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Complemented Infinite Type Power Series Subspaces of Nuclear Fréchet Spaces

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The space of rapidly decreasing sequences s plays a prominent role in the theory of nuclear Fréchet spaces. In this article, we prove among other things, that if s is isomorphic to a subspace of a nuclear Fréchet space E, then E has a complemented subspace isomorphic to s. We note that we do not assume the existence of a basis in E. In fact, with the additional assumption that E has a strong finite dimensional decomposition, Holmström [8] obtained the same result.

We deal with the problem in a more general setting by assuming that there is what we call a local imbedding $i:\lambda(A)\to E$ and an imbedding $j:E\to\lambda(A)^{\mathbb{N}}$, where $\lambda(A)$ is a stable, nuclear G_∞-space. In this setting we prove that E has a complemented subspace which is isomorphic to $\lambda(A)$. By Vogt's characterization of subspaces of s [23], any subspace of s, which has a regular basis, can be expressed as a nuclear G_∞-space.

Pelczynski's decomposition method [13] has been adopted by Vogt [24] to apply to nuclear, stable power series spaces, so that if a Fréchet space E is isomorphic to a complemented subspace of a stable nuclear power series space $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\alpha)$ in turn isomorphic to a complemented subspace of E, one concludes that E and $\Lambda_\infty(\alpha)$ are in fact isomorphic. This powerful method has been used extensively by Vogt in [24, 25]. As a corollary of our result we improve this method so that one can reach the same conclusion by only requiring that there is a local imbedding of $\Lambda_\infty(\alpha)$ into E and, as before, that E is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$.

By the well-known Komura-Komura imbedding theorem [9], every nuclear Fréchet space is isomorphic to a subspace of $s^{\mathbb{N}}$. Even in the case of a nuclear, stable G_∞-space, the existence of an imbedding $j:E\to\lambda(A)^{\mathbb{N}}$ can be expressed in terms of the diametral dimension simply as $\Delta(\lambda(A))\subset\Delta(E)$ [16, 17]. In order to apply our version of the decomposition method effectively, in the second section we deal with the problem of the existence of a local imbedding $i:\lambda(A)\to E$. In [14] Pelczynski asked whether a complemented subspace E of a nuclear Köthe space has a basis. In its generality this is still an open problem. For a complemented subspace E of s, Wagner [31] has proved that if E is isomorphic to $E\times E$, then it has a basis. As an important application of the improved decomposition method, we show that if E

and $E \times E$ have equal diametral dimensions, then E is isomorphic to a power series space $\Lambda_\infty(\alpha)$. For other positive answers to Pelczynski's problem in the case of complemented subspaces of s, we refer to [5] and [11]. Using our result, it is a simple matter to conclude that the space of analytic functions $O(M)$ on a Stein manifold M of dimension d has property (DN) of Vogt [23] if and only if $O(M)$ is isomorphic to the space of entire functions $O(\mathbb{C}^d)$.

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We use the standard terminology and notation of the theory of locally convex spaces as in [10]. For nuclear spaces we refer to [15]. Throughout E will denote a Fréchet space over the real $\mathbb{R}$ or the complex field $\mathbb{C}$ with a fundamental sequence of seminorms $\| \ \|_0 \leqq \| \ \|_1 \leqq \ldots$ and $U_k = \{x \in E : \|x\|_k \leqq 1\}$. $L(E,F)$ is the space of continuous linear maps from E into F and $B(E,F)$ the closed unit ball of $L(E,F)$ provided E and F are Banach spaces.

A Fréchet space E is said to have property (DN) if it has a fundamental sequence of seminorms such that for each k there is a p and $C>0$ with $\|x\|_k \leqq r\|x\|_0 + (C/r)\|x\|_p$ for all $x \in E$ and $r>0$. A nuclear Fréchet space has (DN) if and only if it is isomorphic to a subspace of the space of rapidly decreasing sequences s [23]. E has property (Ω) if for every p there is a q such that for every k there is a j and $C>0$ with

$$U_q \subset C r^j U_k + \frac{1}{r} U_p$$

for all $r>0$. For nuclear Fréchet spaces, the condition (Ω) characterizes quotient spaces of s [29].

The diametral dimension $\Delta(E)$ of E is the set of all sequences (ξ_n) such that for every k there is a p with $\lim \xi_n d_n(U_p, U_k) = 0$, where $d_n(U_p, U_k)$ denotes the n-th Kolmogorov diameter of U_p with respect to U_k [15, 20]. For the calculation of $\Delta(E)$ in case E has (DN) or (Ω) we shall refer to [22]. In particular if a nuclear Fréchet space E has (DN) and (Ω), for $p=0$ we find q_0 as in the (Ω) condition and set $\alpha_n = -\log d_n(U_{q_0}, U_0)$. We then have $\Delta(E) = \Delta(\Lambda_\infty(\alpha))$ [22, *Sect.* 3, (2)].

A Köthe space $\lambda(A)$ which satisfies the following conditions is called a G_∞-space [4, 20]

(1) $a_n^0 = 1$ and $a_n^k \leqq a_{n+1}^k$

(2) for every k there is a p with $((a_n^k)^2 / a_n^p) \in \ell_\infty$.

Power series spaces of infinite type are certainly the best known examples of G_∞-spaces. $\lambda(A)$ is nuclear if and only if $(1/a_n^k) \in \ell_1$ for some k [20]. Any subspace of s, which has a regular basis, can be expressed as a G_∞-space [23] (cf. also [18]). A Fréchet space E is called *stable* if it is isomorphic to $E \times E$. Stability of a G_∞-space is simply equivalent to: for each k there is a p with $(a_{2n}^k / a_n^p) \in \ell_\infty$ [21]. We shall use the following equivalent fundamental sequence of norms for a nuclear G_∞-space:

$$|x|_k = \sum_{n=1}^{\infty} |x_n| a_n^k \quad \text{and} \quad \|x\|_k = \left(\sum_{n=1}^{\infty} (|x|_n a_n^k)^2 \right)^{1/2} .$$

The diametral dimension is a complete isomorphic invariant for the class of nuclear G_∞-spaces, since $\Delta(\lambda(A))=\lambda(A)'$ [20].

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A continuous linear map $i:\lambda(A)\to E$ will be called a *local imbedding* if there is a continuous seminorm $\|\ \|$ on E such that $|x|_0\leqq \|ix\|$ holds. Certainly an imbedding of $\lambda(A)$ is a local imbedding and a local imbedding is one-to-one. The map which sends each $x\in\Lambda_\infty(\alpha)$ to $(x_nR^{\alpha_n})$ for some fixed $R>1$ is a local imbedding of $\Lambda_\infty(\alpha)$ into the finite type power series space $\Lambda_1(\alpha)$ and it is easily seen to be compact. However, a power series space $\Lambda_\infty(\alpha)$ of infinite type is not necessarily isomorphic to a subspace of $\Lambda_1(\alpha)$. Even this simple example shows that a local imbedding can be quite different than an imbedding.

If E is isomorphic to a subspace of a nuclear locally convex space F, we have $\Delta(F)\subset\Delta(E)$ [20]. Here F need not be metrisable. Our first result indicates that in terms of diametral dimension a local imbedding has the same effect as an imbedding.

1.1. Proposition. *If there is a local imbedding of a nuclear G_∞-space $\lambda(B)$ into a nuclear locally convex space F, then $\Delta(F)\subset\Delta(\lambda(B))$.*

Proof. Let $i:\lambda(B)\to F$ be a local imbedding. We have $|x|_0\leqq\|ix\|$ for some continuous semi-norm $\|\ \|$ on F. Let $U=\{y\in i(\lambda(B)):\|y\|\leqq 1\}$, and $(\xi_n)\in\Delta(F)$. Since $\Delta(F)\subset\Delta(i(\lambda(B))$ there is a neighborhood V of $i(\lambda(B))$ with $\lim\xi_n d_n(V,U)=0$. By continuity of i we find k and $C_k>0$ with $P_V(ix)\leqq C_k|x|_k$, $x\in\lambda(B)$. Now if $V\subset dU+L$ where L is a subspace of $i(\lambda(B))$ with dimension not exceeding n and $d>0$, using the fact that i is $1-1$, we find a subspace $\tilde{L}$ of $\lambda(B)$ of dimension not exceeding n, $i(\tilde{L})=L$ and so get

$$U_k\subset C_k dU_0+\tilde{L}.$$

Hence

$$(b_n^k)^{-1}=d_n(U_k,U_0)\leqq C_k d_n(V,U)$$

and therefore $(\xi_n)\in\lambda(B)'=\Delta(\lambda(B))$.

Throughout the rest of this section we let $\lambda(A)$ stand for a *stable, nuclear G_∞-space*. We note that even in case of an imbedding $i:\lambda(A)\to\lambda(A)$, it may happen that $i(\lambda(A))$ is not complemented. For example, we have an exact sequence

$$0\longrightarrow s\xrightarrow{\ i\ }s\longrightarrow s^{\mathbb{N}}\longrightarrow 0$$

[23] and here $i(s)$ certainly not complemented in s.

For the canonical basis (e_n^j) of $\lambda(A)^{\mathbb{N}}$ we have

$$\|e_n^j\|_k=\begin{cases}a_n^k & \text{for } j\leqq k\\ 0 & \text{for } j>k,\end{cases}$$

where $(\|\ \|_k)$ is the sequence of standard Hilbertian seminorms on the product space $\lambda(A)^{\mathbb{N}}$. With the bijection $\beta:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ defined by $\beta(j,n)=2^{j-1}(2n-1)$ we set $f_{\beta(j,n)}=e_n^j$. Hence $\|f_\nu\|_k\leqq a_\nu^k$. Using the stability of $\lambda(A)$ we find r_k, $D_k>0$ with

$$a_{2^k\nu}^k\leqq D_k a_\nu^{r_k}.$$

So if $\|f\|_k \neq 0$, then $f_\nu = e_n^j$ with $j \leqq k$ and

$$a_\nu^k = a_{2^{j-1}(2n-1)}^k \leqq D_k a_n^{r_k} = D_k \|f_\nu\|_{r_k}.$$

Hence we either have $\|f_\nu\|_k = 0$ or $a_\nu^k \leqq D_k \|f_\nu\|_{r_k}$. These simple observations will be used in the proof of the following result which is crucial in the subsequent development.

1.2. Proposition. *Let $i: \lambda(A) \to \lambda(A)^{\mathbb{N}}$ be a local imbedding. Then there is a complemented subspace G of $\lambda(A)^{\mathbb{N}}$ which is isomorphic to $\lambda(A)$ and contained in $\overline{i(\lambda(A))}$.*

Proof. For each k, we find $m(k)$, $C_k > 0$ so that for all $x \in \lambda(A)$ we have

$$\|ix\|_k \leqq C_k |x|_{m(k)}$$

and

$$|x|_0 \leqq C_0 \|ix\|,$$

where $\| \ \|$ is a suitable semi-norm on the nuclear space $\lambda(A)^{\mathbb{N}}$ defined by a scalar product $(\ , \ )$ and we may initially arrange things so that $\|y\| = \|y\|_{m(0)}$ holds for $y \in \lambda(A)^{\mathbb{N}}$. We note that $\| \ \|$ is in fact a norm on $i(\lambda(A))$. Let Q be the projection on $\lambda(A)^{\mathbb{N}}$ with $Q^{-1}(0) = \{y \in \lambda(A)^{\mathbb{N}} : \|y\| = 0\}$. By stability of $\lambda(A)$, the range of Q is isomorphic to $\lambda(A)$ itself.

We now construct a basic sequence (g_n) in $i(\lambda(A))$. We choose $g_1 \in \mathrm{sp}\{ie_1, ie_2\}$ with $(g_1, f_1) = 0$ and $\|g_1\| = 1$. We want to select g_n with the following properties:

(i) $g_n \in \mathrm{sp}\{ie_1, \ldots, ie_{2n}\}$.
(ii) $(g_n, f_\nu) = (g_n, g_j) = 0$ for $\nu = 1, \ldots, n$ and $j = 1, \ldots, n-1$.
(iii) $\|g_n\| = 1$.

Suppose we have already determined $g_1, \ldots, g_{n-1}$. Since $\mathrm{sp}\{ie_1, \ldots, ie_{2n}\}$ is a $2n$-dimensional space, we can find $\tilde{g}_n \neq 0$ in this space with $(\tilde{g}_n, f_\nu) = 0$ for $\nu = 1, \ldots, n$ and $(\tilde{g}_n, g_j) = 0$ for $j = 1, \ldots, n-1$. We simply let $g_n = (1/\|\tilde{g}_n\|)\tilde{g}_n$. Further if

$$g_n = i\left(\sum^{2n} \mu_j^n e_j\right)$$

we have

$$\sum^{2n} |\mu_j^n| = \left|\sum^{2n} \mu_j^n e_j\right|_0 \leqq C_0 \|g_n\| = C_0$$

and therefore

$$\|g_n\|_k \leqq C_k \left|\sum^{2n} \mu_j^n e_j\right|_{m(k)} \leqq C_k \sum^{2n} |\mu_j^n| a_j^{m(k)} \leqq C_k C_0 a_{2n}^{m(k)}.$$

Using the stability of $\lambda(A)$, for each k we determine s_k, $\varrho > 0$ such that the inequality

(iv) $\|g_n\|_k \leqq \varrho a_n^{s_k}$

holds for all $n \in \mathbb{N}$.

At this stage we note that we can replace g_n by Qg_n in (ii), (iii), and (iv), because $(Qg_n, y) = (g_n, y)$ for all $y \in \lambda(A)^{\mathbb{N}}$, and $\|Qg_n\|_k \leqq \|g_n\|_k$.

For $x=\sum x_\nu f_\nu \in \lambda(A)^{\mathbb{N}}$ we have

$$\begin{aligned}|(g_n, x)|\,\|g_n\|_k &\leqq \sum_{\nu>n} |(g_n, f_\nu)|\,|x_\nu|\,\|g_n\|_k\\ &\leqq \varrho \sum_{\nu>n} |x_\nu|\,\|f_\nu\|_{m(0)} a_n^{s_k}\\ &\leqq \varrho \sum_{\nu>n} |x_\nu|\,\|f_\nu\|_{m(0)} a_\nu^{s_k}.\end{aligned}$$

For n fixed, let $I(j)=\{\nu:\nu>n, \|f_\nu\|_j \neq 0\}$. From above we get the estimate

$$|(g_n, x)|\,\|g_n\|_k \leqq \varrho \sum_{\nu\in I(m(0))} |x_\nu|\, a_\nu^{m(0)} a_\nu^{s_k}.$$

Using the fact that $\lambda(A)$ is a nuclear G_∞-space, we now determine $m_k > m(0)$, σ_k such that

$$\varrho a_n^{m(0)} a_n^{s_k} \leqq \sigma_k n^{-2} a_n^{m_k}$$

holds and from $I(m(0)) \subset I(m_k)$ obtain

$$|(g_n, x)|\,\|g_n\|_k \leqq \sigma_k n^{-2} \sum_{\nu\in I(m_k)} |x_\nu|\, a_\nu^{m_k}.$$

If we select r_k and $C>0$ such that $a_j^{m_k} \leqq C\|f_\nu\|_{r_k}$ whenever $\|f_\nu\|_{r_k} \neq 0$, the above estimate yields

(v) $|(g_n, x)|\,\|g_n\|_k \leqq \sigma_k C n^{-2} |x|_{r_k}$.

Hence we can define a continuous operator P on $\lambda(A)^{\mathbb{N}}$ by

$$Px=\sum_{n=1}^{\infty} (g_n, x) g_n.$$

We have $P(g_n)=g_n \in i(\lambda(A))$ and so P is a projection. Its range G is contained in $\overline{i(\lambda(A))}$ and (g_n) is a basis of G.

To conclude the proof it remains to show that G is isomorphic to $\lambda(A)$. Since (Qg_n) satisfies (ii), (iii), (iv), and (v), we can define another projection P_0 on $\lambda(A)^{\mathbb{N}}$ by

$$P_0 x=\sum_{n=1}^{\infty} (Qg_n, x) Qg_n$$

so that the range G_0 of P_0 has (Qg_n) as a basis and it is contained in $Q(\lambda(A)^{\mathbb{N}})$, which is isomorphic to $\lambda(A)$. Hence, as a complemented subspace of $\lambda(A)$, G_0 is isomorphic to some G_∞-space $\lambda(B)$ [21, Theorem 3.1]. So $\lambda(A)'=\Delta(\lambda(A)) \subset \Delta(G_0) =\lambda(B)'$. Since $(Qg_n, y)=(g_n, y)$ for all $y\in\lambda(A)^{\mathbb{N}}$, P_0 and P have equal kernels and therefore G_0 is isomorphic to G. Hence it remains to show that G_0 is isomorphic to $\lambda(A)$. For this purpose we must prove $\lambda(B)'=\Delta(G_0) \subset \lambda(A)'$, but this is an immediate consequence of 1.1 Proposition.

We would like to note that if $i:\lambda(A)\to\lambda(A)^{\mathbb{N}}$ were to be an imbedding, then the space G would be contained in $i(\lambda(A))$ and one could show that it is isomorphic to $\lambda(A)$ itself without having to introduce G_0. It should also be noticed how extensively the stability of $\lambda(A)$ is used in the proof.

A Fréchet space E is said to be $\lambda(A,\mathbb{N})$-nuclear if $\lambda(A)' \subset \Delta(E)$. This generalization of $\Lambda_{\mathbb{N}}(\alpha)$-nuclearity [17] was introduced by Ramanujan and Rosenberger [16].

1.3. Theorem. *If E is $\lambda(A, \mathbb{N})$-nuclear and if there is a local imbedding of $\lambda(A)$ into E, then E has a complemented subspace isomorphic to $\lambda(A)$.*

Proof. Let $i: \lambda(A) \to E$ be a local imbedding and $j: E \to \lambda(A)^{\mathbb{N}}$ an imbedding, whose existence is equivalent to the $\lambda(A, \mathbb{N})$-nuclearity of E [16, 17]. Then $ji: \lambda(A) \to \lambda(A)^{\mathbb{N}}$ is a local imbedding and so by 1.2 Proposition there is a subspace G isomorphic to $\lambda(A)$ which is contained in $\overline{ji(\lambda(A))} = \overline{ji(\overline{\lambda(A)})} \subset j(E)$ and G is complemented in $\lambda(A)^{\mathbb{N}}$.

With the additional assumptions that E has a basis and there is an imbedding of $\lambda(A)$ into E, Holmström [7] reached the same conclusion as in 1.3 Theorem. In a subsequent work [8] he obtained the following corollary, by assuming E has a strong finite dimensional decomposition and a subspace isomorphic to s.

1.4. Corollary. *If there is a local imbedding of s into a nuclear Fréchet space E, then E has a complemented subspace which is isomorphic to s.*

Let $\Lambda_\infty(\alpha)$ be a stable nuclear power series space of infinite type. Complemented subspaces of $\Lambda_\infty(\alpha)$ have been characterized by Vogt and Wagner [30] as those $\Lambda_{\mathbb{N}}(\alpha)$-nuclear Fréchet spaces which have the properties (DN) and (Ω). Vogt's decomposition theorem [24] in this case states that if $\Lambda_\infty(\alpha)$ is isomorphic to a complemented subspace of E, where E is a complemented subspace of $\Lambda_\infty(\alpha)$, then E must be isomorphic to $\Lambda_\infty(\alpha)$. An immediate consequence of 1.3 Theorem is the following improvement of Vogt's decomposition method.

1.5. Corollary. *Let $\Lambda_\infty(\alpha)$ be nuclear and stable. Let E be a $\Lambda_{\mathbb{N}}(\alpha)$-nuclear Fréchet space with (DN) and (Ω). If there is a local imbedding of $\Lambda_\infty(\alpha)$ into E, then E is isomorphic to $\Lambda_\infty(\alpha)$.*

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As we have already pointed out, the existence of an imbedding $j: E \to \lambda(A)^{\mathbb{N}}$ can be expressed in terms of the diametral dimensions of $\lambda(A)$ and E. In order to exploit the decomposition method given in the previous section more fully, we need to know more about the existence of local imbeddings. We start with an example. We believe that this can serve as a model for constructing a local imbedding into a space of functions. In fact, quite a number of Fréchet spaces of functions can be represented as power series spaces [19, 24, 25].

Let M be an irreducible Stein space of dimension d and let $T: \Lambda_\infty(n^{1/d}) \to O(\mathbb{C}^d)$ be an isomorphism [19]. We determine $C>0$ and $R_0>1$ so that the inequality

$$|x|_0 \leqq C \sup\{|Tx(z)| : z \in R_0 \Delta^d\}$$

holds, where $R_0 \Delta^d$ is the polydisc in $\mathbb{C}^d$ around zero with multiradius R_0. Fix $R_1 > R_0$. Choose a regular point $\xi_0 \in M$ and find $f_i \in O(M)$. $i = 1, \ldots, d$, such that $f_i(\xi_0) = 0$ and $F = (f_1, \ldots, f_d): M \to \mathbb{C}^d$ has rank d at ξ_0 [6, p. 209]. By composing F with a linear transformation of $\mathbb{C}^d$ if necessary, we can determine a relatively compact neighborhood U_0 of ξ_0 such that F maps U_0 onto $R_1 \Delta^d$ biholomorphically. We define $F_*: O(\mathbb{C}^d) \to O(M)$ by $F_*(f)(\xi) = f(F(\xi))$, $\xi \in M$. F_* is in fact a continuous algebra homomorphism. Furthermore we have

$$\sup |f(z)| : z \in R_0 \Delta^d\} \leqq \sup\{|g(F(\xi))| : \xi \in U_0\}.$$

Hence $F_*T: \Lambda_\infty(n^{1/d}) \to O(M)$ is a local imbedding.

Let M be now a Stein manifold of dimension d which is always assumed to be *connected*. By the Oka-Cartan theorem [6], $O(M)$ has property (Ω) and $O(M)$ is isomorphic to a subspace of $\Lambda_1(n^{1/d})$ [2]. If $O(M)$ has property (DN), then it is already $\Lambda_{\mathbb{N}}(n^{1/d})$-nuclear [22, 30]. Now that we know the existence of a local imbedding $i: \Lambda_\infty(n^{1/d}) \to O(M)$, 1.5 Corollary yields the following result.

2.1. Proposition. *Let M be a Stein manifold of dimension d. The space of analytic functions $O(M)$ has property (DN) if and only if it is isomorphic to $O(\mathbb{C}^d)$.*

We note that this proposition was already proved in [2] under the additional assumption that $O(M)$ has a basis. In [34] Zaharjuta states that $O(M)$ is isomorphic to $O(\mathbb{C}^d)$ if it satisfies (DN), but in the discussion he also seems to assume that $O(M)$ has a basis. In the case of $d=1$, our proposition was proved in [1] and [11] by different methods. Zaharjuta [32] gave another characterization of $O(\mathbb{C})$. Aytuna and Vogt showed that $O(M)$ has property (DN) if and only if every bounded plurisubharmonic function on M is constant [1]. In case M is an algebraic variety, Mitiagin and Henkin [12] asked whether $O(M)$ is isomorphic to $O(\mathbb{C}^d)$. This was answered positively by Zaharjuta [33], Djakov and Mitiagin [3] and also by Vogt [24] as an application of the decomposition method.

A nuclear Fréchet space E is isomorphic to a complemented subspace of s if and only if it has the properties (DN) and (Ω) [29, 1.10 Satz]. In this case $\Delta(E)$ is equal to the diametral dimension of some $\Lambda_\infty(\alpha)$ [22]. Also, if a complemented subspace of s has a basis, it is isomorphic to some $\Lambda_\infty(\alpha)$ [29, 2.9 Satz]. However, whether a complemented subspace of a nuclear Köthe space has a basis, is an open problem which has been posed by Pelczynski [14]. Wagner [31] has proved that a complemented subspace E of s, which is stable, (i.e. E is isomorphic to $E \times E$), has a basis. Krone [11] reached the same conclusion under the assumption $\Delta(E \times E) = \Delta(E)$ and $\alpha_n \geqq n$ where $\Delta(E) = \Delta(\Lambda_\infty(\alpha))$. In contrast to some kind of stability which is assumed in these and in the following, Dubinsky and Vogt [5] (cf. also [27, 7.2, 7.3] have proved that a complemented subspace of an unstable power series space of infinite type always has a basis. We note that we can obtain 2.1 Proposition also as a corollary of the following theorem.

2.2. Theorem. *Let E be a nuclear Fréchet space with (DN) and (Ω). If $\Delta(E \times E) = \Delta(E)$, then E is isomorphic to some $\Lambda_\infty(\alpha)$.*

Proof. We have $\Delta(E) = \Lambda_\infty(\alpha)'$, $\Delta(E \times E) = \Delta(\Lambda_\infty(\alpha) \times \Lambda_\infty(\alpha))$ and so $\Lambda_\infty(\alpha)$ is stable. Since E is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$ [30, 3.5 Satz], from 1.5 Corollary and the following lemma we reach the conclusion.

2.3. Lemma. *Let E be a nuclear space with (DN) and (Ω). If $\Delta(E) \subset \Delta(\lambda(B))$, then there is a local imbedding of the G_∞-space $\lambda(B)$ into E.*

Proof. Without loss of generality we may assume that E has a fundamental sequence of norms $\| \;\|_k$, where each $\| \;\|_k$ is defined by an inner product. Since $\lambda(B)$ has (DN) and E has (Ω), by various results of Vogt [28, 5.1 Theorem 3.3 Lemma, 3.4

Proposition] we have that for every μ there is an m such that for all k and $r>0$ the following holds:

$$L(\lambda(B), E_m) \subset L(\lambda(B), E_k) + rB(\lambda(B)_0, E_\mu).$$

In this condition, called $(\tilde{S}_1)$ by Vogt [26, 2.2 Theorem], E_k is the Hilbert space obtained by completing $(E, \| \ \|_k)$ and $\lambda(B)_0$ the completion of $(\lambda(B), | \ |_0)$. We choose integers (m_k) increasing to infinity with $m_0=0$ such that $(\tilde{S}_1)$ holds for $\mu=m_{k-1}, m=m_k, k=m_{k+1}$ and further, we may arrange things so that for every i we have a j, $C>0$ with

$$U_{m_1} \subset Cr^j U_i + \frac{1}{r} U_0, \ r>0.$$

To simplify notation we set $k=m_k$ and so we have

$$L(\lambda(B), E_k) \subset L(\lambda(B), E_{k+1}) + rB(\lambda(B)_0, E_{k-1})$$

and $\Delta(E)=\Lambda_\infty(\alpha)'=\Delta(\Lambda_\infty(\alpha))$ where $\alpha_n=-\log d_n(U_1, U_0)$. Since $(e^{\alpha_n}) \in \lambda(B)'$ $=\Delta(\lambda(B))$, for some j and $C>0$ the inequality $(1/d_n(U_1, U_0)) \leqq Cb_n^j$ holds for all n. By nuclearity of $\lambda(B)$ we find $C_0>0$ and k_0 with $|x|_0 \leqq C_0 \|x\|_{k_0}$. So there is a k_1 with $b_n^{k_0} \leqq Cb_n^{k_1} d_n(U_1, U_0)$. Since $\lim d_n(U_1, U_0)=0$, the linking map $\varrho_{1,0}: E_1 \to E_0$ is compact and hence it can be written in the form

$$\varrho_{1,0} y = \sum d_n(U_1, U_0)(y|f_n) g_n,$$

where (f_n) and (g_n) are orthonormal sequences in E_1 and E_0 respectively. We define $T_1: \lambda(B) \to E_1$ by

$$T_1 x = \sum b_n^{k_0} (d_n(U_1, U_0))^{-1} x_n f_n.$$

Then $\|T_1 x\|_1 \leqq C \|x\|_{k_1}$ and also

$$|x|_0 \leqq C_0 \|x\|_{k_0} = C_0 (\sum |x_n|^2 (b_n^{k_0})^2)^{1/2} = C_0 \|T_1 x\|_0.$$

We choose $\varepsilon_i>0$ with $\sum \varepsilon_i \leqq (1/2C_0)$. By $(\tilde{S}_1)$ we choose $T_2: \lambda(B) \to E_2$ such that

$$\|T_1 x - T_2 x\|_0 \leqq \varepsilon_1 |x|_0.$$

Then

$$\left(\frac{1}{C_0} - \varepsilon_1\right) |x|_0 \leqq \|T_2 x\|_0.$$

Applying $(\tilde{S}_1)$ repeatedly, for each k we find $T_k \in L(\lambda(B), E_k)$ such that

$$\|T_k x - T_{k+1} x\|_{k-1} \leqq \varepsilon_k |x|_0$$

and

$$\left(\frac{1}{C_0} - \sum_{i=1}^{k-1} \varepsilon_i\right) |x|_0 \leqq \|T_k x\|_0$$

hold for all $x \in \lambda(B)$. So for each k we have a map $S_k \in L(\lambda(B), E_k)$ defined by $S_k x = \lim\limits_i T_{k+i} x$ and $\varrho_{k+1,k} S_{k+1} = S_k$. Thus we have obtained a continuous linear map $T: \lambda(B) \to E$ such that $\varrho_k T = S_k$, where $\varrho_k: E \to E_k$ is the canonical inclusion.

Further T satisfies

$$\frac{1}{2C_0}|x|_0 \leqq \|Tx\|_0$$

and therefore it is a local imbedding.

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