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Mazur's Intersection Property for Finite Dimensional Sets

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Introduction

In this paper we consider only real Banach spaces. This is not a restriction since the properties we consider in this paper depend only on the real structure of the space.

In Theorem 1 we give dual characterizations for the properties (I_n) defined by:

> Every convex compact set with *affine* dimension at most n, is an intersection of balls.

This will be used to characterize the property:

> Every finite dimensional convex compact set is an intersection of balls. $\Big\}$ ($\mathrm{I}_{f,d}$)

We also prove that for n-dimensional Banach spaces, property (I_{n-1}) implies property (I_n), and examples are constructed to show that this result is the best possible.

At the end we give (without proofs) some stability results for property ($\mathrm{I}_{f,d}$), and we ask whether every Banach space can be renormed to have property (I_1). We also mention an application of Theorem 1 to spaces of compact operators.

Notation

A point x of a Banach space X is said to be an extreme point if $x=0$ or if $x/\|x\|$ is an extreme point of $B(X)$, the unit ball of X. The set of extreme points of X will be denoted by $\mathrm{Ext}(X)$.

For a finite dimensional set C, $\dim C$ will always mean the *affine* dimension of C.

A slice of a bounded set C (in some Banach space X) is a subset of C of the form:

$$S(C,f,\delta)=\left\{x\in C: f(x)>\left(\sup_C f\right)-\delta\right\}$$

for some $f\in X^*$, $\delta>0$.

The closed ball [resp. open ball, resp. sphere] centered at x and with radius r will be denoted by $B(x,r)$ [resp. $\mathring{B}(x,r)$; resp. $S(x;r)$].

Results

Our main result is the following theorem, which is analogous in its spirit to Theorem 1 of [4].

Theorem 1. *For every Banach space X, and every natural number n, the following properties are equivalent:*

(1) *Every compact convex subset C of X with* $\dim C \leqq n$, *is an intersection of balls.*

(2) *For every $f \in X^*$, every $(n+1)$-points $(x_i)_{0 \leqq i \leqq n} \in X$, and every $\varepsilon > 0$, there exists $g \in \mathrm{Ext}(X^*)$ such that* $\sup_{0 \leqq i \leqq n} |x_i(f-g)| < \varepsilon$.

This theorem will be an immediate consequence of the more precise result stated in the next lemma. But we need first to introduce some notation.

For a bounded convex set C, let $\hat{C}$ denote the intersection of the balls containing C, and define $\varrho(C) = \sup_{x \in \hat{C}} \operatorname{dist}(x, C)$.

If X is a Banach space, and n an integer, let $\Lambda = \Lambda_n = \sup\{\varrho(C) : C \subset B(X), \dim C \leqq n\}$. [Hence X has (I_n) if and only if $\Lambda_n = 0$.] We also define

$$\lambda = \lambda_n = \inf \Big\{ \mu > 0 \text{ such that} : \forall f \in S(X^*),$$
$$\forall C \subset B(X), \dim C \leqq n, \exists g \in \mathrm{Ext}(X^*) : \sup_C |f-g| \leqq \mu \Big\}.$$

With these notations, a "quantitative version" of Theorem 1 is given by:

Lemma 2. $\dfrac{\lambda}{2} \leqq \Lambda \leqq 2\lambda$.

Proof of $\lambda \leqq 2\Lambda$. It is enough to prove that given $C \subset B(X)$, $\dim C \leqq n$, and $f \in S(X^*)$, there exists for every $\eta > 0$, an element $g \in \mathrm{Ext}(X^*)$ such that $\sup_C |f-g| \leqq 2(\Lambda + \eta)$.

Let C, f, and η be as before.

If $m = \sup_C |f| \leqq 2(\Lambda + \eta)$, take $g = 0$.

If $m \geqq 2(\Lambda + \eta)$, choose $u_0 \in K = cv(\pm C)$ such that $f(u_0) = m$, and let $u = \dfrac{\Lambda + \eta}{m} u_0$. Define also $D = K \cap \ker f$.

It is clear that $D \subset B(X)$, $\dim D \leqq n$, and $\operatorname{dist}(u, D) \geqq \Lambda + \eta$. By definition of Λ, there exists a Ball $B(z, r)$ containing D and not containing u.

Let w be the unique element of $S(z, r) \cap cv[z, u]$, x the norm one vector $\dfrac{w-z}{r}$, and choose an extreme point h of $B(X^*)$ such that $h(x) = 1$.

One can easily check that $0 \leqq \sup_D h \leqq \sup_{B(z,r)} h < h(u)$. So there exists $\alpha > 0$, such that $\sup_K \alpha h = 1$, and from the above inequalities we deduce that $\sup_D \alpha h < \alpha h(u) \leqq \dfrac{\Lambda + \eta}{m}$.

Then, by Phelps' lemma [applied to the Banach space $sp(K)$ with K as a unit ball], we obtain $\left\|\frac{f}{m} \pm \alpha h\right\|_K \leqq 2\,\frac{\Lambda+\eta}{m}$. This concludes the proof of "$\lambda \leqq 2\Lambda$" since both $\pm\alpha m h$ are in $\mathrm{Ext}(X^*)$.

Proof of $\Lambda \leqq 2\lambda$. It is enough to prove that given $C \subset B(X)$, $\dim C \leqq n$, then $x \in X$ can be separated from C by a ball whenever $\mathrm{dist}(x, C) > 2\lambda$.

Let C and x be as above. We can suppose that $x \in B(X)$, since if not $B(X)$ separates C and x.

Let $K = \dfrac{C-x}{2} \subset B(X)$, and observe that to separate x and C, it is enough to separate 0 and K.

Since $\mathrm{dist}(x, C) > 2\lambda$, we have $\mathrm{dist}(0, K) > \lambda$, which means that $K \cap (\lambda B(X)) = \emptyset$. This implies that we can find $f \in S(X^*)$ such that $\inf_K f > \lambda$, and by the definition of λ, we can find $g \in \mathrm{Ext}(B(X^*))$ such that $\inf_K g = 3\varepsilon > 0$.

Since g is an extreme point of $B(X^*)$, by a well known result due to Choquet, we can find $x \in S(X)$, $\delta > 0$ such that:

$$g \in S(B(X^*); x, \delta) \subset \left\{h \in B(X^*) : \sup_K |g-h| < \varepsilon\right\}.$$

We are going to prove that there exists an $r > 0$ such that K is included in $D_r = B(r\varepsilon x, (r-1)\varepsilon)$. This will conclude the proof, since none of the balls D_r contain 0.

Indeed, if not, by a compacity argument, and since the balls D_r are increasing (with r), the set $L = \bigcap_{r>0} (K \backslash \mathring{D}_r)$ will be non empty.

Take an element $y \in L$, and for every $r > 0$, let $g_r \in S(X^*)$ be such that $g_r(r\varepsilon x - y) = \|r\varepsilon x - y\| \geqq (r-1)\varepsilon$. This inequality implies easily that $\lim_{r=\infty} g_r(x) = 1$, and so $g_r \in S(B(X^*), x, \delta)$ for r large enough.

On the other hand, it is not difficult to see that $(g - g_r)(y) \geqq 2\varepsilon$, and so $\sup_K |g - g_r| \geqq 2\varepsilon$, for every $r > 0$. This conclusion contradicts the preceding one by the choice of x and δ.

This completes the proof of the lemma. □

That Lemma 2 implies Theorem 1 is an immediate consequence of the following geometrical observation: If C is a compact convex set with $\dim C \leqq n$, there exists $(n-1)$-points $(x_i)_{0 \leqq i \leqq n}$ such that $C \subset K = cv\{x_i : 0 \leqq i \leqq n\}$ (and K also satisfies $\dim K \leqq n$).

In the sequel we will list some consequences of Theorem 1.

Corollary 3. *For every Banach space, the following properties are equivalent:*

(1) *Every finite dimensional compact convex subset of X is an intersection of balls.*

(2) *The set* $\mathrm{Ext}(X^*)$, *of extreme points of X^*, is w^*-dense in X^*.*

Proposition 4. *Let X be a Banach space such that for every $n \geqq 1$, X has an equivalent norm $|\cdot|_n$ which satisfies property* (I_n). *Then X has a equivalent norm $[\![\cdot]\!]$ which satisfies property* $(\mathrm{I}_{f,d})$.

Proof. Let us denote by $\|\cdot\|$ the original norm of X. It is easy to see that we can suppose that $|\cdot|_n \geqq \|\cdot\|$ for every n, and hence $|\cdot|_n^* \leqq \|\cdot\|^*$ (on X^*).

Define on X^* an equivalent dual norm by:

$$[\![x^*]\!] = \left(\sum_{n=1}^{\infty} \frac{1}{2^n} |x^*|_n^{*2}\right)^{1/2}$$

and let us prove that its predual norm works.

An easy (and standard) convexity argument shows that $\mathrm{Ext}(X^*_{[\![\cdot]\!]}) \supset \bigcup_{n \geqq 1} \mathrm{Ext}(X^*_{|\cdot|_n})$. This implies by Theorem 1 that $\mathrm{Ext}(X^*_{[\![\cdot]\!]})$ is w^*-dense in X^*, and the conclusion follows by Corollary 3. □

Proposition 5. *Let E be an n-dimensional Banach space with property* (I_{n-1}). *Then E has property* (I_n).

Proof. We will prove that under the hypothesis of Proposition 5, the set of extreme points of $B(E^*)$ is norm dense in $S(E^*)$. This clearly implies the conclusion in view of Theorem 1 (see also [1, 3]).

Let $(e_i)_{1 \leqq i \leqq n}$ be a basis of E, then for every $f \in S(E^*)$, and every $\varepsilon > 0$, there exists $\delta > 0$, such that for every $g \in E^*$, $\|f-g\| < \varepsilon$ whenever $\sup_{1 \leqq i \leqq n} |e_i(f-g)| < \delta$.

Assuming (I_{n-1}), for every $f \in S(E^*)$, we can find $g \in \mathrm{Ext}(E^*)$ such that $\sup_{1 \leqq i \leqq n} |e_i(f-g)| < \delta$, hence $\|f-g\| < \varepsilon$, and then $\left\| f - \frac{g}{\|g\|} \right\| < 2\varepsilon$. □

The result of Proposition 5 cannot be improved as it is shown by the following:

Proposition 6. *For every natural numbers k and n such that $n \geqq k+2$, there exists on $\mathbb{R}^n$ an equivalent norm $|\cdot|_{n,k}$ satisfying* (I_k) *but not* (I_{k+1}).

Remark. The norm $|\cdot|_{n,k}$ (we will define) is given by $|x|_{n,k} = \left(\sum_{i=1}^{k+1} |\tilde{x}_i|^2\right)^{1/2}$, where $(\tilde{x}_i)_{1 \leqq i \leqq n}$ is the decreasing rearrangement of $(|x_i|)_{1 \leqq i \leqq n}$. But this formula is of no help in proving the proposition.

Proof of Proposition 6. Let n and k be fixed natural numbers such that $n \geqq k+2$.

We need first to introduce some notation and to prove a preliminary result.

Let $(e_i)_{1 \leqq i \leqq n}$ be the natural basis of $\mathbb{R}^n$, $\|\cdot\|$ be the Euclidean norm, $\langle \cdot, \cdot \rangle$ the Euclidean scalar product, and S the Euclidean unit sphere.

Let $\mathscr{P} = \{A \subset [1, n] : \operatorname{card} A = k+1\}$, and for every $A \in \mathscr{P}$ define $H_A = sp[e_i : i \in A]$. Define also the sets $\mathscr{E} = \bigcup_{A \in \mathscr{P}} (S \cap H_A)$, and $C = cv(\mathscr{E})$, and let us prove that $\mathrm{Ext}(C) = \mathscr{E}$.

By the Krein-Milman theorem, since $\mathscr{E}$ is closed, it is enough to prove that $\mathscr{E} \subset \mathrm{Ext}(C)$, and to do this, it is again enough to prove that if $x, x_1, \ldots, x_p \in \mathscr{E}$, $\lambda_1, \ldots, \lambda_p \in \mathbb{R}^+$ are such that $x = \sum_{i=1}^{p} \lambda_i x_i$ and $\sum_{i=1}^{p} \lambda_i = 1$, then $x = x_1 = \ldots = x_p$.

Let us prove that the above statement is true. Let $A \in \mathscr{P}$ be such that $x \in H_A$, and denote by P_A the orthogonal projection on H_A.

From $x = \sum_{i=1}^{p} \lambda_i x_i$ we deduce that $x = \sum_{i=1}^{p} \lambda_i P_A(x_i)$ which implies that $x = P_A(x_1) = \ldots = P_A(x_p)$ by the properties of the Euclidean norm.

In particular we have $\|P_A(x_i)\| = 1$, for every i, and so $P_A(x_i) = x_i$ (since $\|x_i\| = 1$). This proves that $x = x_i = \ldots = x_p$, and concludes the proof of $\mathscr{E} = \operatorname{Ext}(C)$.

Let us return now to the proof of Proposition 6. Since C is convex, symmetric, closed, and with no empty interior, C defined a (dual) norm on $\mathbb{R}^n$, the (pre-) dual of which we will denote by $|\cdot|_{n,k} = |\cdot|$, i.e., $B(\mathbb{R}^n_{|\cdot|}) = C^0$ (*the polar set of* C).

But what we have proved in the preliminary part we have that

$$\operatorname{Ext}(\mathbb{R}^n_{|\cdot|*}) = \mathbb{R} \cdot \mathscr{E} = \bigcup_{A \in \mathscr{P}} H_A .$$

Let now $f \in \mathbb{R}^n$, $(x_i)_{0 \leqq i \leqq k} \in \mathbb{R}^n$, and suppose that $(x_i)_{0 \leqq i \leqq l}$ is a *maximal* linearly independent subfamily of $(x_i)_{0 \leqq i \leqq k}$. Then choose an $A \in \mathscr{P}$ such that $((x_i)_{0 \leqq i \leqq l}; (e_j)_{j \notin A})$ is still linearly independent, and find $g \in \mathbb{R}^n$ such that $\langle g, x_i \rangle = \langle f, x_i \rangle$ for $0 \leqq i \leqq l$, and $\langle g, e_j \rangle = 0$ for every $j \notin A$.

Such a g is in H_A, hence $g \in \operatorname{Ext}(\mathbb{R}^n_{|\cdot|*})$, and also is such that $\sup\limits_{0 \leqq i \leqq k} |x_i(f-g)| = 0$ (by the "maximality" of the chosen subfamily).

Theorem 1 implies then that $\mathbb{R}^n_{|\cdot|}$ has (I_k).

On the other hand let $f \in \mathbb{R}^n$ be such that $\langle f, e_i \rangle = 1$ for $1 \leqq i \leqq k+2$. Then there is no element $g \in \operatorname{Ext}(\mathbb{R}^n_{|\cdot|*})$ such that $\sup\limits_{1 \leqq i \leqq k+2} |e_i(f-g)| < 1$.

Indeed for every $A \in \mathscr{P}$, the set $[1, n] \backslash A$ intersects $[1, k+2]$ (cardinality argument), then for every $g \in H_A$, we have $\sup\limits_{1 \leqq i \leqq k+2} |e_i(f-g)| \geqq 1$.

Theorem 1 again implies that $\mathbb{R}^n_{|\cdot|}$ fails (I_{k+1}). □

Remark. Using the same proofs* as in [4], one can obtain the following results:

1) If $T: X \to Y$ is such that T and T^* are injective, and if Y has an equivalent $(\mathrm{I}_{f,d})$-norm then X has also an equivalent $(\mathrm{I}_{f,d})$-norm.

2) Every Banach space has an equivalent $(\mathrm{I}_{f,d})$-norm if and only if the above result is true without the hypothesis "T^* injective".

3) If $(P_\alpha)_{0 \leqq \alpha \leqq \mu}$ is a Schauder decomposition for the Banach space X, such that for every α, $0 \leqq \alpha \leqq \mu$, the space $(P_{\alpha+1} - P_\alpha)(X)$ has an equivalent $(\mathrm{I}_{f,d})$-norm, then X has an equivalent $(\mathrm{I}_{f,d})$-norm.

Using the Hahn-Banach theorem, one can easily see that for every Banach space X, the set $\operatorname{Ext}(X^*)$ of extreme points of X^* intersects all the *affine*, w^*-closed, 1-codimensional subspaces of X^*.

In view of this one can ask the following:

Problem. Does every Banach space X have an equivalent norm such that $\operatorname{Ext}(X^*)$ intersects all the *affine*, w^*-closed, 2-codimensional subspaces of X^*?

A positive answer to this will imply that every Banach space has an equivalent (I_1)-norm. Up to now it is unknown if this conclusion holds even for Asplund spaces.

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* We cannot reproduce these proofs because of their length

Remark. In [5], Theorem 1 is used to prove that the spaces $K(X, Y)$ and $X \underset{\varepsilon}{\otimes} Y$ (with their usual norms) never have property (I_2) if $\dim X \geqq 2$ and $\dim Y \geqq 2$, and Theorem 1 is also used to show that the space $K(l_2^2) = l_2^2 \underset{\varepsilon}{\otimes} l_2^2$ has property (I_1).

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