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Asymptotics for some Green Kernels on the Heisenberg Group and the Martin Boundary

H. Hueber and D. Müller

Fakultät für Mathematik, Universität Bielefeld, Postfach 8640, D-4800 Bielefeld, Federal Republic of Germany

Introduction

The Heisenberg group and especially the sub- (or Kohn-) Laplacian Δ_K on this group has been studied extensively by many authors during at least the last decade, and it would be beyond the scope of this article to even try to list the possible applications and results related to this study. Let us only note that Δ_K is a prototype of a so-called "subelliptic" operator, a notion which is perhaps not well-defined but commonly used for operators such as the "sum of squares" studied by Hörmander [10], or for the so-called "Rockland operators" studied by Rockland [13], Helffer-Nourrigat [7] and others. Therefore, it is desirable to have as much explicit information about Δ_K as possible, and quite a bit is in fact known by now. For example, Folland was the first to calculate an explicit fundamental solution for Δ_K, and Gaveau (see [6]), using stochastic integration, as well as Hulanicki [11] and Cygan [3], making use of the representation theory of the Heisenberg group, derived a formula for the fundamental solution of the corresponding heat operator $\partial/\partial s - \Delta_K$. However, although this formula suffices to give a canonical way of calculating a fundamental solution for Δ_K, it has one decisive drawback: it is explicit only up to the partial Fourier transform along the center of the Heisenberg group, and it seems very unlikely that one might be able to carry through this Fourier transform explicitly. Therefore the best one might hope for is to be able to describe the asymptotic behaviour of this fundamental solution. A partial solution to this problem has already been given by Gaveau ([6]; compare also Theorem 1.1), but his results did not cover regions which are in some sense "close" to the center of the Heisenberg group.

In this article, we shall present a complete picture of the asymptotics for the fundamental solution of the heat operator. Moreover, we shall apply these results resp. methods in order to solve the analogous problem for the operator $\Delta_K - \mu$, μ a positive real number. Finally, in Sect. 3 we shall use these descriptions of the asymptotic behaviour in order to prove that the so-called Martin boundary corresponding to $\Delta_K - 2$ of the Heisenberg group is homeomorphic to the closed unit disk in $\mathbb{C}$. Moreover, we can show that the minimal Martin boundary, that is the space of extremal rays of the cone of all positive solutions h of the equation

$(\Delta_K - 2)h = 0$, is homeomorphic to that part of the Martin boundary that corresponds to the unit circle. These results are in sharp contrast to the classical situation of the Laplace operator on $\mathbb{R}^n$, where the Martin boundary corresponding to $\Delta - 2$ is homeomorphic to the unit sphere S^{n-1} in $\mathbb{R}^n$. A consequence of our result is that each positive eigenfunction of Δ_K corresponding to a positive eigenvalue is independent of the central variable of the Heisenberg group.

0. Preliminaries

(a) In order to avoid a few additional technical problems which arise for higher-dimensional Heisenberg groups and to fix the ideas, we shall only deal with the 3-dimensional Heisenberg group H_1.

H_1 is the 2-step nilpotent Lie group whose underlying real manifold is $\mathbb{C} \times \mathbb{R}$, and where multiplication is given by

$$(z, u)(z', u') = (z + z', u + u' + 2\,\mathrm{Im}\, z \cdot \bar{z}')$$

for (z, u), $(z', u') \in \mathbb{C} \times \mathbb{R}$. We introduce real coordinates (x, y, u) for H_1 by writing $z = x + iy$. Let X, Y, U denote the left-invariant vector fields on H_1 whose values at the neutral element 0 are given by $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial u$ respectively, i.e.:

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial u}, \qquad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial u}, \qquad U = \frac{\partial}{\partial u}.$$

Then we have $[X, Y] = -4U$, and hence it follows for example from [10] that the sub-Laplacian

$$\Delta_K = X^2 + Y^2$$

is hypoelliptic. With respect to the coordinates (x, y, u), Δ_K is explicitly given by

$$\Delta_K = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4\left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right]\frac{\partial}{\partial u} + 4(x^2 + y^2)\frac{\partial^2}{\partial u^2}.$$

The group of automorphisms of H_1 contains two canonical subgroups of outermorphisms, namely the maximal compact subgroup $SU(2) \simeq \mathbb{T}$, which acts on H_1 by ${}^{\varphi}(z, u) := (e^{i\varphi}z, u)$, $0 \leqq \varphi < 2\pi$, and the group $\mathbb{R}^+$, which acts by dilations $D_r(z, u) := (rz, r^2u)$, $r > 0$. It is easily seen that Δ_K commutes with the action of $SU(2)$, and that

$$\Delta_K(f \circ D_r) = r^2(\Delta_K f) \circ D_r$$

for all $f \in C^2(H_1)$ and $r > 0$. This shows especially that, by a suitable dilation, the operator $\Delta_K - 2$ can be transformed into a multiple of the operator $\Delta_K - \mu$ for any $\mu > 0$. Therefore, it will be no restriction that in Sect. 2 we shall only consider the operator $\Delta_K - 2$ instead of $\Delta_K - \mu$.

(b) We shall frequently use the following standard notation from asymptotic analysis:

If X is a locally compact Hausdorff space, and if A is a subset of X, then we say that two complex functions f and g on X are *asymptotically equivalent* for x in A as x tends to infinity, if for any $\varepsilon > 0$ there exists a compact subset B_ε of X, such that $|f(x) - g(x)| \leqq \varepsilon \min(|f(x)|, |g(x)|)$ for every $x \in A \backslash K$. In this case we write

$$f(x) \sim g(x)$$

for $x \in A$ as $x \to \infty$. Note that if $f(x)$ and $g(x)$ are nonzero for sufficiently large x, then this just means that the quotient $f(x)/g(x)$ tends to 1 as $x \in A$ tends to infinity. In some statements, we shall pose some additional restrictions on the range of validity for an asymptotic equivalence by demanding that certain functions of x tend to certain limits as x tends to infinity.

Note that in case of the Heisenberg group $X = H_1$, there is a natural topology on H_1 which is defined by any homogeneous norm (in the sense of [5]) on H_1, for example by

$$|(z,u)| = (|z|^2 + |u|)^{1/2} \quad \text{or by} \quad \|(z,u)\| = (|z|^4 + u^2)^{1/4}.$$

This topology agrees with the Euclidean topology of the space $\mathbb{C} \times \mathbb{R}$. Let us mention here that the "norm" $\|\cdot\|$ is of special importance, because Folland's fundamental solution for Δ_K is just given by $-\dfrac{1}{4\pi}\|(z,u)\|^{-2}$.

1. Asymptotic Estimates for the Heat-Semigroup

Let us denote by $p_s(z,u)$ the positive fundamental solution of the heat operator $\dfrac{\partial}{\partial s} - \dfrac{1}{2}\Delta_K$ on $\mathbb{R} \times H_1$. It is well-known (cf. [6] or [11]) that p_s is given explicitly (for $s > 0$) by

$$p_s(z,u) = \frac{1}{2(2\pi s)^2} \int_{\mathbb{R}} \exp\left(i\frac{xu}{2s} - \frac{|z|^2}{2s} x \coth x\right) \frac{x}{\sinh x}\, dx,$$

and that $\{p_s\}_{s>0}$ is a probability semigroup on H_1.

If we set $p := p_1$, we have especially

$$p_s(z,u) = \frac{1}{s^2} p\left(\frac{z}{\sqrt{s}}, \frac{u}{s}\right). \tag{1.1}$$

We set $R := |z|^2$, and

$$h(R,u) := \int_{\mathbb{R}} e^{iux - Rx\coth x} \frac{x}{\sinh x}\, dx, \tag{1.2}$$

hence

$$p(z,u) = \frac{1}{2(2\pi)^2} h\left(\frac{R}{2}, \frac{u}{2}\right). \tag{1.3}$$

We shall first describe the asymptotic behaviour of $p(z,u)$ as $|z|^2 + |u|$ tends to infinity for the case, that the ratio $\omega = |u|/|z|^2$ stays bounded from above.

As in [6] we define a function

$$\theta :]-\pi, \pi[\to \mathbb{R} \tag{1.4}$$

by $\theta(y) = (2y - \sin 2y)/(2\sin^2 y)$. Obviously, θ is an odd function, and in [6] it has been shown that θ is a strictly increasing diffeomorphism. Let $\tau : \mathbb{R} \to]-\pi, \pi[$ be the inverse function of θ, i.e.

$$\tau = \theta^{-1}. \tag{1.5}$$

The following theorem follows by a somewhat technical, but straight-forward modification of the proof of Theorem 2 (3°) in [6] (and a correction by a factor 1/2):

Theorem 1.1. *If the ratio* $\omega = |u|/|z|^2$ *tends towards a limit* $\omega_\infty < \infty$, *then* $p(z,u)$ *is asymptotically given by*

$$p(z,u) \sim \Phi(\omega_\infty)\frac{1}{|z|}e^{-\gamma^2\left(\frac{|u|}{|z|^2}\right)\frac{|z|^2}{2}}$$

as $|z|^2 + |u| \to \infty$.

Here, γ *and* Φ *are given by*

$$\gamma(\omega) = \tau(\omega)/\sin\tau(\omega),$$

$$\Phi(\omega) = \frac{1}{2(2\pi)^{3/2}}|\tau(\omega)|\left[\frac{\sin\tau(\omega)}{\sin\tau(\omega) - \tau(\omega)\cos\tau(\omega)}\right]^{1/2}, \tag{1.6}$$

if $\omega \neq 0$, *and* $\gamma(0) = 1$, $\Phi(0) = \dfrac{3^{1/2}}{2(2\pi)^{3/2}}$.

Now we turn to the case where the ratio $|u|/|z|^2$ tends to infinity, or, equivalently, where

$$\delta := \sqrt{R/(\pi|u|)} \tag{1.7}$$

tends to zero.

Note, that we may assume $u > 0$, since $p(z,u) = p(z,-u)$, and that $\delta \to 0$ and $R + |u| \to \infty$ imply $|u| \to \infty$.

So let us assume that $u \to +\infty$, and $\delta \to 0$. In this case, the proof of Theorem 2 (3°) in [6] breaks down for the following reason: The proof is based on the stationary phase method. However, in case that $\delta \to 0$, the critical point in the complex plane of the phase $iuz - Rz\coth z$ of the integral (1.2) approaches the point $i\pi$, where this phase has a simple pole. But, as we shall see, it is possible to transform the integral (1.2), modulo an error term, into another one, to which the stationary phase method is again applicable.

In order to prepare the next theorem, let us introduce the following functions:

For complex $z \in \mathbb{C}$ with $|z| < 1$ let

$$r(z) := 1 + \frac{1}{z} - (z+1)\pi\cot\pi z, \tag{1.8}$$

if $z \neq 0$, and $r(0) = 0$. Since $\pi\cot\pi z$ has only one simple pole with residue 1 at $z = 0$ in the open disc $|z| < 1$, r is holomorphic in this unit disc. We can even describe the Taylor series of r around $z = 0$:

The function $\pi\cot\pi z$ has the well-known Laurent-expansion

$$\pi\cot\pi z = \frac{1}{z} - \sum_{n=1}^{\infty} d'_n z^{2n-1},$$

where

$$d'_n = \frac{(2\pi)^{2n}}{(2n)!}B_{2n} = 2\sum_{k=1}^{\infty}\frac{1}{k^{2n}},$$

B_j denoting the j-th Bernoulli number (cf. [2]). Especially one has

$$2 < d'_n < 4 \tag{1.9}$$

for all $n \in \mathbb{N}$. This implies

$$(z+1)\pi \cot \pi z = 1 + \frac{1}{z} - \sum_{n=1}^{\infty} d'_n z^{2n-1} - \sum_{n=1}^{\infty} d'_n z^{2n},$$

hence, for $|z| < 1$,

$$r(z) = \sum_{n=1}^{\infty} d_n z^n, \tag{1.10}$$

where

$d_n = d'_{[(n+1)/2]}$, $[x]$ denoting the integer part of x. Note that consequently (1.9) implies

$$|r(z)| \leqq 4|z|/(1-|z|), \tag{1.11}$$

if $|z| < 1$.

Next, for any real ε with $0 \leqq \varepsilon < 1/6$, define the holomorphic function q_ε on $|z| < 1$ by

$$q_\varepsilon(z) = \cosh z + \frac{\varepsilon}{2} r(-\varepsilon e^{-z}). \tag{1.12}$$

We shall sometimes also write $q(\varepsilon, z)$ instead of $q_\varepsilon(z)$.

Lemma 1.2. *Let* $0 \leqq \varepsilon < 1/6$.

(i) *The restriction of* q_ε *to the real numbers* $\mathbb{R}$ *is a real valued function.*

(ii) *There exists exactly one critical point* $\sigma = \sigma(\varepsilon)$ *of* q_ε *in* $\{|z| < 10\varepsilon^2\}$. *If* ε *is sufficiently small, then* $\sigma(\varepsilon)$ *is real, and the function* $\varepsilon \to \sigma(\varepsilon)$ *is smooth.*

Proof. (i) is clear by the definition of r. In order to prove (ii), we consider the derivative

$$q'_\varepsilon(z) = \sinh z + \frac{\varepsilon^2}{2} e^{-z} r'(-\varepsilon e^{-z}).$$

Now, by (1.10),

$$|r'(z)| \leqq \sum_{n=1}^{\infty} n d_n |z|^{n-1} \leqq 4 \sum_{n=1}^{\infty} n|z|^{n-1} = \frac{4}{(1-|z|)^2}.$$

This implies

$$|q'_\varepsilon(z) - \sinh z| \leqq 2\varepsilon^2 e^{|z|}/(1-\varepsilon e^{|z|})^2 \leqq 6\varepsilon^2,$$

if $|z| = 10\varepsilon^2$. On the other hand, one easily shows that $|\sinh z| \geqq 8\varepsilon^2$ for $|z| = 10\varepsilon^2$. Therefore

$$|q'_\varepsilon(z) - \sinh z| < |\sinh z| + |q'_\varepsilon(z)|$$

for $|z|=10\varepsilon^2$, and so Rouché's theorem implies that q'_ε has, similarly as sinh, exactly one zero in $|z|<10\varepsilon^2$.

The last statement of (ii) follows easily from the implicit function theorem, applied to the real function $(\varepsilon,t)\to q'_\varepsilon(t)$ near $(\varepsilon,t)=(0,0)$. □

Let finally I_0 denote the modified Bessel function

$$I_0(x)=\frac{1}{\pi}\int_0^\pi e^{-x\cos\varphi}d\varphi \quad (\text{cf. [4]}).$$

Theorem 1.3. (i) *If $\delta=\sqrt{|z|^2/(\pi|u|)}$ tends to zero, and if $|u|\,|z|^2\to+\infty$, then $p(z,u)$ is asymptotically given by*

$$p(z,u)\sim\frac{1}{4(2\pi)^{1/2}(\pi|u|\,|z|^2)^{1/4}}e^{-\frac{|z|^2}{2}-\frac{\pi}{2}|u|+\varrho(\delta)\sqrt{\pi|u|\,|z|^2}}$$

as $|z|^2+|u|\to\infty$, where (with σ as in Lemma 1.2)

$$\varrho(\delta)=q(\delta,\sigma(\delta)).$$

Moreover, $\varrho(\delta)=1+O(\delta^2)$.

(ii) *If $|z|^2/|u|$ tends to zero, and if $|u|\,|z|^2\leqq C$ for some positive constant C, then for $|z|^2+|u|\to\infty$*

$$p(z,u)\sim\tfrac{1}{4}I_0(\sqrt{\pi|u|\,|z|^2})e^{-\frac{|z|^2}{2}-\frac{\pi}{2}|u|}.$$

Remark. There is an explicit formula for p if $z=0$, namely

$$p(0,u)=\tfrac{1}{16}[\cosh\tfrac{\pi}{2}u]^{-2}$$

(cf. [6], Theorem 2 (2°), where a factor 2 had been omitted). The proof of this theorem is based on

Lemma 1.4. *If $0\leqq R<u$, then*

$$h(R,u)=\pi e^{-R-\pi u}\int_{-\pi}^{\pi}e^{2\sqrt{\pi Ru}\cos t+Rr(-\delta e^{it})}\varphi_\delta(t)dt+g(R,u),$$

where δ is given by $\delta=\sqrt{R/(\pi u)}$, and

$$\varphi_\delta(t)=\frac{\pi\delta e^{it}}{\sin(\pi\delta e^{it})}(1-\delta e^{it}),$$

and where g can be estimated by

$$|g(R,u)|\leqq\frac{40\pi}{1+\sqrt{R}}e^{-\frac{3\pi}{2}u}.$$

Proof. Let $f(z)=e^{iuz-Rz\coth z}\dfrac{z}{\sinh z}$.

f has exactly one (essential) singularity within the region $0<\operatorname{Im}z<3\pi/2$, namely at $z=\pi i$. Therefore, the theorem of residues easily implies

$$h(R,u)=\int_{\mathbb{R}}f(x)dx=\int_{\mathbb{R}}f\left(x+\frac{3\pi}{2}i\right)dx+2\pi i\operatorname{Res}_{z=\pi i}f.$$

Let $g(R,u)=\int_{\mathbb{R}} f\left(x+\frac{3\pi}{2}i\right)dx$. Since

$$\left|f\left(x+\frac{3\pi}{2}i\right)\right|=e^{-\frac{3\pi}{2}u}\left|\frac{x+3\pi i/2}{\cosh x}e^{iux-R\left(x+\frac{3\pi}{i}i\right)\tanh x}\right|$$

$$\leqq e^{-\frac{3\pi}{2}u}\frac{|x|+3\pi/2}{\cosh x}e^{-Rx\tanh x},$$

we have for $R\leqq 1$

$$|g(R,u)|\leqq e^{-\frac{3\pi}{2}u}\int_{\mathbb{R}}\frac{|x|+3\pi/2}{\cosh x}dx\leqq 4\pi e^{-\frac{3\pi}{2}u}.$$

And, if $R>1$, we get

$$|g(R,u)|\leqq 4\pi e^{-\frac{3\pi}{2}u}\int_{\mathbb{R}}e^{-Rx\tanh x}dx$$

$$\leqq 4\pi e^{-\frac{3\pi}{2}u}\left[\int_0^1 e^{-Rx^2/3}dx+\int_1^\infty e^{-Rx/2}dx\right],$$

since for $0<x<1$ one has

$$\frac{\sinh x}{\cosh x}\geqq 2\frac{x}{2e^x}\geqq\frac{x}{3},$$

and for $x>1$ one has $\tanh x>1/2$. This implies

$$|g(R,u)|\leqq 4\pi e^{-\frac{3\pi}{2}u}\left[\sqrt{\frac{3}{2R}}\pi+\frac{2}{3}\right]$$

if $R>1$. Combination of the estimates of $g(R,u)$ for the two cases $R\leqq 1$ and $R\geqq 1$ yields the desired estimate for $g(R,u)$.

Next, in order to calculate the residue of f at $z=\pi i$, we substitute z by $\pi iz+\pi i$ and get

$$\mathrm{Res}_{z=\pi i}f(z)=\pi i\,\mathrm{Res}_{z=0}f(\pi i(z+1))$$

$$=-\pi ie^{-\pi u}\mathrm{Res}_{z=0}\left[e^{-\pi uz-R(z+1)\pi\cot(\pi z)}\frac{\pi(z+1)}{\sin(\pi z)}\right].$$

Now, by the definition of the function r, we have

$$-\pi uz-R(z+1)\pi\cot\pi z=-R-\pi uz-\frac{R}{z}+Rr(z).$$

We intend to calculate the residue by integrating along a circle with center $z=0$. The radius δ of this circle has to be less then 1, since $(z+1)\cot(\pi z)$ has a pole at $z=1$, and, because of the estimate (1.11) for $r(z)$, it is clear that in the case $u\gg R$ only the terms $-\pi uz$ and $-\frac{R}{z}$ from above have a strong contribution to this integral. Therefore, we choose δ so, that for $|z|=\delta$ we have $|\pi uz|=|R/z|$. This implies

$$\delta=\sqrt{R/(\pi u)}.$$

Thus, if $R>0$, integration along the circle δe^{it}, $0\leqq t<2\pi$, yields

$$2\pi i \operatorname{Res}_{z=\pi i} f = -\pi i e^{-\pi u} \int_0^{2\pi} \exp\left[-R-2\sqrt{\pi R u}\cos t + Rr(\delta e^{it})\right] \times \left[i\frac{\pi\delta e^{it}}{\sin(\pi\delta e^{it})}(1+\delta e^{it})\right]dt.$$

Substituting t by $t-\pi$ yields the desired formula for $h(R,u)-g(R,u)$.

Finally, the case $R=0$ follows from the case $R>0$ by continuity. □

Proof of Theorem 1.3. ad *(i)*: We may assume $u>0$, and set $R=|z|^2$. We set $\kappa=2\sqrt{\pi R u}$, so that $\kappa\to+\infty$. Let

$$H(R,u)=\int_{-\pi}^{\pi} e^{2\sqrt{\pi R u}\cos t + Rr(-\delta e^{it})}\varphi_\delta(t)dt,$$

so that by Lemma 1.4

$$h(R,u)=\pi e^{-R-\pi u}H(R,u)+g(R,u).$$

For $|z|<1$ and $0\leqq\varepsilon<1/6$ we set

$$\tilde q_\varepsilon(z)=\cos z+\frac{\varepsilon}{2}r(-\varepsilon e^{iz}).$$

Then, for $\varepsilon=\delta:=\sqrt{R/(\pi u)}$, we have

$$2\sqrt{\pi R u}\cos t + Rr(-\delta e^{it})=\kappa\tilde q_\delta(t),$$

hence

$$H(R,u)=\int_{-\pi}^{\pi} e^{\kappa\tilde q_\delta(t)}\varphi_\delta(t)dt.$$

Now, we have

$$\tilde q_\varepsilon(z)=q_\varepsilon(-iz),$$

and so Lemma 1.2 implies that $\tilde q_\varepsilon$ has a unique critical point in $\{|z|<10\varepsilon^2\}$, namely the point $i\sigma(\varepsilon)$. Moreover, if δ is small enough, the function $z\to e^{\kappa\tilde q_\delta(z)}\varphi_\delta(z)$ is holomorphic in the strip $|\operatorname{Im} z|<20\delta^2$. Therefore, we choose a path $\gamma:[-\pi,\pi]\to\mathbb{C}$ such that $\gamma(-\pi)=-\pi$, $\gamma(\pi)=\pi$, $|\operatorname{Im}\gamma(t)|\leqq 19\delta^2$ for all $t\in[-\pi,\pi]$, and

$$\gamma(t)=t+i\sigma(\delta)$$

for $t\in[-\frac{\pi}{2},\frac{\pi}{2}]$, and clearly have

$$H(R,u)=\int_{-\pi}^{\pi} e^{\kappa(\tilde q_\delta\circ\gamma)(t)}\varphi_\delta\circ\gamma(t)\gamma'(t)dt.$$

If we set $F_\varrho(t)=\tilde q_\delta\circ\gamma(t)-\tilde q_\delta\circ\gamma(0)$, we obtain

$$\begin{aligned}H(R,u)&=e^{\kappa q_\delta(\sigma(\delta))}\int_{-\pi}^{\pi} e^{\kappa F_\delta(t)}\varphi_\delta\circ\gamma(t)\gamma'(t)dt\\&=\frac{e^{\kappa\varrho(\delta)}}{\sqrt{\kappa}}\int_{-\pi\sqrt{\kappa}}^{\pi\sqrt{\kappa}} e^{\kappa F_\delta(s/\sqrt{\kappa})}\psi_\delta(s/\sqrt{\kappa})ds,\end{aligned}$$

where
$$\psi_\delta(t)=\varphi_\delta\circ\gamma(t)\gamma'(t).$$

One verifies easily that
$$F_\delta(0)=F'_\delta(0)=0,$$
$$F''_\delta(0)=-\cos(i\sigma(\delta))+O(\delta^2),$$
and
$$\left|F_\delta(t)-F''_\delta(0)\frac{t^2}{2}\right|\leqq Ct^3,$$
where C does not depend on δ.

Together, this implies
$$\lim_{\substack{\kappa\to\infty\\ \delta\to 0}}\kappa F_\delta(s/\sqrt{\kappa})=-s^2/2$$
for every $s\in\mathbb{R}$. Also, since $\gamma'(0)=1$, we have
$$\lim_{\substack{\kappa\to\infty\\ \delta\to 0}}\psi_\delta(s/\sqrt{\kappa})=1.$$

Moreover, it is not difficult to show that, for δ sufficiently small, one has
$$|e^{\kappa F_\delta(s/\sqrt{\kappa})}\psi_\delta(s/\sqrt{\kappa})|\leqq Ce^{-s^2/10}$$
for $|s|\leqq\pi\sqrt{\kappa}$, uniformly in δ and κ. So, the dominated convergence theorem implies
$$\lim_{\substack{\kappa\to\infty\\ \delta\to 0}}\int_{-\pi\sqrt{\kappa}}^{\pi\sqrt{\kappa}} e^{\kappa F_\delta(s/\sqrt{\kappa})}\psi_\delta(s/\sqrt{\kappa})ds=\int_{-\infty}^{\infty}e^{-s^2/2}ds=\sqrt{2\pi},$$
hence
$$H(R,u)\sim(2\pi)^{1/2}\frac{e^{\kappa\varrho(\delta)}}{\sqrt{\kappa}}$$
as $\kappa\to\infty$ and $\delta\to 0$. This implies
$$h(R,u)\sim\frac{\pi(2\pi)^{1/2}}{2^{1/2}(\pi Ru)^{1/4}}e^{-R-\pi u+2\sqrt{\pi Ru}\varrho(\delta)}+g(R,u)$$
as $\kappa\to\infty$ and $\delta\to 0$.

Moreover, (1.12) and Lemma 1.2 imply
$$\varrho(\delta)=q(\delta,\sigma(\delta))=1+O(\delta^2).$$
This implies
$$-R-\pi u+2\sqrt{\pi Ru}\,\varrho(\delta)=-\pi[1+\delta^2-2\delta\varrho(\delta)]u=-\pi(1+O(\delta))u.$$

Since $u\to\infty$ and $\frac{R}{u}\to 0$, this and the estimate of $g(R,u)$ in Lemma 1.4 show that $g(R,u)$ is negligible for the asymptotics of $h(R,u)$, and by (1.3), we finally get
$$p(z,u)\sim\frac{\pi(2\pi)^{1/2}}{2(2\pi)^2(\pi Ru)^{1/4}}e^{-\frac{R}{2}-\frac{\pi}{2}u+\sqrt{\pi Ru}\varrho(\delta)},$$
if $u>0$ and $R=|z|^2$.

ad *(ii)*: We adopt the notation of the proof of (i), and write

$$H(R,u)=\int_{-\pi}^{\pi} e^{2\sqrt{\pi Ru}\cos t}\psi(t)dt,$$

where

$$\psi(t)=e^{Rr(-\delta e^{it})}\varphi_\delta(t).$$

For $\delta<1/2$, estimate (1.11) implies

$$|Rr(-\delta e^{it})|\leqq 8R\delta=\frac{8}{\sqrt{\pi}}\left(\frac{R^3}{u}\right)^{1/2}\leqq\frac{8C^{3/2}}{\sqrt{\pi}}\frac{1}{u^2},$$

since $Ru\leqq C$. And, if $R/u\to 0$ and $R+u\to\infty$, then $u\to\infty$, and so $Rr(-\delta e^{it})$ tends to zero uniformly. Moreover, since $\delta\to 0$, φ_δ converges uniformly to the constant function 1 on $[-\pi,\pi]$. Consequently, $\psi\to 1$ uniformly on $[-\pi,\pi]$ as $R+u\to\infty$ and $R/u\to 0$.

But, since $p(z,u)$ and consequently $h(R,u)$ is positive, we have

$$h(R,u)=\pi e^{-R-\pi u}\int_{-\pi}^{\pi} e^{2\sqrt{\pi Ru}\cos t}\,\mathrm{Re}\,\psi(t)dt+\mathrm{Re}\,g(R,u),$$

where also $\mathrm{Re}\,\psi\to 1$ uniformly on $[-\pi,\pi]$.

But, since $e^{2\sqrt{\pi Ru}\cos t}$ is positive, this clearly implies

$$h(R,u)\sim\pi e^{-R-\pi u}\int_{-\pi}^{\pi} e^{2\sqrt{\pi Ru}\cos t}dt+\mathrm{Re}\,g(R,u)$$

as $R+u\to\infty$, $R/u\to 0$, and $Ru\leqq C$.

Moreover

$$\int_{-\pi}^{\pi} e^{2\sqrt{\pi Ru}\cos t}dt>\int_{-\pi/2}^{\pi/2} e^0 dt=\pi,$$

and so the estimate of $g(R,u)$ in Lemma 1.4 shows that the term $\mathrm{Re}\,g(R,u)$ is negligible for the asymptotic behaviour of $h(R,u)$. Thus in combination with (1.3), we get

$$p(z,u)\sim\frac{1}{8\pi}e^{-\frac{R}{2}-\frac{\pi}{2}u}\int_{-\pi}^{\pi} e^{\sqrt{\pi Ru}\cos t}dt.$$

Since

$$\int_{-\pi}^{0} e^{x\cos t}dt=\int_{0}^{\pi} e^{x\cos t}dt,$$

hence

$$\int_{-\pi}^{\pi} e^{x\cos t}dt=2\pi I_0(-x)=2\pi I_0(x),$$

we obtain

$$p(z,u)\sim\tfrac{1}{4}I_0(\sqrt{\pi Ru})e^{-\frac{R}{2}-\frac{\pi}{2}u},\quad\text{q.e.d.}\quad\square$$

Remark 1.5. The proof of Theorem 1.3 (i) shows that the exponent in the asymptotic formula for $p(z,u)$ is just the value of the phase $\zeta \to iu\zeta - R\eta \coth \zeta$ of the integral (1.2) at the (unique) critical value $\zeta_0 \in \mathbb{C}$ of this phase. The same is true for the exponent for the asymptotics of $p(z,u)$ in Theorem 1.1 (compare the proof of Theorem 2 (3°) in [6]). Therefore we have

$$-\frac{|z|^2}{2} - \frac{\pi}{2}|u| + \varrho(\delta)\sqrt{\pi|u|\,|z|^2} = -\gamma^2\left(\frac{|u|}{|z|^2}\right)\frac{|z|^2}{2}. \tag{1.13}$$

Moreover, a careful analysis of the function $\phi(\tau)$ near $\tau = \pi$ and of the function $\tau(\omega)$ for $\omega \to \pm\infty$ shows that, with $\omega = |u|/|z|^2$,

$$\frac{\phi(\tau(\omega))}{|z|} \sim \frac{1}{4(2\pi)^{1/2}(\pi|u|\,|z|^2)^{1/4}}$$

as

$$\delta = \sqrt{|z|^2/(\pi|u|)} \to 0.$$

Thus, in the case of Theorem 1.3 (i) we have the same asymptotics

$$p(z,u) \sim \frac{\phi(\omega)}{|z|} e^{-\gamma^2\left(\frac{|u|}{|z|^2}\right)\frac{|z|^2}{2}} \tag{1.13}$$

as in the case of Theorem 1.1.

However, the right-hand side of (1.13) does no longer describe the asymptotics of p in the case of Theorem 1.3 (ii), and moreover the formula of Theorem 1.3 (i) is more informative than (1.13).

2. Asymptotic Estimates for the Fundamental Solution of $-\Delta_K + 2$

For $(z,u) \in H_1 \setminus \{0\}$ let

$$K(z,u) := \tfrac{1}{2}\int_0^\infty p_s(z,u)e^{-s}ds. \tag{2.1}$$

Since p is a Schwartz-class function (see e.g. [5]), it is clear that this integral converges for every $(z,u) \neq 0$. In fact, the same observations in combination with the homogeneity of p_s as expressed by (1.3) even imply that K vanishes at infinity.

Obviously K is positive and integrable, since $\int_{H_1} p_s(z,u)dzdu = 1$. K is a fundamental solution for $-\Delta_K + 2$. This is indicated by the following formal calculation, which can easily be made precise in the sense of distributions:

$$\Delta_K K = \frac{1}{2}\int_0^\infty (\Delta_K p_s)e^{-s}ds = \int_0^\infty \frac{\partial p_s}{\partial s}e^{-s}ds$$

$$= p_s e^{-s}\Big|_0^\infty + \int_0^\infty p_s e^{-s}ds;$$

but $p_0 = \delta_0$ is the Dirac distribution at the origin, and so

$$\Delta_K K = -\delta_0 + 2K,$$

hence

$$(-\Delta_K+2)K=\delta_0\,. \tag{2.2}$$

Of course, all this is well-known.

Moreover, by a result of Hervé ([9], p. 142), K is in fact the unique positive fundamental solution of $-\Delta_K+2$ which vanishes at infinity.

We are now going to describe the asymptotic behaviour of $K(z,u)$, and again we start with the case where $\omega=|u|/|z|^2$ stays bounded. We adopt the notation of the preceding paragraph.

Theorem 2.1. *If the ratio $\omega=|u|/|z|^2$ tends towards a limit $\omega_\infty<\infty$, then $K(z,u)$ is asymptotically given by*

$$K(z,u)\sim \Psi(\omega_\infty)\frac{1}{|z|^2}e^{-\sqrt{2}\gamma\left(\frac{|u|}{|z|^2}\right)|z|}$$

as $|z|^2+|u|\to\infty$, where

$$\Psi(\omega)=\frac{1}{8\pi}\left[\frac{\sin^3\tau(\omega)}{\sin\tau(\omega)-\tau(\omega)\cos\tau(\omega)}\right]^{1/2}.$$

Proof. Assume that $\lim|u|/|z|^2=\omega_\infty<\infty$ as $|z|^2+|u|\to\infty$. Then, by Theorem 1.1, for every $\varepsilon>0$, there exist $N(\varepsilon)>0$ and $\delta(\varepsilon)>0$, such that

$$\left|1-\frac{p(z,u)}{\tilde p(z,u)}\right|<\varepsilon, \tag{2.3}$$

if $|z|^2+|u|>N(\varepsilon)$ and $\left|\frac{u}{|z|^2}-\omega_\infty\right|<\delta(\varepsilon)$; here $\tilde p(z,u)$ denotes the function

$$\tilde p(z,u)=\Phi(\omega_\infty)\frac{1}{|z|}e^{-\gamma^2\left(\frac{|u|}{|z|^2}\right)\frac{|z|^2}{2}}. \tag{2.4}$$

Now, fix $\varepsilon>0$, and assume that

$$|z|^{2/3}>N(\varepsilon)+1\,,\qquad \left|\frac{|u|}{|z|^2}-\omega_\infty\right|<\delta(\varepsilon)\,.$$

Then (2.3) implies, since p and $\tilde p$ are positive,

$$(1-\varepsilon)\tilde p\left(\frac{z}{\sqrt{s}},\frac{u}{s}\right)\leqq p\left(\frac{z}{\sqrt{s}},\frac{u}{s}\right)\leqq(1+\varepsilon)\tilde p\left(\frac{z}{\sqrt{s}},\frac{u}{s}\right)$$

if $s<|z|^{4/3}$, for in this case

$$\left|\frac{z}{\sqrt{s}}\right|^2+\left|\frac{u}{s}\right|\geqq\frac{|z|^2}{s}=\frac{|z|^{4/3}}{s}\cdot|z|^{2/3}>|z|^{2/3}>N(\varepsilon)\,;$$

and of course $\left|\frac{|u/s|}{|z/\sqrt{s}|^2}-\omega_\infty\right|=\left|\frac{|u|}{|z|^2}-\omega_\infty\right|<\delta(\varepsilon).$

Set

$$K_I(z,u)=\tfrac{1}{2}\int_0^{|z|^{4/3}} p_s(z,u)e^{-s}ds,$$

$$K_{II}(z,u)=\tfrac{1}{2}\int_{|z|^{4/3}}^{\infty} p_s(z,u)e^{-s}ds,$$

and form $\tilde{K}_I$ and $\tilde{K}_{II}$ analogously by replacing p by $\tilde{p}$. Then clearly

$$(1-\varepsilon)\tilde{K}_I(z,u)\leqq K_I(z,u)\leqq(1+\varepsilon)\tilde{K}_I(z,u). \tag{2.5}$$

Moreover, if we set $C_1=\sup\{|p(z,u)|:(z,u)\in H_1\}$, then

$$\begin{aligned}2K_{II}(z,u)&=\int_{|z|^{4/3}}^{\infty} p\left(\frac{z}{\sqrt{s}},\frac{u}{s}\right)\frac{e^{-s}}{s^2}ds\\ &\leqq C_1\int_{|z|^{4/3}}^{\infty} e^{-s}ds\\ &=C_1e^{-|z|^{4/3}}.\end{aligned}$$

Similarly,

$$\begin{aligned}2\tilde{K}_{II}(z,u)&=\frac{\Phi(\omega_\infty)}{|z|}\int_{|z|^{4/3}}^{\infty} s^{-3/2}e^{-\gamma^2\left(\frac{|u|}{|z|^2}\right)\frac{|z|^2}{2s}-s}ds\\ &\leqq\frac{\Phi(\omega_\infty)}{|z|}\int_{|z|^{4/3}}^{\infty} e^{-s}ds\\ &\leqq\Phi(\omega_\infty)e^{-|z|^{4/3}}.\end{aligned}$$

Thus, with $2C_2=\max(C_1,\Phi(\omega_\infty))$, we have

$$K_{II}(z,u)\leqq C_2e^{-|z|^{4/3}},\qquad \tilde{K}_{II}(z,u)\leqq C_2e^{-|z|^{4/3}}. \tag{2.6}$$

Next, we can calculate explicitly $\tilde{K}=\tilde{K}_I+\tilde{K}_{II}$: From ([4], p. 82 (23) and p. 10 (42)), it follows that

$$\int_0^{\infty} s^{-3/2}e^{-\frac{a}{s}-s}ds=\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{a}},$$

if $a>0$. Now

$$\tilde{K}(z,u)=\frac{\Phi(\omega_\infty)}{2|z|}\int_0^{\infty} s^{-3/2}e^{-\gamma^2\left(\frac{|u|}{|z|^2}\right)\frac{|z|^2}{2s}-s}ds,$$

hence

$$\tilde{K}(z,u)=\Psi(\omega_\infty)\frac{1}{|z|^2}\,\frac{\gamma(\omega_\infty)}{\gamma\left(\frac{|u|}{|z|^2}\right)}\,e^{-\sqrt{2}\gamma\left(\frac{|u|}{|z|^2}\right)|z|}. \tag{2.7}$$

The estimates in (2.6) and (2.7) show, that there exists an $N'(\varepsilon)>N(\varepsilon)+1$, such that

$$K_{II}(z,u)/\tilde{K}(z,u)<\varepsilon,\qquad \tilde{K}_{II}(z,u)/\tilde{K}(z,u)<\varepsilon,$$

if $|z|^{2/3} > N'(\varepsilon)$. But this, together with (2.5), easily implies

$$(1-2\varepsilon)\tilde{K}(z,u) \leqq K(z,u) \leqq (1+2\varepsilon)\tilde{K}(z,u),$$

if $|z|^{2/3} > N'(\varepsilon)$ and $\left|\frac{|u|}{|z|^2} - \omega_\infty\right| < \delta(\varepsilon)$.

This clearly implies the theorem. $\square$

Finally, we deal with the case $|z|^2/|u| \to 0$:

Theorem 2.2. (i) *If* $\delta = \sqrt{|z|^2/(\pi|u|)}$ *tends to zero, and if* $|z| \to +\infty$, *then* $K(z,u)$ *is asymptotically given by*

$$K(z,u) \sim \tfrac{1}{8} \frac{e^{-[2(\pi|u| + |z|^2 - 2\varrho(\delta)\sqrt{\pi|u|\,|z|^2})]^{1/2}}}{(\pi|u|)^{3/4}|z|^{1/2}}$$

as $|z|^2 + |u| \to +\infty$, *where* ϱ *is defined as in Theorem 1.3.*

(ii) *If* $\delta = \sqrt{|z|^2/(\pi|u|)}$ *tends to zero, and if there exists a constant* $C > 0$, *such that* $|z| \leqq C$, *then*

$$K(z,u) \sim \frac{2^{3/4}\sqrt{\pi}}{8} I_0(\sqrt{2}|z|) \frac{e^{-\sqrt{2\pi|u|}}}{(\pi|u|)^{3/4}}$$

as $|z|^2 + |u| \to \infty$.

Proof. We may again assume that $u > 0$.

We set

$$F(R,u) = \int_0^\infty \frac{1}{s^2}(h-g)\left(\frac{R}{s}, \frac{u}{s}\right) e^{-s} ds,$$

$$G(R,u) = \int_0^\infty \frac{1}{s^2} g\left(\frac{R}{s}, \frac{u}{s}\right) e^{-s} ds,$$

where h and g are defined as in Lemma 1.4. Then (1.2) implies with $R = |z|^2$, that

$$K(z,u) = \frac{1}{4(2\pi)^2}\left[F\left(\frac{R}{2}, \frac{u}{2}\right) + G\left(\frac{R}{2}, \frac{u}{2}\right)\right]. \tag{2.8}$$

We first estimate $G(R,u)$:

Lemma 1.4 implies

$$|G(R,u)| \leqq 40\pi \int_0^\infty \frac{1}{1 + \left(\frac{R}{s}\right)^{1/2}} \frac{1}{s^2} e^{-\frac{3\pi}{2}\frac{u}{s} - s} ds$$

$$\leqq 40\pi \int_0^\infty \frac{1}{s^2} e^{-\frac{3\pi}{2}\frac{u}{s} - s} ds.$$

Moreover, by ([4], p. 82 (23)),

$$\int_0^\infty \frac{1}{s^2} e^{-\frac{a}{s} - s} ds = \frac{2}{\sqrt{a}} K_1(2\sqrt{a}), \quad \text{if} \quad \operatorname{Re} a > 0,$$

where K_1 denotes the modified Hankel function of order 1, and ([4], p. 86 (7)) implies

$$K_1(z)=\sqrt{\frac{\pi}{2z}}\,e^{-z}[1+O(|z|^{-1})]\,,\quad \text{if} \quad \operatorname{Re} z>0\,,$$

which together yields

$$\int_0^\infty \frac{1}{s^2}\,e^{-\frac{a}{s}-s}\,ds=\frac{\pi^{1/2}}{a^{3/4}}\,e^{-2\sqrt{a}}[1+O(a^{-1/2})]\,, \tag{2.9}$$

if $\operatorname{Re} a>0$.

Thus there exists $C>0$ such that for $|u|>C$

$$|G(R,u)|\leqq 80\pi^{3/2}\,\frac{e^{-\sqrt{6\pi u}}}{|u|^{3/4}}\,. \tag{2.10}$$

Next, we are going to determine the asymptotics for $F(R,u)$: By Fubini's theorem, we have

$$F(R,u)=\pi\int_{-\pi}^{\pi}\int_0^\infty \frac{1}{s^2}\,e^{-\frac{a(t)}{s}-s}\,ds\,\varphi_\delta(t)\,dt$$

where

$$a(t)=\pi u+R-2\sqrt{\pi R u}\,\cos t-Rr(-\delta e^{it})\,. \tag{2.11}$$

Since we may assume $u\gg R$, we have $\operatorname{Re} a(t)>0$ for every $t\in[-\pi,\pi]$, and thus (2.9) implies

$$F(R,u)=\pi^{3/2}\int_{-\pi}^{\pi}\frac{e^{-2\sqrt{a(t)}}}{[a(t)]^{3/4}}\,[1+\xi(t)]\varphi_\delta(t)\,dt\,, \tag{2.12}$$

where ξ is smooth, depends also on u and R, but where the supremum norm $\|\xi\|_\infty$ of ξ on $[-\pi,\pi]$ satisfies

$$\|\xi\|_\infty=O(|u|^{-1/2})\,. \tag{2.12}$$

For the sequel, it is also important to note, that for the asymptotics considered in Theorem 2.2 we have, uniformly on $[-\pi,\pi]$,

$$a(t)\sim\pi u\,. \tag{2.13}$$

In order to prove (i), let us now assume $R\to+\infty$: The complex critical points of the phase $-2\sqrt{a(z)}$ are given by $a'(z)=0$, or, equivalently, by $\tilde{q}_\delta'(z)=0$, where $\tilde{q}_\delta$ is defined as in the proof of Theorem 1.3 (i). Thus, again the only critical point of this phase in $\{|z|<10\delta^2\}$ is the point $i\sigma(\delta)$. So, introducing a path γ as in the proof of Theorem 1.3 (i) and arguing like there, we obtain

$$F(R,u)=\pi^{3/2}\,\frac{e^{-2\sqrt{a(i\sigma(\delta))}}}{(\pi u)^{3/4}}\int_{-\pi}^{\pi} e^{\sqrt{R}b(t)}\psi(t)\left[\frac{\pi u}{a(t)}\right]^{3/4}dt\,,$$

where

$$b(t)=[2\sqrt{a(i\sigma(\delta))}-2\sqrt{a\circ\gamma(t)}]/\sqrt{R}\,,$$

$$\psi(t)=[1+\xi\circ\gamma(t)]\varphi_\delta\circ\gamma(t)\gamma'(t)$$

depend also on R and u.

Now, clearly $b(0)=b'(0)=0$, and

$$b''(0)=-\frac{a''(i\sigma(\delta))}{[a(i\sigma(\delta))]^{1/2}\sqrt{R}}$$
$$=\frac{-2\cos(i\sigma)+\delta^2e^{-\sigma}r'(-\delta e^{-\sigma})-\delta^3e^{-2\sigma}r''(-\delta e^{-\sigma})}{[1+\delta^2-2\delta\cosh\sigma-\delta^2r(-\delta e^{-\sigma})]^{1/2}},$$

where we set $\sigma=\sigma(\delta)$. Since $|\sigma|\leqq 10\delta^2$, this implies, for $\delta\to 0$

$$b''(0)\to -2. \tag{2.14}$$

Similarly, but with a bit more technical effort, one can show that b'' is uniformly bounded on $[-\pi,\pi]$, independently of u and R, if δ is sufficiently small and u sufficiently large. In fact, for $1/\delta$ and u sufficiently large, one even has

$$|b'''(t)|\leqq 4|\sin t|, \qquad t\in[-\tfrac{\pi}{2},\tfrac{\pi}{2}].$$

So we can apply similar arguments as in the proof of Theorem 1.3 to show that, if also $R\to\infty$,

$$R^{1/4}\int_{-\pi}^{\pi} e^{\sqrt{R}b(t)}\psi(t)\left[\frac{\pi u}{a(t)}\right]^{3/4}dt=\int_{-\pi R^{1/4}}^{\pi R^{1/4}} e^{\sqrt{R}b(s/R^{1/4})}\psi(s/R^{1/4})\left[\frac{\pi u}{a(s/R^{1/4})}\right]^{3/4}ds$$

tends towards $\int_{-\infty}^{\infty}\exp(-s^2)ds=\sqrt{\pi}$.

Here, one should also note that, as $u\to+\infty$, $1+\xi(t)$ tends towards 1 uniformly on $[-\pi,\pi]$ because of (2.12), and that $\pi u/a(t)$ tends towards 1 uniformly on $[-\pi,\pi]$ because of (2.13). Thus we have shown that

$$F(R,u)\sim\pi^2e^{-2\sqrt{a(i\sigma(\delta))}}/[(\pi u)^{3/4}R^{1/4}] \tag{2.15}$$

as $u\to+\infty$, $\delta\to 0$ and $R\to+\infty$.

But, since

$$a(i\sigma(\delta))=\pi u+R-\varrho(\delta)2\sqrt{\pi Ru},$$

this and (2.10) show that $G(R,u)$ is negligible for the asymptotics of $K(R,u)$, and so (2.8) and (2.15) imply (i).

It remains to prove (ii). So, assume that $|z|$ is bounded by some constant $C>0$. This case is similar to case (ii) of Theorem 1.3, and therefore we abbreviate the argument. Consider formula (2.12). Since

$$|\sqrt{a(t)}-(\pi u+R-2\sqrt{\pi Ru}\cos t)^{1/2}|=O\left(\frac{R|r(-\delta e^{it})|}{\sqrt{\pi u}}\right)=O\left(\frac{R^{3/2}}{\sqrt{u}}\right)=O\left(\frac{1}{\sqrt{u}}\right)$$

because of (1.11), and since $\xi(t)\to 0$ and $\varphi_\delta(t)\to 1$ uniformly on $[-\pi,\pi]$ as $u\to+\infty$ and $\delta\to 0$, we easily see that

$$F(R,u)\sim\pi^{3/2}\int_{-\pi}^{\pi}\frac{e^{-2[\pi u+R-2\sqrt{\pi Ru}\cos t]^{1/2}}}{(\pi u)^{3/4}}dt,$$

where once again we have also made use of (2.13) and the positivity of K. Moreover,

$$[\pi u+R-2\sqrt{\pi R u}\cos t]^{1/2}=(\pi u)^{1/2}[1+\delta^2-2\delta\cos t]^{1/2}$$
$$=(\pi u)^{1/2}[1-\delta\cos t+O(\delta^2)],$$

and since

$$(\pi u)^{1/2}\cdot\delta^2=\frac{R}{\sqrt{\pi u}}\leqq\frac{C^2}{\sqrt{\pi u}}\to 0,$$

the term $O(\delta^2)$ is negligible.

Thus we have

$$F(R,u)\sim\frac{\pi^{3/2}}{(\pi u)^{3/4}}e^{-2\sqrt{\pi u}}\int_{-\pi}^{\pi}e^{2\sqrt{R}\cos t}dt,$$

hence

$$F(R,u)\sim\frac{2\pi^{5/2}}{(\pi u)^{3/4}}I_0(2\sqrt{R})e^{-2\sqrt{\pi u}}.$$

Again, this shows that $G(R,u)$ may be neglected, and together with (2.8) this implies (ii). □

Remark 2.3. Formula (1.8) shows that the function $-a(t)$ is just the phase function $\zeta\to iu\zeta-R\zeta\cot\zeta$ of (1.2) composed with the function $t\to\pi i(1+\delta e^{i(t-\pi)})$. Therefore, by the proof of Theorem 2.2 (i), it is clear that $-a(i\sigma(\delta))$ is nothing but the value of this phase function at it's critical point, that is $-\gamma^2\left(\frac{|u|}{|z|^2}\right)\frac{|z|^2}{2}$ (compare Remark 1.5). Therefore we have

$$-[2(\pi|u|+|z|^2-2\varrho(\delta)\sqrt{\pi|u|\,|z|^2})]^{1/2}=-\sqrt{2}\gamma\left(\frac{|u|}{|z|^2}\right)|z|. \tag{2.16}$$

Moreover, similarly as in Remark 1.5, one can show that, with $\omega=|u|/|z|^2$,

$$\frac{\psi(\omega)}{|z|^2}\sim\frac{1}{8(\pi|u|)^{3/4}|z|^{1/2}}$$

as $\delta\to 0$, and thus we also have

$$K(z,u)\sim\frac{\psi(\omega)}{|z|^2}e^{-\sqrt{2}\gamma\left(\frac{|u|}{|z|^2}\right)|z|} \tag{2.16}$$

in the case of Theorem 2.2 (i). But again the analog is no longer true in case of Theorem 2.2 (ii).

3. The Martin Boundary

We are now going to determine the Martin compactification of H_1. As references to the notions and results from potential theory which we shall use, we recommend the classical book of Helms [8], where the theory is developped for the Laplacian on $\mathbb{R}^n$ in such a way that it could easily be extended to our situation, and Brelot's book [1]. The latter book presents a more abstract potential theory which is already sufficiently general to cover our situation.

By definition, the *Martin compactification* $M(H_1)$ of H_1 corresponding to the operator $-\Delta_K+2$ is the unique (up to homeomorphisms) compactification of H_1, such that each of the functions

$$F^y(x):=\frac{K(y^{-1}x)}{K(x)}, \quad y\in H_1,$$

which may be considered as a continuous function from H_1 to $[0,\infty]$, can be extended continuously to $M(H_1)$, and such that the set of all those extended functions separates the points of $\Delta:=M(H_1)\backslash H_1$. The set Δ is called the *Martin boundary* of H_1.

Note, that if the extension of F^y to $M(H_1)$ is again denoted by F^y, then the function $F(x,y):=F^y(x)$ is continuous with values in $]0,\infty[$ on $\Delta\times H_1$; moreover, for any fixed $x\in\Delta$, the function $F_x(y):=F(x,y)$ is $(-\Delta_K+2)$-harmonic and positive.

Next, let Δ_1 denote the set of all $x\in\Delta$ such that the function F_x lies on an extremal ray of the cone Γ of all positive $(-\Delta_K+2)$-harmonic functions on H_1. Δ_1 is called the *minimal Martin boundary*. Then the representation theorem of Martin/Choquet ([1] Theorem XIV, 4) states that every $(-\Delta_K+2)$-harmonic function $h\geqq 0$ admits a representation

$$h(y)=\int_\Delta F_x(y)d\mu(x), \tag{3.1}$$

where μ is a positive measure on Δ with $\mu(\Delta\backslash\Delta_1)=0$, and μ is uniquely determined by the properties. Note that (3.1) implies that every extremal ray of Γ is of the form $\mathbb{R}^+F_x$ for some $x\in\Delta_1$.

Now we are going to describe a compactification M of H_1 which will turn out to be the Martin compactification.

As a set, let M be the disjoint union of H_1, the complex plane $\mathbb{C}$ and the unit circle S^1 in $\mathbb{C}$, that is $M=H_1\dot\cup\mathbb{C}\dot\cup S^1$. The topology on M is defined as follows:

The neighborhoods of a point in H_1 are just the neighborhoods of the Euclidean topology on H_1.

If ζ_0 is a point in $\mathbb{C}$, then a basis for the system of neighborhoods of ζ_0 is given by the sets

$$U_\varepsilon(\zeta_0)=\left\{(z,u)\in H_1:|z-\zeta_0|<\varepsilon,|u|>\frac{1}{\varepsilon}\right\}\cup\{\zeta\in\mathbb{C}:|\zeta-\zeta_0|<\varepsilon\}\ , \quad \varepsilon>0.$$

If $e^{i\varphi_0}$ is a point in S^1, then a basis of neighborhoods of $e^{i\varphi_0}$ is given by the sets

$$\begin{aligned}V_\varepsilon(e^{i\varphi_0})=&\left\{(re^{i\varphi},u)\in H_1:r>\frac{1}{\varepsilon},\left|e^{i\left[\tau\left(\frac{u}{r^2}\right)+\varphi\right]}-e^{i\varphi_0}\right|<\varepsilon\right\}\\ &\cup\left\{\varrho e^{i\psi}\in\mathbb{C}:\varrho>\frac{1}{\varepsilon},|e^{i\psi}+e^{i\varphi_0}|<\varepsilon\right\}\\ &\cup\{e^{i\varphi}\in S^1:|e^{i\varphi}-e^{i\varphi_0}|<\varepsilon\}\ , \quad \varepsilon>0.\end{aligned}$$

Here τ denotes the diffeomorphism $\tau:\mathbb{R}\to]-\pi,\pi[$ from Sect. 1. Note that

$$\lim_{\kappa\to\pi}\tau^{-1}(\kappa)=+\infty\,,\qquad \lim_{\kappa\to-\pi}\tau^{-1}(\kappa)=-\infty\,.$$

It is easy to see that these properties of τ imply that M thus becomes a compact Hausdorff space, and that M is a compactification of H_1. The following lemma shows that M is even much nicer and more symmetric than it might appear from the very definition:

Lemma 3.1. *Let $\tilde{M}'$ denote the "solid torus" $\tilde{M}'=D\times S^1$, where $D=\{z\in\mathbb{C}:|z|\leqq 1\}$ denotes the closed unit disk in $\mathbb{C}$. Introduce the structure of a fibre bundle $\pi:S^1\times S^1\to S^1$ on the boundary $S^1\times S^1$ of $\tilde{M}'$ whose base projection is given by $\pi(e^{i\varphi},e^{i\psi})=e^{i(\varphi-\psi)}$ (note that in fact this defines a principal fibre bundle over the multiplicative group $S^1\subset\mathbb{C}^*$).*

Define a closed equivalence relation R on $\tilde{M}'$ whose classes are the fibres

$$\pi^{-1}(e^{i\varphi})=\{(e^{i(\varphi+t)},e^{it}):t\in\mathbb{R}\}$$

and the one-point sets $\{v\}$, $v\in D^0\times S^1$. Then M is homeomorphic to the quotient space $M'=\tilde{M}'/R$.

Proof. Since $\tilde{M}'$ is compact and R is closed, $M'=\tilde{M}'/R$ is compact too.

Define a mapping $\tilde{\phi}:M\to\tilde{M}'$ by

$$\tilde{\phi}(z,u)=\left(\frac{z}{1+|z|},e^{-i\tau(u/(1+|z|^2))}\right),\qquad (z,u)\in H_1\,,$$

$$\tilde{\phi}(\zeta)=\left(\frac{\zeta}{1+|\zeta|},-1\right),\qquad \zeta\in\mathbb{C}\,,$$

$$\tilde{\phi}(e^{i\varphi})=(e^{i\varphi},1)\,,\qquad e^{i\varphi}\in S^1\,,$$

and let ϕ denote the corresponding mapping into $M'=\tilde{M}'/R$. Then it is easy to check that ϕ is bijective and continuous. But, since M is compact, ϕ is necessarily even a homeomorphism. □

Theorem 3.2. (i) *M is the Martin compactification of H_1. Especially, the Martin boundary of H_1 is homeomorphic to the closed unit disk in $\mathbb{C}$, and the minimal Martin boundary is homeomorphic to the unit circle S^1 in $\mathbb{C}$.*

(ii) *If ζ is a point of the part $\mathbb{C}$ of Δ, then*

$$F_\zeta(z,u)=\frac{I_0(\sqrt{2}|z-\zeta|)}{I_0(\sqrt{2}|\zeta|)}\,,$$

and if $e^{i\varphi}$ is a point of the part S^1 of Δ, then

$$F_{e^{i\varphi}}(z,u)=e^{\sqrt{2}\,\mathrm{Re}(e^{i\varphi}\bar{z})}\,.$$

Proof. First we shall show that for any point η in the boundary $\mathbb{C}\cup S^1$ of M and any sequence $\{(z_n,u_n)\}_n$ in H_1 which converges to η in M, we have

$$\lim_{n\to\infty}F_{(z_n,u_n)}(z,u)=F_\eta(z,u)\tag{3.2}$$

pointwise for all $(z,u)\in H_1$. In fact, we shall show that any sequence $\{(z_n,u_n)\}_n$ which converges to η contains a subsequence $\{(z'_m,u'_m)\}_m$ such that

$$\lim_{m\to\infty} F_{(z'_m,u'_m)}(z,u)=F_\eta(z,u)\,. \tag{3.2$'$}$$

This clearly is equivalent to (3.2).

In order to prove (3.2)′, assume first that $\eta=\zeta\in\mathbb{C}$. Then we have $z_n\to\zeta$, and $|u_n|\to+\infty$. Hence, the asymptotics of Theorem 2.2 (ii) applies to the sequence $\{(z_n,u_n)\}_n$ as well as to the sequence $\{(z,u)^{-1}\cdot(z_n,u_n)\}_n$, and thus

$$F_{(z_n,u_n)}(z,u)\sim\frac{I_0(\sqrt{2}|z_n-z|)}{I_0(\sqrt{2}|z_n|)}a_n\,,$$

where

$$a_n=\left[\frac{|u_n|}{|u_n-u-2\,\mathrm{Im}(z\bar z_n)|}\right]^{3/4}\exp\left[\sqrt{2\pi|u_n|}-\sqrt{2\pi|u_n-u-2\,\mathrm{Im}(z\bar z_n)|}\right].$$

Since obviously $\lim\limits_{n\to\infty}a_n=1$, and since $z_n\to\zeta$, we hence obtain

$$\lim_{n\to\infty}F_{(z_n,u_n)}(z,u)=\frac{I_0(\sqrt{2}|\zeta-z|)}{I_0(\sqrt{2}|\zeta|)}\,.$$

Next assume that $\eta=e^{i\varphi}\in S^1$. Then, if $z_n=r_ne^{i\varphi_n}$, we have $r_n\to\infty$, and

$$e^{i[\tau(u_nr_n^{-2})+\varphi_n]}\to e^{i\varphi}\,.$$

Passing to a subsequence, if necessary, we may either assume that the sequence $\{u_nr_n^{-2}\}_n$ converges to a real number $\omega_\infty\in\mathbb{R}$, or that $|u_n|r_n^{-2}\to+\infty$.

In the first case, Theorem 2.1 applies to the arguments $\{(z_n,u_n)\}_n$ as well as to $\{(z'_n,u'_n)\}_n$, where

$$(z'_n,u'_n)=(z,u)^{-1}\cdot(z_n,u_n)=(z_n-z,u_n-u-2\,\mathrm{Im}(z\bar z_n))\,.$$

Then it is clear that

$$F_{(z_n,u_n)}(z,u)\sim e^{\sqrt{2}\Omega_n}\,,$$

where

$$\Omega_n=\gamma\left(\frac{u_n}{|z_n|^2}\right)|z_n|-\gamma\left(\frac{u'_n}{|z'_n|^2}\right)|z'_n|\,.$$

Now, with $\omega_n=\dfrac{u_n}{|z_n|^2}$ and $\omega'_n=\dfrac{u'_n}{|z'_n|^2}$, we have

$$\gamma(\omega'_n)=\gamma(\omega_n)+\gamma'(\omega_n)(\omega'_n-\omega_n)+O(|\omega'_n-\omega_n|^2)\,,$$

since $\lim\omega_n=\lim\omega'_n=\omega_\infty$.

Then a simple estimate shows that

$$\begin{aligned}\omega'_n-\omega_n&=\frac{u_n-u-2\,\mathrm{Im}(z\bar z_n)}{|z_n-z|^2}-\frac{u_n}{|z_n|^2}\\&=2\left[\frac{u_n\,\mathrm{Re}(z_n\bar z)}{|z_n|^4}-\frac{\mathrm{Im}(z\bar z_n)}{|z_n|^2}\right]+O(|z_n|^{-2})\,.\end{aligned}$$

Especially we have

$$|\omega_n' - \omega_n|^2 = O(|z_n|^{-2}).$$

One can also easily show that

$$|z_n'| = |z_n - z| = |z_n| - \frac{\mathrm{Re}(\bar{z}z_n)}{|z_n|} + O(|z_n|^{-1}).$$

So, together we obtain

$$\Omega_n = \gamma(\omega_n)\frac{\mathrm{Re}(\bar{z}z_n)}{|z_n|} - 2\gamma'(\omega_n)\left[\frac{u_n \mathrm{Re}(\bar{z}z_n)}{|z_n|^4} - \frac{\mathrm{Im}(z\bar{z}_n)}{|z_n|}\right] + O(|z_n|^{-1}).$$

Since $\frac{\bar{z}z_n}{|z_n|} = e^{i\varphi_n}\bar{z}$, and since $e^{i\varphi_n} \to e^{i[-\tau(\omega_\infty)+\varphi]}$, we get

$$\begin{aligned}\lim_{n\to\infty} \Omega_n = {} & \gamma(\omega_\infty)\,\mathrm{Re}(e^{i[-\tau(\omega_\infty)+\varphi]}\bar{z}) \\ & - 2\omega_\infty \gamma'(\omega_\infty)\,\mathrm{Re}(e^{i[-\tau(\omega_\infty)+\varphi]}\bar{z}) \\ & - 2\gamma'(\omega_\infty)\,\mathrm{Im}(e^{i[-\tau(\omega_\infty)+\varphi]}\bar{z}).\end{aligned}$$

But,

$$\gamma'(\omega) = \frac{\tau'(\omega)}{\sin^2\tau(\omega)}(\sin\tau(\omega) - \tau(\omega)\cos\tau(\omega)),$$

and

$$\theta'(\tau) = 2\frac{\sin\tau - \tau\cos\tau}{\sin^3\tau},$$

hence, since $\tau = \theta^{-1}$,

$$\gamma'(\omega) = \tfrac{1}{2}\sin\tau(\omega).$$

This yields, by definition of θ and τ,

$$\gamma(\omega) - 2\omega\gamma'(\omega) = \cos\tau(\omega),$$

hence

$$\begin{aligned}\lim_{n\to\infty} \Omega_n &= \cos\tau(\omega_\infty)\,\mathrm{Re}(e^{i[-\tau(\omega_\infty)+\varphi]}\bar{z}) - \sin\tau(\omega_\infty)\,\mathrm{Im}(e^{i[-\tau(\omega_\infty)+\varphi]}\bar{z}) \\ &= \mathrm{Re}[e^{i\tau(\omega_\infty)}e^{i[-\tau(\omega_\infty)+\varphi]}\bar{z}] \\ &= \mathrm{Re}[e^{i\varphi}\bar{z}].\end{aligned}$$

Thus we obtain

$$\lim_{n\to\infty} F_{(z_n,u_n)}(z,u) = e^{\sqrt{2}\,\mathrm{Re}(e^{i\varphi}\bar{z})}.$$

It remains the case where $|u_n|r_n^{-2} \to \infty$. Since $\lim_{\kappa\to\pm\infty} \tau(\kappa) = \pm\pi$, we have

$$\lim_{n\to\infty} e^{i\varphi_n} = -e^{i\varphi}.$$

Since in this case the asymptotics of Theorem 2.2 (i) applies, we have

$$F_{(z_n,u_n)}(z,u) \sim b_n e^{\sqrt{2}\Omega'_n},$$

where

$$b_n = \left[\frac{|u_n|}{|u_n - u - 2\,\mathrm{Im}(z\bar{z}_n)|}\right]^{3/4} \cdot \left[\frac{|z_n|}{|z - z_n|}\right]^{1/2},$$

$$\Omega'_n = [\pi|u_n| + |z_n|^2 - 2\varrho(\delta_n)\sqrt{\pi|u_n|\,|z_n|^2}]^{1/2} - [\pi|u'_n| + |z'_n|^2 - 2\varrho(\delta'_n)\sqrt{\pi|u'_n|\,|z'_n|^2}]^{1/2}.$$

Here we set $\delta_n = \sqrt{|z_n|^2/(\pi|u_n|)}$, $\delta'_n = \sqrt{|z'_n|^2/(\pi|u'_n|)}$, and (z'_n, u'_n) is defined as before.

Since $|z_n| \to \infty$ and $\delta_n \to 0$, we also have $|u_n| \to \infty$, and together this implies $b_n \to 1$. Moreover, a tedious, but straight-forward calculation similar to those before shows that

$$\Omega'_n = -\varrho(\delta_n)\frac{\mathrm{Re}(\bar{z}z_n)}{|z_n|} + \xi_n,$$

where $\lim_{n\to\infty} \xi_n = 0$. Since, by Theorem 1.3, $\varrho(\delta_n) = 1 + O(\delta_n^2)$, and since $e^{i\varphi_n} \to -e^{i\varphi}$, this implies

$$\lim_{n\to\infty} F_{(z_n,u_n)}(z,u) = e^{\sqrt{2}\,\mathrm{Re}(e^{i\varphi}\bar{z})}.$$

The proof of (3.2) is now complete.

But, since H_1 is dense in M, it is easy to see that (3.2) implies that the functions $F^{(z,u)}$ [extended to M by (ii)] are continuous with values in $[0, \infty]$ on the whole of M. Moreover, since for different $\eta_1, \eta_2 \in \mathbb{C} \cup S^1$ the functions F_{η_1} and F_{η_2} are obviously different, it is clear that the functions $F^{(z,u)}$, $(z,u) \in H_1$, separate the points of $M \backslash H_1$. Thus we have shown that M is the Martin compactification of H_1. Moreover, it is clear (for example by Lemma 3.1), that $\Delta = M \backslash H_1$ is homeomorphic to the closed unit disk in $\mathbb{C}$. Finally, by (ii) all functions F_η with $\eta \in \Delta$ are independent of the variable u. Thus, if we consider them as functions of the variable z, they are just eigenfunctions of the "classical" Laplacian on $\mathbb{C} \cong \mathbb{R}^2$, and so the well-known classical theory implies that the extremal rays of the cone Γ are just the rays $\mathbb{R}F_{e^{i\varphi}}$, $e^{i\varphi} \in S^1$. This shows that $\Delta_1 = S^1 \subset M$ and concludes the proof of Theorem 3.2. □

The following corollary to Theorem 3.2 is clear by (3.1):

Corollary 3.3. *If $h \geqq 0$ is a $(-\Delta_K + 2)$ harmonic function on H_1, then there exists a unique positive measure μ on $[0, 2\pi]$, such that*

$$h(x,y,u) = \int_0^{2\pi} e^{\sqrt{2}(x\cos\varphi + y\sin\varphi)}\,d\mu(\varphi)$$

for all $(x,y,u) \in H_1$. Especially, h does not depend on the variable u.

The readers who are only interested in a proof of this corollary should note, that one could proceed almost word by word as in [12] to derive this statement. In fact, Margulis' argument covers a large class of harmonic spaces on nilpotent Lie groups. Especially it shows that the immediate generalization of our corollary to sub-Laplacians on stratified Lie groups holds.

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