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Compactifications of $\mathbb{C}^3$. II

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1. Introduction

This is the announced second part to the joint paper "Compactifications of $\mathbb{C}^3, I$" with M. Schneider.

In order to state the results of this paper and to put them in connection with the general theory remember that a compactification of $\mathbb{C}^3$ is a compact complex manifold X with a divisor $Y \subset X$ such that $X \setminus Y \simeq \mathbb{C}^3$ biholomorphically. We assume always $b_2(X)=1$ (i.e. Y is irreducible) and X projective (for the non-projective case see [P-S]). Remember that X is a Fano 3-fold of index r, i.e. ω_X^{-1} is ample and there is a generator $\mathscr{L} \in \mathrm{Pic}(X) \simeq \mathbb{Z}$ such that $\mathscr{L}^r \simeq \omega_X^{-1}$. One knows $r \leqq 4$ and the cases $r=4$ (resp. $r=3$) give the classical compactifications $\mathbb{P}^3$ (resp. Q_3, the 3-dimensional quadric) with divisors Y at infinity $\mathbb{P}_2$ resp. the quadric cone.

If $r=2$ Furushima [Fu 1] constructed a new compactification X with two possible divisors at infinity – one normal and one non-normal. By [Fu 1] and [P-S] these are the only ones for $r=2$. Observe that X is rational and has $b_3(X)=0$. Also by [P-S] and [Fu 2], if $r=1$ and Y is normal then X has to be rational with $b_3(X)=0$; i.e. X is a Fano 3-fold of "genus 12". Two such X are known, one constructed by Iskovskij [Is 1], one by Mukai-Umemura [M-U]. Probably these are the only ones (recently proved by Mukai as I understand). Both cases cannot be a compactification with normal divisor at infinity, as proved by Furushima [Fu 3]; the non-normal case is still undecided.

This paper now deals with the case $r=1$ and Y non-normal. The main result is the

Theorem. *Let X be a compactification of $\mathbb{C}^3$ with $b_2(X)=1$ and divisor Y at infinity. Assume that X is a Fano 3-fold of index 1 and Y non-normal.*

Then the genus $g(X) = -\dfrac{c_1(\omega_X)^3}{2} + 1 = 12$, *in particular X is rational and* $b_3(X)=0$.

Even in the case $g(X)=12$ we get some information: the non-normal locus E of Y consists of one or two smooth rational curves meeting transversely in one point;

the conductor ideal is reduced. If $f: \tilde{Y} \to Y$ is the normalization then $\tilde{E}$ – the analytic preimage of E – is reduced too and consists of two smooth rational curves meeting of order two in exactly one point. In particular, $b_3(\tilde{Y})=0$ and hence $\tilde{Y}$ and Y are rational.

In conclusion one can state now the following

Theorem. *Any projective compactification X of $\mathbb{C}^3$ with $b_2(X)=1$ is rational with $b_3(X)=0$.*

The only remaining open problems – besides a problem in the non-algebraic case (cp. [P-S]) – are the questions of existence for the Mukai-Umemura example (normal case) and the Iskovskij and the Mukai-Umemura example (non-normal case) – provided these are the only Fano 3-folds with $r=1$, $b_2(X)=1$, $g(X)=12$.

Some remark to the proof of the main theorem.

Let X be a compactification of $\mathbb{C}^3$ with $b_2(X)=1$ which is a Fano 3-fold of index 1. Let Y be the (irreducible) divisor at infinity. Assume that Y is non-normal. Let $f: \tilde{Y} \to Y$ be the normalization and $\pi: \hat{Y} \to \tilde{Y}$ a desingularization. It is easy to see $\kappa(\hat{Y}) = -\infty$. In order to show $b_3(Y)=0$ we have to do two things. First, we have to control the topology of $\hat{Y}$, namely, we want to prove $b_3(\tilde{Y}) = b_3(Y)$. Second, we must prove the rationality of $\hat{Y}$ (i.e. the rationality of Y). Then $b_3(\hat{Y}) = b_3(\tilde{Y}) = 0$, hence $b_3(Y)=0$.

The first problem is solved by analyzing carefully the map f, i.e. the non-normal locus $E \subset Y$ and its analytic preimage $\tilde{E} \subset \tilde{Y}$. The second one is treated by very special hyperplane sections (and by using the Iskovskij classification for X).

2. Preliminaries

(2.1) We recollect some notations and facts from [P-S]. A compactification of $\mathbb{C}^3$ is a pair (X, Y) consisting of a compact complex manifold X and an analytic subset $Y \subset X$ such that $X \setminus Y \simeq \mathbb{C}^3$ (biholomorphically). Necessarily Y is of pure dimension 2. We are only interested here in the case $b_2(X)=1$ which is the same as to say Y is irreducible. We also assume that X is projective. Then X is a Fano 3-fold, i.e. the canonical sheaf ω_X is negative. We treat in this paper the case of index 1, i.e. ω_X generates $\mathrm{Pic}(X) \simeq \mathbb{Z}$ (cp. [P-S], Sect. 0). For more properties of (X, Y) see again [P-S], Sect. 0.

(2.2) For a Fano 3-fold X the genus $g(X)$ is defined by

$$g(X) = -\frac{c_1(\omega_X)^2}{2} + 1 .$$

Iskovskij [Is 1, 2] proved that for a Fano 3-fold X of index 1 with $b_2(X)=1$ one has always $2 \leqq g(X) \leqq 12$ and $g(X) \neq 11$. Moreover these 3-folds can be classified (see [Is 1, 2]). The anticanonical bundle is always very ample except for two cases (cp. 3.14). If ω_X^{-1} is very ample, X is said to be of the principal series.

We need to know the Betti numbers $b_3(X)$, which are given by the following table (see [Is-Šo])

$g(X)$	$\frac{1}{2}b_3(X)$
2	52
3	30
4	20
5	14
6	10
7	5
8	4
9	3
10	2
12	0

(2.3) **Lemma.** *Let X be a purely 1-dimensional reduced projective complex space. Let $X' \subset X$ be an irreducible component with the reduced structure. Then $\omega_{X'}$ is a subsheaf of ω_X (ω denotes the dualizing sheaf), moreover*

$$\omega_X | \omega_{X'} \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_{X'/X}, \omega_X).$$

Proof. It is well known (see e.g. [A-K]) that

$$\omega_{X'} \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X'}, \omega_X). \tag{1}$$

Let $\mathcal{I}$ be the ideal sheaf of X' in X. Then from the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$ we obtain:

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X'}, \omega_X) \to \omega_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \omega_X) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{X'}, \omega_X). \tag{*}$$

So by (1) it is sufficient to show:

$$\mathcal{E} = \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{X'}, \omega_X) = 0. \tag{2}$$

Let $\mathcal{O}_X(1)$ be an ample line bundle on X.

Take $v \gg 0$ such that $\mathcal{E} \otimes \mathcal{O}_X(v)$ is globally generated.

Then it is sufficient to show

$$H^0(\mathcal{E}(v)) = 0. \tag{2'}$$

But $H^0(\mathcal{E}(v)) \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{X'}, \omega_X(v))$.

Since X is 1-dimensional, it is Cohen-Macaulay. So by Serre duality:

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{X'}, \omega_X(v)) \simeq H^0(\mathcal{O}_{X'}(-v)) = 0$$

which proves (2'), hence (2).

(2.4) **Lemma.** *Let X be a purely 1-dimensional projective Cohen-Macaulay space. Let $\mathscr{n}$ be the sheaf of nilpotent functions on X. Then there is an exact sequence*

$$0 \to \omega_{\mathrm{red}\,X} \to \omega_X \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{n}, \omega_X) \to 0 .$$

Proof. Same as that of (*) and (2) in the proof of (2.3).

3. The Main Result

Let (X, Y) always denote a smooth projective compactification of $\mathbb{C}^3$ with $b_2(X)=1$. Then X is a Fano 3-fold of genus $g(X)$. We assume that X is of index 1, i.e. ω_X generates $\mathrm{Pic}(X) \simeq \mathbb{Z}$. Moreover let Y be non-normal.

(3.1) **Theorem.** *X has genus $g(X)=12$, moreover $b_3(X)=0$ and X is rational.*

The proof will follow from several propositions of this section.

(3.2) We denote by E the non-normal locus of Y equipped with the structure given by the conductor ideal. Let $\tilde{E}$ be the analytic preimage of E with respect to the normalization $f: \tilde{Y} \to Y$. In order to prove (3.1) it will be sufficient (by the Iskovskij classification, 2.2) to show $g(X) > 10$, equivalently $b_3(X)=0$.

We denote by r (resp. $\tilde{r}$) the number of irreducible components of E (resp. $\tilde{E}$).

(3.3) **Proposition.** 1. *$H^1(\mathcal{O}_E) = H^1(\mathcal{O}_{\mathrm{red}\,E}) = 0$; in particular all components E_i of* red E *are smooth rational curves.*

2) $\omega_{\tilde{E}} \simeq \mathcal{O}_{\tilde{E}}$.

Proof. 1. Using the exact sequence [Mo, 3.34.2]

$$0 \to \mathcal{O}_Y \to f_*(\mathcal{O}_{\tilde{Y}}) \to \omega_E \to 0$$

and $\omega_Y \simeq \mathcal{O}_Y$, we obtain by $H^1(\mathcal{O}_Y)=0$:

$$0 = H^0(\omega_E) \simeq H^1(\mathcal{O}_E),$$

since E is Cohen-Macaulay (see e.g. [Mo, K-W, S]). Letting $\mathscr{n}$ be the sheaf of nilpotent functions on E and taking cohomology from

$$0 \to \mathscr{n} \to \mathcal{O}_E \to \mathcal{O}_{\mathrm{red}\,E} \to 0$$

we obtain $H^1(\mathcal{O}_{\mathrm{red}\,E})=0$.

Last $H^1(\mathcal{O}_{E_i})=0$ follows in the same spirit, so $E_i \simeq \mathbb{P}_1$.

2) This follows from $\omega_{\tilde{E}} \mp f^*(\omega_Y) \otimes \mathcal{O}_{\tilde{E}}$ ([Mo, 3.34.1]) and $\omega_Y \simeq \mathcal{O}_Y$.

If $\tilde{E}$ is reduced we can say immediately a lot on the structure of $\tilde{E}$:

(3.4) **Proposition.** *Assume that $\tilde{E}$ is reduced.*

a) *If a connected component of $\tilde{E}$ consists of exactly one irreducible component $\tilde{E}_i$, then $\tilde{E}_i$ is a torus or a singular rational curve with $\omega_{\tilde{E}_i} \simeq \mathcal{O}_{\tilde{E}_i}$, i.e. a cubic in $\mathbb{P}_2$.*

b) *If a connected component of $\tilde{E}$ consists of more than one irreducible component, then all these components $\tilde{E}_i$ are smooth and rational.*

Proof. a) By (3.3, 1) we have $\omega_{\tilde{E}_i} \simeq \mathcal{O}_{\tilde{E}_i}$.

b) By (2.3) $\omega_{\tilde{E}_i}$ is a proper subsheaf of $\omega_{\tilde{E}}$, i.e. of $\mathcal{O}_{\tilde{E}}$. Hence $H^0(\omega_{\tilde{E}_i})=0$ and $H^1(\mathcal{O}_{\tilde{E}_i})=0$, i.e. $\tilde{E}_i \simeq \mathbb{P}_1$.

(3.5) **Proposition.** $H^1(\mathcal{O}_{\tilde{Y}})=0;\ h^0(\mathcal{O}_E)=h^0(\mathcal{O}_{\tilde{E}})=h^1(\mathcal{O}_{\tilde{E}})=1.$

Proof. Let $g=h^1(\mathcal{O}_{\tilde{Y}})$. By the exact sequence

$$0\to\mathcal{O}_Y\to f_*(\mathcal{O}_{\tilde{Y}})\to\omega_E\to 0$$

we obtain $h^1(\mathcal{O}_{\tilde{Y}})=h^1(\omega_E)-1=h^0(\mathcal{O}_E)-1=\mu-1$ by definition of μ.

Since $\omega_{\tilde{E}}\simeq\mathcal{O}_{\tilde{E}}$, we have $h^0(\mathcal{O}_{\tilde{E}})\simeq h^1(\mathcal{O}_{\tilde{E}})$. Now by the exact sequence [Mo, 3.36.2]

$$0\to\mathcal{O}_E\to f_*(\mathcal{O}_{\tilde{E}})\to\omega_E\to 0$$

we see $h^0(\mathcal{O}_E)=h^0(\mathcal{O}_{\tilde{E}})$ since $h^0(\omega_E)=h^1(\mathcal{O}_E)=0$. Thus:

$$g+1=\mu=h^0(\mathcal{O}_E)=h^0(\mathcal{O}_{\tilde{E}})=h^1(\mathcal{O}_{\tilde{E}}). \tag{1}$$

Hence it is sufficient to prove $g=0$. Assume $g>0$, so $\mu>1$. Let $\pi:\hat{Y}\to\tilde{Y}$ be a minimal desingularization of $\tilde{Y}$, $\sigma:\hat{Y}\to Y_m$ a minimal model. Since $\kappa(Y)=-\infty$ and $g>0$, Y_m is a non-rational ruled surface. Let $p:Y_m\to C_m$ be a ruling. Let Z be the exceptional set of π. Then I claim:

$$\dim p\circ\sigma(Z)=0. \tag{2}$$

Assume $\dim p\circ\sigma(Z)>0$ and let $S_1,\ldots,S_q$ be the irreducible components of Z with $\dim p\circ\sigma(S_i)=1$.

Since $\omega_{\tilde{Y}}$ is a proper subsheaf of $\mathcal{O}_{\tilde{Y}}$ by [Mo, 3.34.3], $H^2(\mathcal{O}_{\tilde{Y}})=0$. Moreover $H^2(\mathcal{O}_{\hat{Y}})=0$. So from the Leray spectral sequence we have

$$H^1(\mathcal{O}_{\hat{Y}})\simeq H^1(\mathcal{O}_{\tilde{Y}})\oplus H^0(R^1\pi_*(\mathcal{O}_{\hat{Y}})). \tag{3}$$

By Riemann-Hurwitz $g(S_i)\geqq\tilde{g}:=g(C_m)$.

Hence $h^1(\mathcal{O}_{S_i})\geqq\tilde{g}$ and consequently $h^1(\mathcal{O}_Z)\geqq q\cdot\tilde{g}$ where Z carries the reduced structure. This last fact is clear by considering the normalization of Z.

Let $\pi(Z)=\{y_1,\ldots,y_t\}$. Define λ_i by

$$R^1\pi_*(\mathcal{O}_{\hat{Y}})_{y_i}\simeq\mathbb{C}^{\lambda_i}.$$

Then $\sum_{i=1}^{t}\lambda_i\geqq q\cdot\tilde{g}$ because the restriction map $H^1(\mathcal{O}_{\hat{Z}})\to H^1(\mathcal{O}_Z)$, $\hat{Z}$ the completion of Z, is surjective, then use Grauert's comparison theorem. Hence (3) implies: $\tilde{g}=g+\sum\lambda_i\geqq g+q\cdot\tilde{g}$. So $q=0$ since $g>0$, i.e. $\dim p\circ\sigma(Z)=0$ and (2) is proved.

We thus find $x_1,\ldots,x_k\in C_m$ such that

$$Z\subset\bigcup_{i=1}^{k}(p\circ\sigma)^{-1}(x_i).$$

Hence from $H^1((p\circ\sigma)^{-1}(x_i),\mathcal{O})=0$, we obtain $H^1(\mathcal{O}_Z)=0$. Since $R^1(p\circ\sigma)_*(\mathcal{O}_{\hat{Y}})=0$, even $H^1((p\circ\sigma)^{-1}(x_i)_\mu,\mathcal{O})=0$ for any infinitesimal neighborhood $(p\circ\sigma)^{-1}(x_i)_\mu$, thus $H^1(\mathcal{O}_{Z_\mu})=0$ for any infinitesimal neighborhood Z_μ of Z, consequently $H^1(\mathcal{O}_{\hat{Z}})=0$.

This proves $R^1\pi_*(\mathcal{O}_{\hat{Y}})=0$ and $\tilde{g}=g$ by (3). We next prove

$$\dim p\circ\sigma(\hat{E})=1, \tag{4}$$

where $\hat{E}$ is the strict transform of $\tilde{E}$ in $\hat{Y}$. In fact, otherwise we would get [using (2)] a lot of smooth rational curves l in $Y\backslash E$. Then l is a Cartier divisor in Y, so

$\mathcal{O}_Y(l) \simeq \mathcal{O}_Y(k)$ for some $k \in \mathbb{N}$. Since $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(Y)$ we can write $l = Y \cap H$ for some hypersurface $H \subset X$. But $(H \cdot E) > 0$, so $H \cap E = \emptyset$ is not possible. This proves (4).

By (4) we find a component, say $\hat{E}_1$, of $\hat{E}$ such $p \circ \sigma(\hat{E}_1) = C_m$. So $g(\hat{E}_1) \geqq \tilde{g} = g$.

Hence $\mathrm{red}\,\tilde{E}$ contains a curve of genus $\geqq g$ implying $h^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}}) \geqq g$. On the other hand by (1): $h^1(\mathcal{O}_{\tilde{E}}) = g+1$. So we obtain the inequality

$$g+1 = h^1(\mathcal{O}_{\tilde{E}}) \geqq h^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}}) \geqq g\,. \tag{5}$$

Now consider $\hat{E}$, the strict transform of E, equipped with the reduced structure. If $g > 1$, the component $\hat{E}_1$ described above is unique because otherwise $h^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}}) \geqq 2g$.

(A) So assume $g > 1$ for the moment. Since all other components of $\hat{E}$ are components of fibers $(p \circ \sigma)^{-1}(x)$, we easily get $h^1(\mathcal{O}_{\hat{E}}) = g$.

Exactly the same arguments apply to $\bar{E} = \pi^{-1}(\tilde{E})$ with the reduced structure [use (2)!]. So $h^1(\mathcal{O}_{\bar{E}}) = g$. Now clearly $\mathcal{O}_{\mathrm{red}\,\tilde{E}} \simeq \pi_*(\mathcal{O}_{\bar{E}})$, so we conclude

$$h^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}}) = g\,, \tag{6}$$

i.e. (5) becomes a strict inequality and (5) excludes the case $\tilde{E} = \mathrm{red}\,\tilde{E}$.

Assume that $\tilde{E}_1$ is non-reduced. Let $\mathcal{J}$ be the ideal sheaf of $\mathrm{red}\,\tilde{E}$ in $\tilde{E}$. Then by (3.7a) below (for $\mu = 1$) we obtain:

$$h^1(\tilde{E}, \mathcal{O}_{\tilde{E}}/\mathcal{J}^2) \geqq 2g\,,$$

since $h^1(\mathcal{J}/\mathcal{J}^2) \geqq g$. Namely, $\mathcal{J}/\mathcal{J}^2 | \mathrm{red}\,\tilde{E}_1 \simeq \mathcal{O}_{\mathrm{red}\,\tilde{E}_1}$ modulo torsion (3.7).

Hence $h^1(\mathcal{O}_{\tilde{E}}) \geqq 2g$, contradiction.

So $\tilde{E}_1$ is reduced. But then clearly $\omega_{\tilde{E}_1}$ is a subsheaf of $\omega_{\tilde{E}} | \tilde{E}_1 \simeq \mathcal{O}_{\tilde{E}_1}$ [cf. (2.3), $\tilde{E}_1$ is smooth, so $\omega_{\tilde{E}_1}$ is locally free!]

Hence $g(\tilde{E}_1) \leqq 1$, contradiction.

(B) So we are reduced to the case $g = 1$. Then

$$1 \leqq h^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}}) \leqq 2\,.$$

(B_1) First let $h^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}}) = 2$.

Letting $\mathcal{O}_{\tilde{E}_{(1)}} = \mathcal{O}_{\tilde{E}}/\mathcal{J}^2$ we have the exact sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_{\tilde{E}_{(1)}} \to \mathcal{O}_{\mathrm{red}\,\tilde{E}} \to 0\,.$$

Taking cohomology gives the exact sequence (5):

$$\begin{aligned} 0 \to H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{O}_{\tilde{E}_{(1)}}) \xrightarrow{\gamma} H^0(\mathcal{O}_{\mathrm{red}\,\tilde{E}}) \to H^1(\mathcal{J}/\mathcal{J}^2) \\ \to H^1(\mathcal{O}_{\tilde{E}_{(1)}}) \to H^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}}) \to 0\,. \end{aligned}$$

Since $h^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}}) = 2$ and since $h^1(\mathcal{O}_{\tilde{E}}) = 2 \geqq h^1(\mathcal{O}_{E_{(1)}})$ we obtain $h^1(\mathcal{O}_{\tilde{E}_{(1)}}) = 2$.

So γ being surjective, $h^1(\mathcal{J}/\mathcal{J}^2) = 0$.

We see that the components $\tilde{E}_i$ of $\tilde{E}$ of genus 1 (there are exactly one or two!) must be reduced because otherwise again $\mathcal{J}/\mathcal{J}^2 | \mathrm{red}\,\tilde{E}_i \simeq \mathcal{O}_{\mathrm{red}\,\tilde{E}_i}$ modulo torsion by (3.7a), hence $h^1(\mathcal{J}/\mathcal{J}^2) > 0$. But then $\omega_{\tilde{E}_i}$ is again a subsheaf of $\mathcal{O}_{\tilde{E}_i}$. $\tilde{E}_i$ being of genus 1, we conclude by Sect. 2 that $\tilde{E}_i$ is a connected component of $\tilde{E}$.

First assume that there are two elliptic components, say $\tilde{E}_1$ and $\tilde{E}_2$. Since $h^0(\mathcal{O}_{\mathrm{red}\,\tilde{E}}) = 2$, we conclude $\tilde{E} = \tilde{E}_1 \cup \tilde{E}_2$, i.e. $\tilde{r} = 2$. So $r \leqq 2$. By the exact sequence

([Ba-Ka, 3.A.7])

$$\begin{array}{ccccccccccc} 0 \to H^2(Y,\mathbb{Z}) & \to & H^2(\tilde{Y},\mathbb{Z})\oplus H^2(E,\mathbb{Z}) & \xrightarrow{\alpha} & H^2(\tilde{E},\mathbb{Z}) & \to & H^3(Y,\mathbb{Z}) & \to & H^3(\tilde{Y},\mathbb{Z}) \to 0 \\ \downarrow\simeq & & & & \downarrow\simeq & & \downarrow\simeq & & \\ \mathbb{Z} & & & & \mathbb{Z}^r & & \mathbb{Z}^2 & & \end{array}$$

we obtain $2\leqq b_3(Y)\leqq b_3(\tilde{Y})+2=4$.

Since $\alpha\neq 0$ (α is the canonical "difference map") we must have $b_3(Y)\leqq 3$, so $b_3(Y)=2$ since $b_3(Y)$ is even. So $b_3(X)=2$. But by Iskovskij there is no Fano 3-fold X with $b_3(X)=2$. Hence there is a unique elliptic component $\tilde{E}_1$. With the same arguments we exclude the case $\tilde{E}=\tilde{E}_1\cup\tilde{E}_2$, $\tilde{E}_2$ a singular rational cubic in $\mathbb{P}_2$. Since $h^0(\mathcal{O}_{\mathrm{red}\,\tilde{E}})=2$, there is a second connected component $\tilde{E}'$ consisting of smooth rational curves. If $\tilde{E}'$ is not reduced we conclude by the method of (3.7) that $H^1(\mathcal{J}^\mu/\mathcal{J}^{\mu+1})=0$ for all μ. Namely, all components of $\tilde{E}'$ have to be smooth [if some is singular then by $h^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}'})=1$ it has to be a singular cubic in $\mathbb{P}_2$ whence $h^1(\mathcal{J}/\mathcal{J}^2)>0$, contradiction]. In fact, since we know $H^1(\mathcal{J}/\mathcal{J}^2)=0$, and all $\tilde{E}_i'\simeq\mathbb{P}_1$ it is an easy exercise to exclude the only other possible case $H^1(\mathcal{J}^\mu/\mathcal{J}^{\mu+1})\simeq\mathbb{C}$ (look at the normalization of $\mathrm{red}\,\tilde{E}'$). Then using the higher analogs of (5) (for the infinitesimal neighborhoods of $\mathrm{red}\,\tilde{E}$ in $\tilde{E}$) we get the contradiction $h^0(\mathcal{O}_{\tilde{E}})\geqq 3$.

(The contradiction can also be derived directly from $h^1(\mathcal{J}^\mu/\mathcal{J}^{\mu+1})\leqq 1$ using (5) since $h^0(\mathcal{J}^\mu/\mathcal{J}^{\mu+1})>0$ as long as $\mathcal{J}^\mu/\mathcal{J}^{\mu+1}\neq 0$.)

So $\tilde{E}'$ is reduced. We know that $H^1(\mathcal{O}_{\tilde{E}'})\simeq\mathbb{C}$. Let $\bar{E}':=\pi^{-1}(\tilde{E}')$. Then $H^1(\mathcal{O}_{\bar{E}'})=0$ since $\dim p\circ\sigma(\bar{E}')=0$ and $\pi_*(\mathcal{O}_{\bar{E}'})\simeq\mathcal{O}_{\tilde{E}'}$. This contradicts $H^1(\mathcal{O}_{\tilde{E}'})\simeq\mathbb{C}$.

(B_2) We are left with the case $h^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}})=1$. Now the elliptic component of $\tilde{E}$ is uniquely determined. Call it $\tilde{E}_1$.

First let us see that $\tilde{E}$ must be connected. In fact, by (3.7a) we see that $h^0(\mathcal{J}^\mu/\mathcal{J}^{\mu+1})>0$ as long as $\mathcal{J}^\mu/\mathcal{J}^{\mu+1}\neq 0$, and that $h^1(\mathcal{J}^\mu/\mathcal{J}^{\mu+1})\leqq 1$. So by $h^0(\mathcal{O}_{\mathrm{red}\,\tilde{E}})\geqq 2$ we would obtain [using (5)] that $h^0(\mathcal{O}_{\tilde{E}_{(\nu)}})=h^0(\mathcal{O}_{\tilde{E}}/\mathcal{J}^{\nu+1})\geqq 3$ for all ν, in particular $h^0(\mathcal{O}_{\tilde{E}})\geqq 3$, contradiction.

Next I claim that

$$E_1=f(\tilde{E}_1) \tag{7}$$

is not reduced.

Assume that E_1 is reduced. Then we find some j such that E_j is non-reduced [otherwise we would find $h^0(\mathcal{O}_E)=1$, E being connected!]. Hence by [K-W] for general $y_0\in E_j$ the formal local ring $\mathcal{O}_{Y,y_0}$ is not of the form $(F_1)\mathbb{C}[\![X,Y]\!]/(X\cdot Y)$ and not of the form $(F_2)\mathbb{C}[\![X,Y]\!]/(X^2-Y^3)$. Now by Iskovskij we find through any y_0 a conic $l\subset X$. Since $(l\cdot Y)=2$, we conclude $l\subset Y$. Namely, assume $l\not\subset Y$. If Y is irreducible at y_0 (for generic y_0), then f is locally around y_0 a homeomorphism and (by [K-W] and [S, 1.2.20]) we are in situation (F_2). Otherwise, if Y is reducible at y_0, we can locally only have two smooth irreducible components of Y meeting transversely, i.e. $\mathcal{O}_{Y,y_0}\simeq\mathbb{C}\{X,Y,Z\}/(X\cdot Y)$. So we are in situation (F_1).

Thus we have $l\subset Y$ and Y is filled up by conics. The strict transforms $\hat{l}$ of those conics l are contracted by $p\circ\sigma$ (since $g=1$). By construction the general $\hat{l}$ meets a fixed component $\hat{E}_k$ with $f\circ\pi(\hat{E}_k)=E_j$. So $p\circ\sigma(\hat{E}_k)=C_m$, hence $\hat{E}_k=\hat{E}_1$, contradiction. This proves (7).

The same argument shows that E_1 is the only non-reduced component of E and that only finitely many conics meet E_j, $j \geqq 2$.

Now let $\tilde{E}_0$ be the non-reduced part of $\tilde{E}$; $\tilde{E}_1 \subset \tilde{E}_0$. The components of $\tilde{E}$ not belonging to $\tilde{E}_0$ are smooth rational curves meeting exactly one component of $\operatorname{red}\tilde{E}_0$ transversely in one point (use the exact sequence

$$0 \to \omega_{\tilde{E}_j} \to \mathcal{O}_{\tilde{E}} \to \mathcal{H}om(\mathcal{I}_{\tilde{E}_j}, \mathcal{O}_{\tilde{E}}) \to 0).$$

Let $\mathcal{J}$ be the ideal of $\tilde{E}_0$ in $\tilde{E}$. Then by

$$0 \to \omega_{\tilde{E}_0} \to \mathcal{O}_{\tilde{E}} \to \mathcal{H}om(\mathcal{J}, \mathcal{O}_{\tilde{E}}) \to 0$$

we obtain

$$\begin{array}{c} 0 \to H^0(\omega_{\tilde{E}_0}) \to H^0(\mathcal{O}_{\tilde{E}}) \to H^0(\mathcal{H}om(\mathcal{J}, \mathcal{O}_{\tilde{E}})) \\ \wr\| \\ \mathbb{C}^2 \\ \to H^1(\omega_{\tilde{E}_0}) \to H^1(\mathcal{O}_{\tilde{E}}) \to H^1(\mathcal{H}om(\mathcal{J}, \mathcal{O}_{\tilde{E}})) \to 0. \\ \wr\| \\ \mathbb{C}^2 \end{array}$$

$\operatorname{supp}\mathcal{H}om \,\ddagger(\mathcal{J}, \mathcal{O}_{\tilde{E}}) = \bigcup_{j \in J} \tilde{E}_j$, where $j \in J \Leftrightarrow \tilde{E}_j$ is reduced.

All these $\tilde{E}_j$ are disjoint, and $\mathcal{J}|\tilde{E}_j \simeq \mathcal{O}(-2)$, so

$$h^0(\mathcal{H}om(\mathcal{J}, \mathcal{O}_{\tilde{E}})) = 3r_0, \quad r_0 = \# J$$

and $h^1(\mathcal{H}om(\mathcal{J}, \mathcal{O}_{\tilde{E}})) = 0$.

Now $\tilde{E}_0$ is Cohen-Macaulay, so

$$h^1(\omega_{\tilde{E}_0}) = h^0(\mathcal{O}_{\tilde{E}_0}) = 2,$$

hence the above sequence gives

$$3r_0 \leqq 2, \quad \text{so} \quad r_0 = 0.$$

But $r_0 = 0$ implies:

$$r = 1: \tag{8}$$

assume that there is an reduced irreducible component $E_2 \subset E$. We have seen above (when we proved reducedness of E_j) that through a general point $y \in E_2$ we cannot find a conic in Y. Moreover either Y is a topological manifold around y or $\mathcal{O}_{Y,y} \simeq \mathbb{C}\{X, Y, Z\}/(X \cdot Y)$ (otherwise we would find conics). But then by [5, 1.18, 1.2.20] we can conclude that generically $f^{-1}(E_2)$ is reduced, hence reduced, contradiction. Thus $r = 1$ and (8) is proved.

As seen above, through any point of E there is a conic in Y. The strict transform of the conics in $\hat{Y}$ are contracted by $p \circ \sigma$. Thus the images of the fibers $(p \circ \sigma)^{-1}(x)$ are just the conics in Y.

We want to prove (9): $\tilde{r} = 1$. Assume $\tilde{r} > 1$. Then take $\tilde{E}_2 \subset \tilde{E}_0$, $\tilde{E}_2 \neq \tilde{E}_1$. Since $\dim p \circ \sigma(\hat{E}_2) = 0$ and since $f(\tilde{E}_2) = E_1 = E$ (set-theoretically), we conclude that E_1 is a conic or a line. Now $E_1 = E$ being the non-normal locus of Y, we have for the conormal bundles of $\operatorname{red}E_1$ in Y resp. X:

$$N^*_{\operatorname{red}E_1|Y} \simeq N^*_{\operatorname{red}E_1|X}$$

(see [P-S, proof of 2.3]).

By [Is 1] we know

$$N^*_{\mathrm{red}\,E_1|X} \simeq \begin{cases} \left.\begin{matrix} \mathcal{O}\oplus\mathcal{O} \\ \mathcal{O}(-1)\oplus\mathcal{O}(1) \\ \mathcal{O}(-2)\oplus\mathcal{O}(2) \\ \mathcal{O}(-4)\oplus\mathcal{O}(4) \end{matrix}\right\} & \text{conic case} \\ \left.\begin{matrix} \mathcal{O}(1)\quad\oplus\mathcal{O} \\ \mathcal{O}(2)\quad\oplus\mathcal{O}(-1) \end{matrix}\right\} & \text{line case.} \end{cases}$$

Let $(\mathrm{red}\,E)_1$ denote the 1st infinitesimal neighborhood of $(\mathrm{red}\,E)$ in Y.

Then by the exact sequence

$$0\to H^0(N^*_{\mathrm{red}\,E|X})\to H^0(\mathcal{O}_{(\mathrm{red}\,E)_1})\to H^0(\mathcal{O}_{(\mathrm{red}\,E)})\to 0$$

we get by the table for $N^*: h^0(\mathcal{O}_{(\mathrm{red}\,E)_1})\geqq 3$, hence $h^0(\mathcal{O}_{(\mathrm{red}\,\tilde{E})_1})\geqq 3$.

Now let $\tilde{\mathcal{J}}$ be the ideal sheaf of $\mathrm{red}\,\tilde{E}$ in $\tilde{E}$. Then consider

$$\begin{aligned} 0&\to H^0(\tilde{\mathcal{J}}^\nu/\tilde{\mathcal{J}}^{\nu+1})\to H^0(\mathcal{O}_{(\mathrm{red}\,\tilde{E})_\nu}\to H^0(\mathcal{O}_{(\mathrm{red}\,\tilde{E})_{\nu-1}}) \\ &\to H^1(\tilde{\mathcal{J}}^\nu/\tilde{\mathcal{J}}^{\nu+1})\to H^1(\mathcal{O}_{(\mathrm{red}\,\tilde{E})_\nu})\to H^1(\mathcal{O}_{(\mathrm{red}\,\tilde{E})_{\nu-1}})\to 0\,. \end{aligned}$$

Here $(\mathrm{red}\,\tilde{E})_\nu$ denotes the ν-th infinitesimal neighborhood of $\mathrm{red}\,\tilde{E}$ in $\tilde{E}$. Since $h^1(\tilde{\mathcal{J}}^\nu/\tilde{\mathcal{J}}^{\nu+1})\leqq 1$ for all ν (3.7) and since $h^0(\mathcal{O}_{\tilde{E}})=h^1(\mathcal{O}_{\tilde{E}})=2$ we conclude $h^0(\mathcal{O}_{(\mathrm{red}\,\tilde{E})_\nu})\leqq 2$ for all ν, contradiction and (9) is shown.

So $\tilde{r}=1$. Similar as in the case $(B\,1)$ we obtain by the exact sequence [Ba-Ka, 3.A.7]:

$$b_3(Y)=b_3(\tilde{Y})=2$$

and a contradiction as in $(B\,1)$.

This ends the proof of (3.5).

(3.6) **Proposition.** *$\tilde{E}$ is non-reduced iff* $H^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}})=0$.

Proof. If $H^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}})=0$, clearly $\tilde{E}\neq\mathrm{red}\,\tilde{E}$ since $H^1(\mathcal{O}_{\tilde{E}})\simeq\mathbb{C}$. So assume $\tilde{E}$ non-reduced. Let $\tilde{n}\subset\mathcal{O}_{\tilde{E}}$ be the sheaf of nilpotent functions on $\tilde{E}$. Then by (2.4) we have the exact sequence

$$0\longrightarrow\omega_{\mathrm{red}\,\tilde{E}}\longrightarrow\mathcal{O}_{\tilde{E}}\xrightarrow{\ \psi\ }\mathcal{H}om_{\mathcal{O}_{\tilde{E}}}(\tilde{n},\mathcal{O}_{\tilde{E}})\longrightarrow 0\,.$$

Taking cohomology and using $H^0(\mathcal{O}_{\tilde{E}})\simeq\mathbb{C}$ (3.5), moreover $H^0(\psi)\neq 0$, it follows $H^0(\omega_{\mathrm{red}\,\tilde{E}})=0$, i.e. $H^1(\mathcal{O}_{\mathrm{red}\,\tilde{E}})=0$, $\mathrm{red}\,\tilde{E}$ being Cohen-Macaulay.

(3.7) **Proposition.** a) *Let $\tilde{E}_j$ be a non-reduced component of $\tilde{E}$ such that* $\mathrm{red}\,\tilde{E}_j$ *is smooth. Then, letting $\mathcal{J}$ be the ideal sheaf of* $\mathrm{red}\,\tilde{E}$ *in $\tilde{E}$,* $(\mathcal{J}/\mathcal{J}^2|\mathrm{red}\,\tilde{E}_j)/_{\mathrm{torsion}}\simeq\mathcal{O}_{\mathrm{red}\,\tilde{E}_j}$, *and* $(\mathcal{J}^\mu/\mathcal{J}^{\mu+1}|\mathrm{red}\,\tilde{E}_j)/_{\mathrm{torsion}}$ *either contains the subsheaf* $\mathcal{O}_{\mathrm{red}\,\tilde{E}_j}$ *or is* 0.

b) *$\tilde{E}$ is reduced.*

Remark. (3.7, a) will be proved independently of (3.5)!

Proof. a) Denote by $\tilde{E}_1,\dots,\tilde{E}_s$ the irreducible components of $\tilde{E}$ with the induced structures (so $\tilde{E}_i=$ the biggest subspace of $\tilde{E}$ with underlying reduced space $\mathrm{red}\,\tilde{E}_i$). By (2.4) there is an exact sequence

$$0\to\omega_{\mathrm{red}\,\tilde{E}}\to\omega_{\tilde{E}}\to\mathcal{H}om_{\mathcal{O}_{\tilde{E}}}(\mathcal{J},\mathcal{O}_{\tilde{E}})\to 0\,. \tag{1}$$

Restricting (1) to $\operatorname{red}\tilde{E}_j$ gives

$$\omega_{\operatorname{red}\tilde{E}}|\operatorname{red}\tilde{E}_j \xrightarrow{\alpha_j} \mathcal{O}_{\operatorname{red}\tilde{E}_j} \longrightarrow \mathcal{H}om(\mathcal{I}, \mathcal{O}_{\tilde{E}})|\operatorname{red}\tilde{E}_j \longrightarrow 0. \tag{2}$$

If $\tilde{E}$ is reduced, α_j is (generically) injective; if $\tilde{E}_j$ is non-reduced, $\alpha_j = 0$ (observe that $\tilde{E}_j$ then is non-reduced everywhere because $\tilde{E}$ is a Weil divisor on the normal surface Y!).

Now take $\tilde{E}_j$ non-reduced.

We consider the canonical map

$$\phi: \mathcal{H}om(\mathcal{I}, \mathcal{O}_E)|\operatorname{red}\tilde{E}_j \to \mathcal{H}om(\mathcal{I}/\mathcal{I}^2|\operatorname{red}\tilde{E}_j, \mathcal{O}_{\operatorname{red}\tilde{E}_j}).$$

Because of (2), $\mathcal{H}om(\mathcal{I}, \mathcal{O}_{\tilde{E}})|\operatorname{red}\tilde{E}_j \simeq \mathcal{O}_{\operatorname{red}\tilde{E}_j}$.

Generically $\mathcal{I}/\mathcal{I}^2$ has rank 1, so $\mathcal{H}om(\mathcal{I}/\mathcal{I}^2|\operatorname{red}\tilde{E}_j, \mathcal{O}_{\operatorname{red}\tilde{E}_j})$ is locally free of rank 1 ($\operatorname{red}\tilde{E}_j$ is smooth). Thus ϕ is injective.

Now it is an easy exercise to show that ϕ is also surjective, i.e. any homomorphism $(\mathcal{I}/\mathcal{I}^2|\operatorname{red}\tilde{E}_j)_x \to \mathcal{O}_x$ can be lifted locally. Namely, it is sufficient to lift homomorphisms $(\mathcal{I}/\mathcal{I}^2|\operatorname{red}\tilde{E}_j)/_{\text{torsion}} \to \mathcal{O}$ locally. The left sheaf being a line bundle, this is clearly possible (for instance lift first to $\tilde{Y}$, then restrict to $\tilde{E}$).

Thus ϕ is an isomorphism, i.e.:

$$\mathcal{H}om(\mathcal{I}, \mathcal{O}_{\tilde{E}})|\operatorname{red}\tilde{E}_j \simeq \mathcal{O}_{\operatorname{red}\tilde{E}_j}. \tag{3}$$

Let $\tilde{E}_0$ be the union of all non-reduced $\tilde{E}_j$ with the induced structure. Then we obtain also:

$$\mathcal{H}om(\mathcal{I}, \mathcal{O}_{\tilde{E}})|\operatorname{red}\tilde{E}_0 \simeq \mathcal{H}om(\mathcal{I}/\mathcal{I}^2|\operatorname{red}\tilde{E}_0, \mathcal{O}_{\operatorname{red}\tilde{E}_0}) \simeq \mathcal{O}_{\operatorname{red}\tilde{E}_0}. \tag{4}$$

This proves the first part of a).

We consider the canonical homomorphism

$$\alpha: S^\mu(\mathcal{I}/\mathcal{I}^2) \to \mathcal{I}^\mu/\mathcal{I}^{\mu+1}.$$

α is an isomorphism on $\operatorname{supp}(\mathcal{I}^\mu/\mathcal{I}^{\mu+1})$ outside a finite set.

By (4), $S^\mu(\mathcal{I}/\mathcal{I}^2|\operatorname{red}\tilde{E}_0)/_{\text{torsion}} \simeq \mathcal{O}_{\operatorname{red}\tilde{E}_0}$ and via α, for any component $\tilde{E}_j$ of $\tilde{E}_0$, $(\mathcal{I}^\mu/\mathcal{I}^{\mu+1}|\operatorname{red}\tilde{E}_j)/_{\text{torsion}}$ contains the subsheaf $\mathcal{O}_{\operatorname{red}\tilde{E}_j}$ or 0. This proves the second part of a).

b) Assume that $\tilde{E}$ is non-reduced. Then from a) and $\operatorname{red}\tilde{E}_j \simeq \mathbb{P}_1$ for any j (use 3.6) we obtain:

$$h^0(\mathcal{I}^\mu/\mathcal{I}^{\mu+1}) > 0 \quad \text{as long as} \quad \mathcal{I}^\mu/\mathcal{I}^{\mu+1} \neq 0; \tag{5a}$$

$$h^1(\mathcal{I}^\mu/\mathcal{I}^{\mu+1}) = 0 \tag{5b}$$

for any μ.

Some explanation for (5b):

Denote by $\tilde{E}_0(\mu)$ the union of those $\tilde{E}_j$ for which $(\mathcal{I}^\mu/\mathcal{I}^{\mu+1}|\operatorname{red}\tilde{E}_j)/_{\text{torsion}} \neq 0$. Then

$$\mathcal{O}_{\operatorname{red}\tilde{E}_0(\mu)} \hookrightarrow (\mathcal{I}^\mu/\mathcal{I}^{\mu+1}|\tilde{E}_0(\mu))|_{\text{torsion}}$$

so by $h^1(\mathcal{O}_{\tilde{E}_0(\mu)}) = 0$ (since $h^1(\mathcal{O}_{\operatorname{red}\tilde{E}}) = 0$) we get our claim (5b).

Let $(\operatorname{red}\tilde{E})_\mu$ be the μ-th infinitesimal neighborhood of $\operatorname{red}\tilde{E}$ in $\tilde{E}$. Then by (5):

$$h^0(\mathcal{O}_{(\operatorname{red}\tilde{E})_\mu}) < h^0(\mathcal{O}_{(\operatorname{red}\tilde{E})_{\mu+1}}),$$

as long as $\mathcal{I}^\mu/\mathcal{I}^{\mu+1} \neq 0$, i.e. $(\operatorname{red}\tilde{E})_\mu \neq (\operatorname{red}\tilde{E})_{\mu+1}$. Since $h^0(\mathcal{O}_{\operatorname{red}\tilde{E}}) = 1 = h^0(\mathcal{O}_{\tilde{E}})$ by (3.5), we deduce $\operatorname{red}\tilde{E} = \tilde{E}$, a contradiction.

(3.8) **Proposition.** *$\tilde{E}$ consists of two smooth rational curves meeting in exactly one point of order two. Moreover $b_3(Y)=b_3(\tilde{Y})$.*

Proof. We have an exact sequence ([Ba-Ka, 3.A.7])

$$0=H^1(Y,\mathbb{Z})\to H^1(\tilde{Y},\mathbb{Z})\oplus H^1(E,\mathbb{Z})\to H^1(\tilde{E},\mathbb{Z})\to H^2(Y,\mathbb{Z})$$
$$\to H^2(\tilde{Y},\mathbb{Z})\oplus H^2(E,\mathbb{Z}).$$

From (3.5) we know $H^1(\tilde{Y},\mathbb{Z})=0$, (via exponential sequence) moreover $H^1(E,\mathbb{Z})=0$ by (3.2).

So $H^1(\tilde{E},\mathbb{Z})=0$.

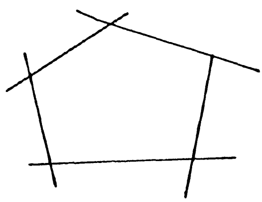
Hence (3.4.a) cannot appear and consequently ($\tilde{E}$ is connected) all irreducible components of $\tilde{E}$ are smooth rational.

Take a component $\tilde{E}_1$. Then we have the exact sequence

$$\begin{array}{l} 0\to\omega_{\tilde{E}_1}\to\omega_{\tilde{E}}\simeq\mathcal{O}_{\tilde{E}}\to\mathcal{H}om(\mathcal{J}_{\tilde{E}_1|\tilde{E}},\mathcal{O}_{\tilde{E}})\to 0 \quad (2.3). \\ \quad\ \wr\!\| \\ \mathcal{O}_{\tilde{E}_1}(-2) \end{array}$$

This sequence immediately implies that either $\tilde{E}_1$ meets exactly two components transversely in a point or meets one component in two points transversely or meets one component in one point of order two.

In the first case $\tilde{E}$ must be a cycle:



But then $H^1(\tilde{E},\mathbb{Z})\simeq\mathbb{Z}$, contradiction.

So $\tilde{E}$ is not a cycle, hence clearly $\tilde{r}\leqq 2$ (since $\tilde{E}_1$ is arbitrary in the above considerations). The case $\tilde{r}=1$ is not possible since $\omega_{\tilde{E}}\simeq\mathcal{O}_{\tilde{E}}$. If $\tilde{r}=2$ and the two curves meet in two points (transversely), then $H^1(\tilde{E},\mathbb{Z})\neq 0$. So we are left with $\tilde{r}=2$ and two smooth rational curves meeting in exactly one point of order two. It remains to prove $b_3(Y)=b_3(\tilde{Y})$. To do this we use another part of the above exact sequence:

$$\begin{array}{ccccccccccc} 0\to & H^2(Y,\mathbb{Z}) & \to H^2(\tilde{Y},\mathbb{Z})\oplus & H^2(E,\mathbb{Z}) & \to & H^2(\tilde{E},\mathbb{Z}) & \to H^3(Y,\mathbb{Z})\to H^3(\tilde{Y},\mathbb{Z})\to 0 \\ & \wr\| & & \wr\| & & \wr\| & \\ & \mathbb{Z} & & \mathbb{Z}^r & & \mathbb{Z}^2 & \end{array}$$

Since $r\leqq 2$, $b_2(\tilde{Y})>0$ and since $b_3(Y)$ and $b_3(\tilde{Y})$ are even ($b_3(Y)=b_3(X)$ and $b_3(\tilde{Y})=b_3(\hat{Y})$!), we obtain $b_3(\tilde{Y})=b_3(Y)$.

In the following we let $\pi:\hat{Y}\to\tilde{Y}$ be a minimal desingularization of $\tilde{Y}$ and let $\sigma:\hat{Y}\to Y_m$ be a minimal model of $\hat{Y}$.

(3.9) **Proposition.** *If $b_3(X)>0$, Y_m is a non-rational ruled surface. If $p: Y_m \to C_m$ denotes the ruling,* $g(C_m)=\frac{b_3(X)}{2}$.

Proof. It is clear that $\kappa(Y_m)=-\infty$ i.e. Y_m is ruled or $\mathbb{P}_2$. So it suffices to prove $g(C_m) = \frac{b_3(X)}{2}$ in case Y_m is ruled and that $Y_m \neq \mathbb{P}_2$. Let Y_m be ruled. Then $b_3(\tilde{Y})=b_3(\hat{Y}) = b_3(Y_m)=2g(C_m)$. By (3.8), $b_3(\tilde{Y})=b_3(Y)$. Since $b_3(Y)=b_3(X)$, we obtain the equation we want. If $Y_m=\mathbb{P}_2$ the same arguments show $b_3(X)=0$, contradiction.

We first consider the case that X is of the principal series, i.e. the canonical divisor is very ample. We take over all notations of Sect. 2 concerning the genus of X etc. We will make heavily use of

(3.10) **Proposition.** *Let Z be the exceptional set of π. If $b_3(X)>0$,* $\dim p \circ \sigma(Z)=1$.

Proof. Assume $\dim p \circ \sigma(Z)=0$. Then obviously $R^1\pi_*(\mathcal{O}_{\tilde{Y}})=0$ (so $\tilde{Y}$ has only rational singularities). Moreover we know $H^1(\mathcal{O}_{\tilde{Y}})=0$ by (3.5), thus $H^1(\mathcal{O}_{\hat{Y}})=0$. Hence $\hat{Y}$ is rational and $b_3(Y)=b_3(\hat{Y})=0$ (3.8).

(3.11) **Proposition.** $g(X)\geqq 8$ *(i.e.* $g(X)\in\{8, 9, 10, 12\}$*).*

Proof. Remember that X is embedded in $\mathbb{P}_{g+1}$ by the canonical divisor. If we take two smooth hyperplane sections $H, H' \subset X$, the resulting smooth curve $C=H\cap H'$ has genus $g(C)=g(X)=g$. But here we want to consider $C:=Y\cap H$. Since $H\cap S(Y) \neq \emptyset$, C becomes singular, possibly reducible. C being connected (since $H^1(\mathcal{O}_Y(-1))=0$), we have $h^1(\mathcal{O}_C)=g(X)$ (since C is a degeneration of curves of the form $H\cap H'$). Let $C_1 \subset C$ be an irreducible component. So $h^1(\mathcal{O}_{C_1})\leqq g(X)$; i.e. $g(C_1)$ (=genus of the normalization) $\leqq g(X) \leqq 7$. Let $\hat{C}_1$ be the strict transform of C_1 in $\hat{Y}$. If H is general, $\dim p\circ\sigma(\hat{C}_1)=1$. Let $\bar{C}_1 \to \hat{C}_1$ be the normalization. Then we apply Riemann-Hurwitz to the map $\bar{C}\to C_m$ which has degree say α:

$$2g(\bar{C})-2=\alpha(2g(C_m)-2)+\deg R, \tag{*}$$

R the ramification divisor.

Now $g(C_m)=\frac{b_3(X)}{2}$ by (3.9), hence by (2.2):

$$g(C_m)\geqq 5.$$

Since $g(\bar{C})\leqq 7$, we obtain from (*): $\alpha=1$ and $g(X)\geqq 5$.

Now for general H, C is irreducible and reduced. Then we obtain $(f\pi(F)\cdot H)=1$ (at least if $C\cap f(S(\tilde{Y}))=\emptyset$). So $f\pi(F)$ is a line in Y. Since $\pi \neq \mathrm{id}$ (otherwise $\dim p\circ\sigma(\hat{E})=1$ and Y would be rational!), by (3.10) all the lines of the form $f\pi(F)$ pass through one fixed point, namely the point $f(\pi(Z_i)$ where $Z_i \subset Z$ is a component with $p\circ\sigma(Z_i)=C_m$.

But every Fano 3-fold X with $g(X)\geqq 4$ (of the principal series) has the property that through any point there are only finitely many lines ([Is 1]), contradiction.

(3.12) **Proposition.** $g(X) \neq 8$.

Proof. Assume the existence of X. Then proceeding as in (3.11) and using the same notations as in (3.11), we obtain now from $g(C_m)=4$ and

$$2g(\bar{C})-2=\alpha(2g(C_m)-2)+\deg R: \tag{*}$$

$\alpha \leqq 2$ and $\alpha=2$ iff $R=0$, $g(C)=7$.

The case $\alpha=1$ is excluded as in (3.11).

So $\alpha=2$ (which means that $f\pi(F)$ is a conic, hence Y is filled up by conics through a fixed point).

Since $R=0$, $\hat{C}$ is smooth, i.e. $\bar{C}=\hat{C}$. Since $\hat{C} \to C_m$ is unramified, $C'=\sigma(\hat{C})$ is smooth. Let C_0 a section of Y_m with minimal self-intersection; G a fiber of p. Define by $C_0^2=-e$ (cp. [Ha, Chap. V, Sect. 2]). Write for numerical equivalence:

$$C' \sim 2C_0+\beta G.$$

Then the adjunction formula gives:

$$12=2g(C')-2=(-2C_0+(6-e)G \cdot 2C_0+\beta G)+C^2=2\beta-2e+12.$$

Hence $\beta=e$.

On the other hand, for general C, $\hat{C}$ is an ample divisor on $\hat{Y}$, hence $\hat{C}^2>0$. So $C'^2 \geqq \hat{C}^2>0$. But $C'^2=4\beta-4e=0$, contradiction.

(3.13) **Proposition.** $g(X) \neq 9$.

Proof. The proof being similar to (3.14) treating the case $g(X)=10$ (and in fact easier) we will omitt it.

(3.14) **Proposition.** $g(X) \neq 10$.

Proof. Assume $g(X)=10$. Then we will make use of the following construction due to Iskovskij ([Is 1]). Take a sufficiently general line $Z \subset X$. Then there are exactly four lines $Z_1, \ldots, Z_4$ meeting Z. Let $\tau_1: X_1 \to X$ be the blow-up of Z in X. Let $\tau_2: X_2 \to X_1$ be the blow-up of the strict transforms $Z_i^{(1)}$ in X_1. Let $Z^{(2)}$ be the strict transform of $\tau_1^{-1}(Z)$ in X_2, let $Z_i^{(2)}$ be the proper transforms of the $Z_i^{(1)}$.

Let $\mathscr{L}:=\tau_2 * \tau_1 * (\mathcal{O}_X(1)) \otimes \mathcal{O}(-2Z^{(2)}) \otimes \mathcal{O}(-\Sigma Z_i^{(2)})$.

Then $\mathscr{L}$ is globally generated and $h^0(X_2, \mathscr{L})=5$. Let $\phi: X_2 \to \mathbb{P}_4$ be the associated morphism. Then $\phi(X_2)$ is a smooth 3-dimensional quadric Q_3.

Moreover ϕ is birational and contracts exactly S_2 and $Z_i^{(2)}$, where S_2 is the strict transform of the surface $S \subset X$ swept out by all conics in X meeting Z. So far Iskovskij's construction.

Now denote by Y_2 the strict transform of Y in X_2 and let $\phi(Y_2)=Y_0 \subset Q_3$. Since Z is general, Z is not contained in Y. Namely, otherwise Y would be filled up lines. So the strict transforms of the lines in $\hat{Y}$ would have to be contracted by $p \circ \sigma$ (since $g(C_m)=2$ in our case!). But then all the lines would have to pass through one and the same point (because of π!) which is not possible by [Is 1]. So $Z \not\subset Y$. Since $(Z \cdot Y=1)$, we conclude $Z \cap S(Y)=\emptyset$, in particular $Z \cap E=\emptyset$. Hence for any $i: Z_i \not\subset E$. From this we deduce at once: $E \not\subset S$ (otherwise E would be a line or a conic meeting Z).

Going into the construction of Iskovskij we see that $\tau_2 \circ \tau_1 | Y_2 \to Y$ and $\phi | Y_2 \to Y_0$ are birational, moreover the set of indeterminacy of $\phi \circ (\tau_2 \circ \tau_1)^{-1}$ does not contain E. Hence Y_0 is non-normal. Now an easy calculation shows that

$$\deg Y_0 = 6 \text{ (in } \mathbb{P}_4).$$

So Y_0 is the intersection of a quadric (Q_3) and a cubic in $\mathbb{P}_4$. Taking the general quadric and the general cubic and looking at its smooth intersection Y_t, the general smooth hyperplane section C_t of Y_t will have degree 6, hence $g(C_t) = 4$ (by adjunction formula).

By degeneration we conclude for the general hyperplane section C_0 of Y_0 (C_0 being singular): $g(C_0) \leqq 3$. Let

$$f_0 : \tilde{Y}_0 \to Y_0$$

be the normalization,

$$\pi_0 : \hat{Y}^0 \to Y_0$$

a minimal desingularization. Let

$$\sigma_0 : \hat{Y}_0 \to Y_{0,m}$$

be a minimal model.

Then $Y_{0,m}$ is a ruled surface over a curve, $C_{0,m}$ of genus 2 (since $g(C_m) = 2$), denote by p_0 the projection. Let C_0 be the strict transform of C_0 in $\hat{Y}_0$, and $\bar{C}_0$ its normalization. Apply Riemann-Hurwitz to $\bar{C}_0 \to C_{0,m}$ to obtain:

$$2g(C_0) - 2 = 2\alpha_0 + \deg R_0,$$

R_0 the ramification divisor, α_0 the degree of $\bar{C}_0 \to C_{0,m}$. Now $g(C_0) \leqq 3$, hence either

a) $g(C_0) = 3$, $\alpha_0 = 1$, $\deg R_0 = 2$
b) $g(C_0) = 3$, $\alpha_0 = 2$, $R_0 = 0$
c) $g(C_0) = 2$, $\alpha_0 = 1$, $R_0 = 0$.

a) cannot occur: because of $\alpha_0 = 1$, $\sigma_0(\hat{C}_0)$ would have to be a section of $Y_{0,m}$, hence smooth. So $\hat{C}_0$ would be smooth, i.e. $\hat{C}_0 = \bar{C}_0$ and $\hat{C}_0 \to \sigma_0(\hat{C}_0)$ would be isomorphic. Hence $R_0 = 0$. Now assume b). Then we proceed as in (3.12): compute $\sigma_0(C_0)$ in $Y_{0,m}$ for numerical equivalence and conclude $\sigma_0(C_0)^2 = 0$, which is impossible (argue as in (3.12)).

So we are left with case c). So Y_0 is filled up by lines. Let α be the degree of the images of the fibers $(p \circ \sigma)^{-1}(x)$ in Y. Since Y and Y_0 are non-rational, we deduce that the images of the curves $(p \circ \sigma)^{-1}(x)$ under our birational map $Y \to Y_0$ are just the lines in Y_0. But then we have $\alpha = 1$! Namely, if Z is general, we can achieve $Z_i \not\subset Y$ for all i (since by Iskowskij any line in X meets only finitely many other lines). But then – letting $l = f\pi(p \circ \sigma)^{-1}(x)$ – we conclude

$$\alpha = (c_1(\mathcal{O}_X(1) \cdot l) = (c_1(\mathcal{L}) \cdot l_2) = (c_1(\mathcal{O}_{Q_3}(1)) \cdot \phi(l_2)) = 1$$

for general l (l_2 is the strict transform in X_2).

Conclusion: Y is filled up by lines which have all to pass through a fixed point (since $\pi \neq \mathrm{id}$ as before). This being impossible by Iskovskij the proof is finished.

(3.14) **Conclusion.** We have now proved: If X is of the principal series, then $g(X) \geqq 11$. Since by [Is 1] $g(X) \neq 11$ and $g(X) \leqq 12$, we obtain $g(X) = 12$. So it remains to exclude the cases where X is not of the principal series. These are the following ([Is 1])

a) $g(X)=2$ and the anti-canonical map $\phi_{K^{-1}}: X \to \mathbb{P}_3$ is $2:1$ and ramified in a sextic

b) $g(X)=3$ and $\phi_{K^{-1}}: X \to Q_2$ (=smooth quadric in $\mathbb{P}_4$) is $2:1$ and ramified in a surface of degree 8.

(3.15) **Proposition.** *The case* $2 \leqq g(X) \leqq 3$ *and* X *not of the principal series does not occur.*

Proof. We use in principal the same method as in (3.11). We have $g(C_m)=52$ (resp. 30) if $g(X)=2$ (resp. 3). Take $s \in H^0(\mathcal{O}_Y(1))$ general. Then $C=\{s=0\}$ is irreducible and reduced. Moreover $g(C) \leqq 2$ (resp. 3); even $g(C) \leqq 1$ (resp. 2) since C is singular. Considering the map $\hat{C} \to C_m$ we obtain a contradiction to $g(C_m)=52$ (resp. 30).

So theorem (3.1) is proved completely.

We cannot decide here whether a compactification X with $g(X)=12$ (and non-normal Y) exists. But we know something on the structure of X if it exists:

(3.16) **Theorem.** *Assume that* X *is a compactification of* $\mathbb{C}^3$ *with non-normal* Y *such that* X *is a Fano-3 fold of the principal series, of index* 1, *with* $g(X)=12$.

Then E *consists either of one smooth rational curve or of two smooth rational curves meeting transversely in one point.* $\tilde{E}$ *consists of two smooth rational curves meeting in one point of order* 2. *Moreover* E *and* $\tilde{E}$ *are reduced. Here we use the notations of* (3.2).

Proof. (3.7), (3.8). The reducedness of E follows from that one of $\tilde{E}$.

(3.17) *Remark.* In the situation of (3.16) one can say move on the singularities of Y and $\tilde{Y}$. Namely, by [K-W] or [S], for general $y \in E$ we have either $\hat{\mathcal{O}}_{Y,y} \simeq \mathbb{C}[\![X, Y]\!]/(X \cdot Y)$ or $\hat{\mathcal{O}}_{Y,y} \simeq \mathbb{C}[\![X, Y]\!]/(X^2+Y^3)$.

Here $\hat{\mathcal{O}}_{Y,y}$ denotes completion of $\mathcal{O}_{Y,y}$.

The first case occurs exactly when E is irreducible, the second when E consists of two components (then f is a homeomorphism).

Moreover the only possible singularity of $\tilde{Y}$ on $\tilde{E}$ is the point where the two components of $\tilde{E}$ intersect ([K-W]). Observe that by [S] Y is weakly normal (sometimes called maximal, cf. [F]). Let us remark that one can show that $Y \backslash E$ is smooth (a priori it could have rational double points), so $\tilde{Y}$ has at most one singularity which must be rational.

(3.18) *Remark.* If Y is assumed normal in (3.12) or (3.13) we can carry out the same construction as in the proof of (3.13) and conclude – with some minor changes in the proof – the non-existence of the compactification (X, Y). This finishes the proof of part I, Theorem 3.5, as promised.

4. A Remark on Compactifications with Index 2

(4.1) This section is joint work with Schneider and gives a supplement to [PS]. We are indebted to Furushima and N. Nakayama for very fruitful discussions.

In [PS] we proved (Theorem 2.4) that two compactifications (X, Y), (X', Y') of $\mathbb{C}^3$ with $b_2(X)=b_2(X')=1$, where X, X' are Fano 3-folds of index 2 and Y, Y' are either both normal or both non-normal are isomorphic. This means precisely the following: there is a biholomorphic map $\phi: X \to X'$ such that $\phi(Y)=Y'$. As promised in [PS] we present here some details which were omitted in [PS].

Note that it is already clear that X and X' are abstractly isomorphic, namely the Fano 3-fold V_5 of Iskovskij (cf. [Fu 1], [PS]). Moreover Y and Y' are abstractly isomorphic and the structure is well-known (see [PS], Theorem 2.4).

(4.2) Iskovskij constructed a birational morphism from the Fano 3-fold X of type V_5 to a 3-dimensional smooth quadric Q_3. This construction has been modified by Furushima [Fu 1] in the following way.

Take points $p, p_0 \in l$, a line in $Q_3 \subset \mathbb{P}_4$. Take tangent hyperplane sections H, H_0 to p, p_0. Let C be a twisted cubic contained in H_0. Necessarily $p_0 \in C$. Let $\pi: X' \to Q_3$ be the blow-up of C in Q_3. Let $\hat{H}_0$ be the strict transform of H_0 in X'. Then $\hat{H}_0 \simeq \Sigma_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ over $\mathbb{P}_1$ and X' can be blow down along the projection $\hat{H}_0 \to \mathbb{P}_1$. We obtain a modification $X' \to X$ and thus a birational map $X \to Q_3$. σ is the just the blowup of a line $l_0 \subset X$ with $N_{l_0|X} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$. If we set $Y = \sigma\pi^{-1}(C)$, then Y is a non-normal hypersurface in X with non-normal locus l_0 and $X \backslash Y \simeq \mathbb{C}^3$. Moreover *all* compactifications (X, Y) (with X of type V_5) arise in this way. Remark that the strict transform of Y in X' is just Σ_3 and that π contracts exactly the strict transforms of the lines in Y (Y can be described as the surface of lines meeting l_0). The last facts follow from [PS].

Now consider the strict transform A' of H in Y'. Let $A := \sigma(A')$. Then (X, A) is a compactification of $\mathbb{C}^3$ with A normal and all "normal" compactifications arise in this manner [Fu 1].

(4.3) Let (X, Y), (X', Y') be two smooth compactifications of $\mathbb{C}^3$ such X, X' is of type V_5. Assume either both Y and Y' are normal or non-normal. Proving the existence of a biholomorphic map $\phi: X \to X'$ such that $\phi(Y)=Y'$ comes down (by (4.2)) to prove the following.

(4.4) **Theorem.** *Let $(Q, \tilde{Q}, C, l, q)$ be a quintuple consisting of a smooth 3-dimensional quadric $Q \subset \mathbb{P}_4$, a twisted cubic curve $C \subset Q$, the uniquely determined quadric cone $\tilde{Q} \subset Q$ containing C, the uniquely determined line $l \subset Q$ such that $l \cap C$ is the vertex of $\tilde{Q}$ and a point $q \in l$. Let $(Q', \tilde{Q}', C', l', q')$ be another quintuple of this type. Then there exists a biholomorphic map $\phi: Q \to Q'$ such that $\phi(\tilde{Q})=\tilde{Q}'$, $\phi(C)=C'$, $\phi(l)=l'$, $\phi(q)=q'$.*

Proof. The proof is given in several steps which are well-known and whose proofs are very easy (thus omitted).

1. We may assume $Q=Q'$ and $\tilde{Q}=\tilde{Q}'$ (since there is $\psi: Q \to Q'$ biholomorphic such that $\psi(\tilde{Q})=\tilde{Q}'$).

2. For any quadric cone $\tilde{Q} \subset \mathbb{P}_3$ and $x \in \tilde{Q}$, $x' \in \tilde{Q}$ there is $\psi \in \mathrm{Aut}(\mathbb{P}_3)$ such that $\psi(x)=x'$, $\psi(C)=C'$.

3. If $C \subset \mathbb{P}_3$ is a twisted cubic, $x \in C$, then there is a uniquely determined quadric cone $\tilde{Q} \subset \mathbb{P}_3$ such that $C \subset \tilde{Q}$ and x is the vertex of $\tilde{Q}$.

4. Put $x :=$ vertex of $\tilde{Q}$ in our situation.

By 2) we find $\psi \in \mathrm{Aut}(\mathbb{P}_3)$ such that $\psi(C)=C'$ and $\psi(x)=x$. By 3) we conclude $\psi(Q)=\tilde{Q}$ since the vertex of $\psi(\tilde{Q})$ is x. Hence we have $\phi \in \mathrm{Aut}(\tilde{Q})$ such that $\phi(C)=C'$. Then automatically $\phi(l)=l'$!

5. Now lift ϕ to an automorphism $\tilde{\phi} \in \mathrm{Aut}(Q)$. This is possible since the restriction map

$$\mathrm{Aut}_{\tilde{Q}}(Q) \to \mathrm{Aut}(\tilde{Q})$$

(from the group of automorphisms of Q fixing $\tilde{Q}$ to $\mathrm{Aut}(\tilde{Q})$) is an isomorphism. In fact, it is sufficient to see $\dim \mathrm{Aut}_{\tilde{Q}}(Q)=\dim \mathrm{Aut}(\tilde{Q})=7$ and injectivity of the restriction map.

6. Still we have to see that we can achieve $\phi(q)=q'$. To do this we just mention that any $\psi \in \mathrm{Aut}(C)$ with $\psi(p)=p$ can be liftet to $\tilde{\psi} \in \mathrm{Aut}(\tilde{Q})$ with $\psi(C)=C$, hence to $\tilde{\tilde{\psi}} \in \mathrm{Aut}(Q)$.

Thus the group of automorphisms ϕ constructed in 5) acts transitively on C, q.e.d.

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Note added in proof. Recently M. Furushima proved that there exists a compactification of $\mathbb{C}^3$ with non-normal boundary at infinity which is a Fano 3-fold of index 1 of "Mukai-Umemura" type.