventually, $P(x_\alpha - x) \leqq 1/2$ hence $P/T(Tx_\alpha - Tx) \leqq 1/2$. From (6.4), P/T is lsc on $T(E)$. Thus, since $Tx_\alpha \to 0$, $P/T(-Tx) \leqq 1/2$. Hence

$$\exists w \in -T^{-1}Tx \quad \text{such that} \quad P(w) \leqq 1.$$

Let $u = x + w$. Then $u \in T^{-1}(0)$ and $P(u - x) \leqq 1$. Since this holds for all P and E is locally convex, $x \in \overline{T^{-1}(0)}$.

It follows from these considerations that if E is locally convex and Pråk and F is barreled then the implication (6.4) $\Rightarrow$ (17.1) can also be deduced from [7, 37.5 (5), p. 104]. Furthermore, the implications (17.1) $\Rightarrow$ (6.1) $\Rightarrow$ (6.2) $\Rightarrow$ (6.3) $\Rightarrow$ (6.4) [which hold even if $T(E)$ is not barreled or metrizable] throw light on the result of [7, 37.4 (1), p. 100] that, in the locally convex case, every open map is weakly singular.

Theorem 20 (Open Mapping and the Other Half of the Closed Range Theorem). *If E is a locally convex Pták space, T is adequate (= weakly singular) and $T(E)$ is barreled then T is an open map from E onto $T(E)$ and $T^t(D)$ is $w(E', E)$-closed.*

Proof. This is immediate from Theorem 11 and Lemma 17.

Theorem 21 (Surjective Open Mapping Theorem). *If E is a locally convex Pták space, F is barreled T is adequate (= weakly singular) and $T(E)=F$ then T is an open map.*

Proof and remarks. This is immediate from Theorem 20. This is a special case of [7, 37.5 (5), p. 104]. If T is continuous we obtain Pták's homomorphism theorem (see [12, IV.8.3, Corollary 1, p. 164]), since a continuous linear map into a locally convex space is adequate.

Theorem 22 (Semi-Open Mapping and Closed Range Theorem). *Let E be a Pták space and D separate the points of $\overline{T(E)}$. If $T(E)$ is barreled or metrizable then*

$$T^t(D) \text{ is } w(E',E)\text{-closed} \Rightarrow T \text{ is semi-open} \Rightarrow T(E) \text{ is closed in } F.$$

If every closed subspace of F is barreled then

$$T(E) \text{ is closed in } F \Leftrightarrow T \text{ is semi-open} \Rightarrow T^t(D) \text{ is } w(E',E)\text{-closed}.$$

Proof and Remarks. This is immediate from Theorem 11 and Theorem 14. When we compare this result with [3, Theorem 2.1, p. 65] and [2, Theorem 8, p. 283], we observe that we do not require E to be locally convex and that Lemma 3 has enabled us to avoid considering "condition (t)." See the discussion in the next section.

8. More General Versions of our Results

The proof of Lemma 3 shows that, in all cases in this paper where we have assumed that $T(E)$ is metrizable, we could equally well have assumed the weaker condition [akin to that of $T(E)$ being bornological]: if Q is a seminorm on $T(E)$ such that $\{Q(z_n): n \geqq 1\}$ is bounded whenever $\{z_n\}_{n\geqq 1} \subset T(E)$ and $z_n \to 0$ then Q is continuous on $T(E)$.

We can obtain a more radical generalization of the first half of the closed range theorem by modifying Lemma 12 rather than Lemma 3. To this end, let us say that a subspace H of F is a *μ-subspace* if every Mackey seminorm on H can be extended to a Mackey seminorm on $\bar{H}$. We can now prove the following modification of Lemma 12: if (6.3) is satisfied, $T(E)$ is a μ-subspace of F and $z \in \overline{T(E)}$ then, for all continuous seminorms P on E, $T^\tau(\Delta_z) \cap E_P^*$ is $w(E',E)$-compact. This leads to the following modification of Theorem 14: let E be a Pták space, Δ separate the points of $\overline{T(E)}$ and $T(E)$ be a μ-subspace of F. Then

$$T^\tau(\Delta) \text{ is } w(E',E)\text{-closed} \Rightarrow T(E) \text{ is closed in } F,$$

with a corresponding restatement in terms of $T^t(D)$ if F is locally convex. In seminorm terms, *F satisfies condition* (t) (see [2] and [3]) if every Mackey seminorm on every subspace of F can be extended to a Mackey seminorm on F. In particular, if F satisfies condition (t) then *every* subspace of F is a μ-subspace. Consequently, the above result generalizes the relevant part of [2, Theorem 8, p. 283] and [3, Theorem 2.1, p. 65]. Of course, it is requiring much less to ask for an extension from $T(E)$ to $\overline{T(E)}$ than it is to ask for an extension from every subspace of F to F. Furthermore, in view of the close relationship between a subspace and its closure, there is some hope of a reasonable internal characterization of μ-subspace.

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