Math. Ann. 283, 211–239 (1989)

Mathematische Annalen

Asymptotic Behavior of Fundamental Solutions and Potential Theory of Parabolic Operators with Variable Coefficients

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1. Introduction

In $\mathbf{R}^{n+1}$ we consider the second order parabolic differential equation

$$Lu = \operatorname{div}(A(x,t)\nabla_x u) - D_t u = 0, \tag{1.1}$$

where $A(x,t) = (a_{ij}(x,t))$ is a real, symmetric, matrix-valued function on $\mathbf{R}^{n+1}$ with C^∞ entries. We assume there exists $\nu \in (0,1]$ such that for every $(x,t) \in \mathbf{R}^{n+1}$ and every $\xi \in \mathbf{R}^n$

$$\nu|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leqq \nu^{-1}|\xi|^2. \tag{1.2}$$

We also assume, as this is not restrictive for our purposes, that there exists a compact set $F_0 \subset \mathbf{R}^{n+1}$ containing the origin such that for $i,j = 1, \ldots, n$

$$a_{ij}(x,t) = \delta_{ij} \quad \text{for} \quad (x,t) \in \mathbf{R}^{n+1} \backslash F_0. \tag{1.3}$$

Then a fundamental solution Γ for (1.1) exists and under the assumptions made on $A(x,t)$ Γ is C^∞ off the diagonal in $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, see e.g., [Fr]. Throughout this paper we will use the letters z, z_0, ζ to denote respectively the points (x,t), (x_0,t_0), (ξ,τ) in $\mathbf{R}^{n+1}$. Then if $z \in \mathbf{R}^{n+1}$ and $r > 0$, we set

$$\Omega_r(z) = \{\zeta \in \mathbf{R}^{n+1} | \Gamma(z;\zeta) > (4\pi r)^{-n/2}\} \tag{1.4}$$

and

$$\Psi_r(z) = \{\zeta \in \mathbf{R}^{n+1} | \Gamma(z;\zeta) = (4\pi r)^{-n/2}\}^-, \tag{1.5}$$

where for a subset $E \subset \mathbf{R}^{n+1}$, E^- denotes its closure. We call $\Omega_r(z)$ and $\Psi_r(z)$ respectively the *parabolic ball* and the *parabolic sphere* "centered" at z and of radius r. If $A(z) = \text{Identity}$, and hence L in (1.1) is the heat operator on $\mathbf{R}^{n+1}$, $H = \Delta - D_t$, then Γ is given by the Gauss-Weierstrass kernel

$$K(x,t;\xi,\tau) = K(x-\xi;t-\tau) = \begin{cases} (4\pi(t-\tau))^{-n/2} \exp\left(-\dfrac{|x-\xi|^2}{4(t-\tau)}\right), & t > \tau, \\ 0, & t \leqq \tau. \end{cases} \tag{1.6}$$

In this case $\Omega_r(z)$ is a football-shaped domain in $\mathbf{R}^{n+1}$ whose intersection with hyperplanes perpendicular to the time axis are n-dimensional balls

$$|x-\xi|^2 \leqq R_r(t-\tau), \qquad t-r<\tau<t, \tag{1.7}$$

where $R_r(t-\tau)=2n(t-\tau)\ln\left(\frac{r}{t-\tau}\right)$. We remark that the "center" z of the parabolic ball $\Omega_r(z)$ lies on the boundary $\Psi_r(z)$ of the ball itself. Throughout the paper dH_n stands for n-dimensional Hausdorff measure. In [GL] we proved

Theorem A. *Let $u \in C^\infty(\mathbf{R}^{n+1})$ and let $z \in \mathbf{R}^{n+1}$. For a.e. $r>0$ we have*

$$\begin{aligned}-\int_{\Psi_r(z)} u(\zeta)A(\zeta)(\nabla_\xi\Gamma(z;\zeta))\cdot\vec{N}_\xi(\zeta)dH_n(\zeta) &= u(z)\\ &+\int_{\Omega_r(z)} Lu(\zeta)[\Gamma(z;\zeta)-(4\pi r)^{-n/2}]d\zeta.\end{aligned} \tag{1.8}$$

For every $r>0$ we have

$$\begin{aligned}(4\pi r)^{-n/2}\int_{\Omega_r(z)} u(\zeta)\frac{A(\zeta)(\nabla_\xi\Gamma(z;\zeta))\cdot\nabla_\xi\Gamma(z;\zeta)}{\Gamma^2(z;\zeta)}d\zeta &= u(z)\\ +\frac{n}{2}r^{-n/2}\int_0^r l^{n/2}\int_{\Omega_l(z)} Lu(\zeta)[\Gamma(z;\zeta)-(4\pi l)^{-n/2}]d\zeta\frac{dl}{l}.&\end{aligned} \tag{1.9}$$

In (1.8) $\vec{N}_\xi(\zeta)$ denotes the spatial component of the outer normal $\vec{N}(\zeta)=(\vec{N}_\xi(\zeta), N_\tau(\zeta))$ in ζ to the surface $\Psi_r(z)$.

It appears clear from Theorem A that in order to fully use (1.8), (1.9) we must have as much information as possible about the kernels appearing in them. This leads to the study of the asymptotic behavior for small times of the fundamental solution Γ and of its derivatives. Section 2 in this paper is devoted to this purpose. Our main result there, Theorem 2.1, reads: *if x is sufficiently close to y and if $\Gamma(x,y,t)=\Gamma(x,t;y,0)$, then as $t\to 0^+$*

$$\Gamma(x,y,t)\sim(4\pi t)^{-n/2}\exp\left(-\frac{d^2(x,y,t)}{4t}\right)\sum_{j=0}^{\infty} t^j u_j(x,y,t). \tag{1.10}$$

A similar result holds for the derivatives of Γ.

For a complete explanation of (1.10) we refer to Sect. 2. We only mention that $d(x,y,t)$ represents the Riemannian distance generated on $\mathbf{R}^n$ at the time t by the metric $g_{ij}(t)dx_i\otimes dx_j$, where $g_{ij}(t)(x)=a^{ij}(x,t)$, and $(a^{ij}(x,t))=A^{-1}(x,t)$. For time-independent parabolic operators (1.10) is a classical result of Minashisundaram and Pleijel, see [BGM]. A different approach based on transmutation formulas is due to Kannai [K]. Theorem 2.1 plays a basic role in this paper. It has also been crucial in our recent work [GL] on Wiener's criterion for the operator L in (1.1) which has extended Evans and Gariepy's previous result for the heat operator [EG]. To provide further motivation for the results in this paper we describe a lemma in [GL] somewhat of a geometric flavor the proof of which ultimately relies on Theorem 2.1. If $K(x,t)=K(x,t;0,0)$ is as in (1.6) and if $E(x,t)=\ln K(x,t)$, then for $t>0$ we have

$$\nabla_x E(x,t)=-\frac{x}{2t}, \qquad D_t E(x,t)=\frac{|x|^2}{4t^2}-\frac{n}{2t}.$$

From this it is then immediate to see that given any $\theta > 1$ the inequality

$$|\nabla_x E(x,t)|^2 \leqq \theta D_t E(x,t) \tag{1.11}$$

holds iff $t \leqq \delta|x|^2$ where $\delta = \delta(\theta) = (\theta - 1)/2n\theta$, i.e., iff (x,t) lies outside a paraboloid with vertex at $(0,0)$ and aperture depending on θ. Let now $\Gamma(x,t) = \Gamma(x,t;0,0)$ be the fundamental solution of (1.1) with pole at $(0,0)$ and let us again set $E(z) = E(x,t) = \ln \Gamma(x,t)$. Then in [GL] we proved the following

Lemma B. *There exist $r_0 > 0$, and for every $\theta > 1$ a $\delta = \delta(\theta) > 0$, such that*

$$A(z)(\nabla_x E(z)) \cdot \nabla_x E(z) \leqq \theta D_t E(z), \tag{1.12}$$

for every $z = (x,t) \in \{\zeta \in \mathbf{R}^{n+1} | \Gamma(\zeta;0) > (4\pi r)^{-n/2}\}$ with $t \leqq \delta|x|^2$, and every $r \leqq r_0$.

We remark that in the intrinsic notation of Theorem 2.1 below (1.12) can be rewritten as

$$|\nabla_{M_t} E(z)|_t^2 \leqq \theta D_t E(z), \tag{1.13}$$

where ∇_{M_t} is the gradient and $|\cdot|_t$ is the norm in the metric $g_{ij}(t)dx_i \otimes dx_j$. A comparison of (1.11) and (1.13) unravels the deep connection between the geometry induced on $\mathbf{R}^n$ by the matrix $A(x,t)$ in (1.1) and the fundamental solution of the operator L. Whereas the proof of (1.11) is a simple calculation based on the explicit knowledge of the fundamental solution K of the heat operator, the proof of Lemma B is quite delicate and uses the full strength of Theorem 2.1. (1.11) was first observed by Evans and Gariepy who used it as a key step in the proof of a strong version of Harnack's inequality for the heat operator, see Lemma 3.2 in [EG]. Roughly speaking, the latter asserts that if u is nonnegative solution of $Hu = 0$ in a heat ball $\Omega_{2r}(0)$, then the infimum of u in a smaller ball "concentric" to $\Omega_{2r}(0)$, say $\Omega_{r/2}(0)$, is strictly positive (and independent of r!) provided that the n-dimensional average of u at the time level $t = -r$ is one. The interest of this result is that it provides a stronger information than the normal Harnack inequality for parabolic equations and cannot be derived from it. This is so because the parabolic ball $\Omega_{r/2}(0)$ contains regions which lie outside any paraboloid with vertex in $0 \in \mathbf{R}^{n+1}$ and aperture in the negative time direction. Using Lemma B we proved a similar result for the operator L in (1.1), see Theorem 1.4 in [GL].

Before proceeding with the plan of the paper we pause to provide some historical background. If in (1.8) we take u such that $Lu = 0$ we obtain

$$u(z) = -\int_{\Psi_r(z)} u(\zeta) A(\zeta)(\nabla_\xi \Gamma(z;\zeta)) \cdot \vec{N}_\xi(\zeta) dH_n(\zeta). \tag{1.14}$$

The ancestor of (1.14) is the well-known formula

$$u(x) = \frac{1}{n\omega_n r^{n-1}} \int_{|x-y|=r} u(y) d\sigma(y), \tag{1.15}$$

valid for any harmonic function u in $\mathbf{R}^n$, any $x \in \mathbf{R}^n$ and $r > 0$ (ω_n = volume of n-dimensional unit ball). (1.15) is the keystone of classical potential theory. Immediate consequences of it are the maximum principle, Harnack's inequality, the smoothness of harmonic functions (Weyl's lemma), just to name a few, see e.g., [H] and [F]. Since in (1.14)

$$\vec{N}_\xi(\zeta) = -\frac{\nabla_\xi \Gamma(z;\zeta)}{|(\nabla_\xi \Gamma(z;\zeta), D_\tau \Gamma(z;\zeta))|},$$

if A in (1.1) is the identity matrix we obtain

$$u(z) = \int_{\Psi_r(z)} u(\zeta) \frac{|\nabla_\xi K(z-\zeta)|^2}{|(\nabla_\xi K(z-\zeta), D_\tau K(z-\zeta))|} dH_n(\zeta). \tag{1.16}$$

(1.16) was first discovered and used by Pini in [P 1] – [P 3] for the heat operator in $\mathbf{R}^2$. Fulks [Fu] subsequently extended Pini's result to any number of variables. Watson [W] starting from Fulks' formula (1.16) obtained the following representation

$$u(z) = (4\pi r)^{-n/2} \int_{\Omega_r(z)} u(\zeta) \frac{|x-\xi|^2}{4(t-\tau)^2} d\zeta, \tag{1.17}$$

for any solution u of the $Hu=0$, any $z=(x,t) \in \mathbf{R}^{n+1}$ and $r>0$. If we observe that

$$|\nabla_\xi K(z-\zeta)|^2 = \frac{|x-\xi|^2}{4(t-\tau)^2} K(z-\zeta)^2,$$

we see that (1.17) is just a special case of

$$u(z) = (4\pi r)^{-n/2} \int_{\Omega_r(z)} u(\zeta) \frac{A(\zeta)(\nabla_\xi \Gamma(z;\zeta)) \cdot \nabla_\xi \Gamma(z;\zeta)}{\Gamma^2(z;\zeta)} d\zeta, \tag{1.18}$$

valid for any solution of (1.1), any $z \in \mathbf{R}^{n+1}$ and $r>0$. (1.14) was found by Fabes and one of us in [FG], and (1.18) was established in the same paper using (1.14) and Federer's coarea formula, see [Fe]. Theorem A above extends all previous results and can be used to study parabolic potential theory.

An unfavorable feature displayed by (1.18) consists in the unboundedness of the kernel appearing in it. This can be easily seen in the case of the heat operator [see (1.17)], where the kernel takes the form $(4\pi r)^{-n/2} \frac{|x-\xi|^2}{4(t-\tau)^2}$. In Sect. 3 we prove some new representation formulas for smooth functions on $\mathbf{R}^{n+1}$ containing kernels which are not only bounded, but also possess a degree of regularity arbitrarily high. These new formulas are based on the following observations. If $m \in \mathbf{N}$ and $y \in \mathbf{R}^m$ we define a parabolic operator on $\mathbf{R}^{n+m+1}$ by setting $\hat{L} = L + \Delta_y$, if L is as in (1.1). Then if u is a function on $\mathbf{R}^{n+1}$ and for $y \in \mathbf{R}^m$ we define

$$\hat{u}(x,y,t) = u(x,t)$$

we have

$$\hat{L}\hat{u}(x,y,t) = Lu(x,t).$$

Moreover, if $\hat{\Gamma}$ and Γ are respectively the fundamental solutions of $\hat{L}$ and L, then it turns out that

$$\hat{\Gamma}(x,y,t;\xi,\eta,\tau) = \Gamma(x,t;\xi,\tau) K_m(y-\eta; t-\tau) \tag{1.19}$$

where K_m is the Gauss-Weierstrass kernel in $\mathbf{R}^{m+1}$, see (1.6). With these observations in mind we apply the representation results in [GL] to the function $\hat{u}$ and the operator $\hat{L}$ on $\mathbf{R}^{n+m+1}$ above defined. Because of (1.19) and the fact that the matrix $\hat{A}$ of $\hat{L}$ is given by the $(n+m)\times(n+m)$ block matrix

$$\hat{A}=\begin{bmatrix} A & 0 \\ 0 & I_y \end{bmatrix},$$

something magic happens and the dependence in the added variable $y\in\mathbf{R}^m$ disappears in all the integrals involved. We refer to Theorem 3.1 below for details. We only quote here a special case. For $m\in\mathbf{N}$ we define the *modified parabolic ball* centered at $z=(x,t)\in\mathbf{R}^{n+1}$ and of radius $r>0$

$$\Omega_r^m(z)=\left\{\zeta=(\xi,\tau)\in\mathbf{R}^{n+1}|(4\pi(t-\tau))^{-m/2}\Gamma(z;\zeta)>(4\pi r)^{-\frac{n+m}{2}}\right\}, \tag{1.20}$$

see (1.4). Then if $u\in C^\infty(\mathbf{R}^{n+1})$ is a solution of $Lu=0$ in $\mathbf{R}^{n+1}$ we have

$$u(z)=\int_{\Omega_r^m(z)} u(\zeta)E_r^{(m)}(z;\zeta)d\zeta. \tag{1.21}$$

If on $\Omega_r^m(z)$ we define the function $\zeta\mapsto R_r(z;\zeta)$ by setting

$$R_r^2(z;\zeta)=4(t-\tau)\ln\left[(4\pi r)^{\frac{n+m}{2}}(4\pi(t-\tau))^{-m/2}\Gamma(z;\zeta)\right],$$

then in Sect. 3 we prove that the kernel $E_r^{(m)}$ in (1.21) is given by

$$E_r^{(m)}(z;\zeta)=(4\pi r)^{-\frac{n+m}{2}}\omega_m R_r^m(z;\zeta)\left[\frac{A(\zeta)(\nabla_\xi\Gamma(z;\zeta))\cdot\nabla_\xi\Gamma(z;\zeta)}{\Gamma^2(z;\zeta)}+\frac{m}{m+2}\frac{R_r^2(z;\zeta)}{4(t-\tau)^2}\right], \tag{1.22}$$

where ω_m denotes the volume of the m-dimensional unit ball. We emphasize that if we agree to set $\omega_0=1$, then we obtain from (1.22)

$$E_r^{(0)}(z;\zeta)=(4\pi r)^{-n/2}\frac{A(\zeta)(\nabla_\xi\Gamma(z;\zeta))\cdot\nabla_\xi\Gamma(z;\zeta)}{\Gamma^2(z;\zeta)},$$

and therefore (1.21) gives (1.18) back if $m=0$.

In Sect. 4 using (1.10) and similar expansions for the derivatives of Γ we prove that if $m\in\mathbf{N}$ and $m>2$, then the kernel $E_r^{(m)}$ in (1.21) is bounded by an appropriate power of r on the modified parabolic ball $\Omega_r^m(z)$. This fact is used in Theorem 4.1 to give an elementary proof of Harnack's inequality modelled on the classical proof via mean value formulas for harmonic functions. The result itself is clearly not new as Moser [M] has proved Harnack's inequality for parabolic operators with bounded measurable coefficients. Our point however is to emphasize the elementary and self-contained character of the proof.

In Sect. 5 we investigate several questions in classical potential theory related to the averaging operators $u\mapsto u_r^{(m)}$ where

$$u_r^{(m)}(z)=\int_{\Omega_r^m(z)} u(\zeta)E_r^{(m)}(z;\zeta)d\zeta, \tag{1.23}$$

for $z \in \mathbf{R}^{n+1}$, $r>0$, $m \in \mathbf{N} \cup \{0\}$. To be more specific, we need to introduce some definition. A bounded open set $U \subset \mathbf{R}^{n+1}$ is said to be *L-regular* if for any $\varphi \in C(\partial U)$ there exists a (unique) $H_\varphi^U \in C^\infty(U) \cap C(\bar{U})$ such that $LH_\varphi^U = 0$ in U and for which

$$\lim_{\substack{z \to z_0 \\ z \in U}} H_\varphi^U(z) = \varphi(z_0)$$

for every $z_0 \in \partial U$. Given an open set $D \subset \mathbf{R}^{n+1}$ a function $w: D \to \bar{\mathbf{R}}$ is said to be *L-superparabolic* in D if: (i) $-\infty < w \leqq +\infty$, $w < +\infty$ in a dense subset of D; (ii) w is lower semi-continuous (l.s.c.); (iii) for every L-regular subset $U \subset \bar{U} \subset D$, and every $\varphi \in C(\partial U)$ if $w|_{\partial U} \geqq \varphi$, then $w \geqq H_\varphi^U$ in U. In Proposition 5.1 we show that L-superparabolic functions are characterized by the super mean value property, i.e., a l.s.c. function on $\mathbf{R}^{n+1}$ is L-superparabolic iff for every $z \in \mathbf{R}^{n+1}$ and r sufficiently small

$$u(z) \geqq u_r^{(m)}(z) \quad \text{for any fixed } m \in \mathbf{N} \cup \{0\}.$$

The rest of Sect. 5 is devoted to proving Proposition 5.2 (see also Corollary 5.1) which states that for a L-superparabolic function on $\mathbf{R}^{n+1}$ the averages $u_r^{(m)}(z)$ increase as $r \to 0$ to the value of the function u at z. Moreover, for any $r>0$ small enough the function $z \mapsto u_r^{(m)}(z)$ is itself L-superparabolic in $\mathbf{R}^{n+1}$. The proof of Proposition 5.2 is accomplished in several steps. The crucial one is to prove the following property of the fundamental solution Γ of L. If for $\zeta \in \mathbf{R}^{n+1}$ we set $w = \Gamma(\cdot\,;\zeta)$ and denote by w_r^σ the surface average of w defined as in (1.14), then for every ϱ, r sufficiently small and $z \in \mathbf{R}^{n+1}$ we have

$$(w_r^\sigma)_\varrho(z) \leqq w_r^\sigma(z). \tag{1.24}$$

$(w_r^\sigma)_\varrho(z)$ denotes the solid average defined by the right-hand side of (1.18) of the function $z \mapsto w_r^\sigma(z)$. (1.24) implies that a similar inequality holds for the solid averages of Γ, i.e.,

$$((\Gamma(\cdot\,;\zeta))_r)_\varrho(z) \leqq (\Gamma(\cdot\,;\zeta))_r(z),$$

see Lemma 5.2. The latter inequality easily leads to the conclusion of the proof of Proposition 5.2.

Section 6 contains one of the main results in this paper. It is well-known that a superharmonic function in $\mathbf{R}^n$ can be approximated by an increasing sequence of smooth superharmonic functions. This can be accomplished by the usual device of mollification. The same device can be applied to supertemperatures in $\mathbf{R}^{n+1}$, i.e., supersolutions of the heat operator. Such an approach does not work, however, for operators with variable coefficients. In this general context the problem of approximating a given supersolution with an increasing sequence of (sufficiently) smooth supersolutions is a rather delicate one. For superharmonic functions in $\mathbf{R}^n$ a different approach is based on the use of the averaging operators $u \mapsto u_r$ where

$$u_r(x) = \frac{1}{\omega_n r^n} \int_{|x-y|<r} u(y)\,dy, \tag{1.25}$$

see [H]. It is well-known that the operator $u \mapsto u_r$ is a smoothing operator which, moreover, preserves superharmonicity, hence one obtains the desired approximation property by successive iterations of (1.25). In Sect. 6 we take up this approach. The main result, Theorem 6.1, reads: *given a L-superparabolic function u in* $\mathbf{R}^{n+1}$ *and* $v \in N$ *there exists a sequence* $(u_j)_{j \in N}$ *with* $u_j \in C^v(\mathbf{R}^{n+1})$, u_j *L-superparabolic in* $\mathbf{R}^{n+1}$, $u_j \leqq u_{j+1}$ *for any* $j \in \mathbf{N}$, *such that* $u_j \to u$ *as* $j \to \infty$. We mention that this result plays an important role in the proof of the sufficiency of Wiener's condition in [GL]. The proof of Theorem 6.1 is based on some mean value operators which are constructed from (1.20) through a process of superposition. These operators which are introduced in (3.20)–(3.22) of Sect. 3, are reminiscent of those in Weyl's classical proof of smoothness of harmonic functions, see [F]. Besides their independent interest they turn out to be extremely useful since to check their smoothing properties is considerably easier than doing the same for the operators $u_r^{(m)}$ defined by (1.23).

Although the results of this paper do apply to time-independent solutions of the Eq. (1.1), thus providing corresponding results for elliptic equations, we have omitted any reference to such a situation. Concerning Theorem 6.1, however, we mention that we have recently become aware of a very interesting paper by Littman [L 2] (dated 1963) which is concerned with the problem of monotonic approximation of supersolutions of elliptic equations. Although Littman's result is not based on exact formulas, it is very similar in spirit to our approach. In fact, in the case of a divergence form elliptic operator with sufficiently smooth coefficients, the conclusion of Theorem A in [L 2], although not the construction of the smooth approximation sequence, is essentially analogous to the conclusion of Theorem 6.1 in this paper. We wish to thank Prof. A. Devinatz for bringing references [L 1], [L 2] to our attention and Prof. W. Littman for kindly discussing with us the results of his mentioned papers.

Finally, we would like to thank the referee, whose constructive criticism has led us to improve the presentation of the proof of Theorem 2.1 and simplify the proof of Lemma 5.2.

2. Asymptotic Behavior of the Fundamental Solution for Small Times

This section is devoted to obtain an asymptotic expansion of the fundamental solution $\Gamma(x, t; y, s)$ of (1.1) together with its derivatives as $t \to s^+$ and x, y vary in a compact neighborhood of zero. Such an estimate plays a fundamental role in the work [GL] on Wiener's criterion as well as in the subsequent sections of this paper. Our approach is modelled on Minakshisundaram and Pleijel's asymptotic evaluation of the fundamental solution of the heat operator on a compact manifold, see e.g., [BGM]. In what follows K will denote a (sufficiently small) connected compact neighborhood of zero in $\mathbf{R}^n$. For a fixed $t \geqq 0$ we let $A(t)$ denote the matrix-valued function on $\mathbf{R}^n$ $A(t)(x) = (a_{ij}(x, t))$, where $A(x, t) = (a_{ij}(x, t))$ is as in (1.1). Also we set $A(t)^{-1} = (a^{ij}(\cdot, t))$. M_t will denote the Riemannian manifold $(K, g_{ij}(t))$, where $g_{ij}(t) = a^{ij}(\cdot, t)$, $g^{ij}(t) = a_{ij}(\cdot, t)$. $d(x, y, t)$ will indicate the distance between two points x, y on M_t.

Theorem 2.1. *For* $x, y \in K, x \neq y$, *we let* $\Gamma(x, y, t) = \Gamma(x, t; y, 0)$ *denote the fundamental solution of* L *in* (1.1) *with pole at* $(y, 0)$. *Then as* $t \to 0^+$ *we have the asymptotic expansion*

$$\Gamma(x, y, t) \sim (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y, t)}{4t}\right) \sum_{j=0}^{\infty} t^j u_j(x, y, t). \tag{2.1}$$

By (2.1) *we mean that there exist a suitably small* $T>0$ *and a sequence* $(u_j)_{j\in N}$, *with* $u_j \in C^\infty(K\times K\times[0,T])$ *such that*

$$\Gamma(x,y,t)-(4\pi t)^{-n/2}\exp\left(-\frac{d^2(x,y,t)}{4t}\right)\sum_{j=0}^{k} t^j u_j(x,y,t)=w_k(x,y,t), \tag{2.2}$$

with

$$w_k(x,y,t)=0\left(t^{k+1-n/2}\exp\left(-\frac{\delta|x-y|^2}{4t}\right)\right), \quad as \quad t\to 0^+, \tag{2.3}$$

uniformly for $x,y\in K$. *In* (2.3) $\delta>0$ *is a number depending on* v *in* (1.2) *and* n. *The function* u_0 *in* (2.1) *can be chosen such that* $u_0(x,x,0)=1$.

An expansion similar to (2.1) *holds for the derivatives of* Γ.

Proof. We fix a small enough neighborhood of the origin, K, and a number $T>0$ sufficient small, so that we can find $\varepsilon_0>0$ with the property that for every $y\in K$ the geodesic ball

$$B^t_{\varepsilon_0}(y)=\{z\in K\,|\,d(z,y,t)<\varepsilon_0\}$$

is a normal, convex neighborhood of y for every $t\in[0,T]$. It is clear that restricting, if needed, T and ε_0, we can determine a cylinder $U_\varepsilon\times(0,T)$, with

$$0<\varepsilon\leqq\varepsilon_0,\qquad U_\varepsilon=\{(x,y)\in K\times K\,|\,d(x,y,t)<\varepsilon\},\qquad t\in[0,T],$$

for which the function $\exp\left(-\dfrac{d^2(x,y,t)}{4t}\right)$ is in $C^\infty(U_\varepsilon\times(0,T))$. After these preliminaries, for every $t\in[0,T]$ and $y\in K$ fixed we denote by $(r(t),\theta(r(t)))$ the intrinsic geodesic polar coordinates on M_t with pole at y. We have $r(t)(x)=d(x,y,t)$. In what follows we let $r(x,t)=r(t)(x)$. We will need to distinguish between the Euclidean gradient on K and the intrinsic gradient on M_t. The former will be denoted by ∇_x, the latter by ∇_{M_t}. If $g(t)=\det(g_{ij}(t))$, the Laplace-Beltrami operator on M_t is

$$\Delta_{M_t}=\frac{1}{\sqrt{g(t)}}\sum_{i,j=1}^{n} D_j(\sqrt{g(t)}g^{ij}(t)D_i),$$

where $D_i=\dfrac{\partial}{\partial x_i}$. The Euclidean inner product on K will be denoted by $\cdot$, whereas the intrinsic inner product on M_t by $\langle\,,\rangle_t$. The symbol $|u|$ will always stand for the Euclidean length of the vector $u\in\mathbf{R}^n$, whereas $|u|_t=(\langle u,u\rangle_t)^{1/2}$. In the geodesic polar coordinates $(r(t),\theta(r(t)))$ the Laplace-Beltrami operator can be written as (see [BGM])

$$\Delta_{M_t}=D_r^2+D_r(\ln\sqrt{A(r(t))})D_r+\Delta_{S_{r(t)}}, \tag{2.4}$$

where $D_r=\dfrac{\partial}{\partial r}$, $S_{r(t)}=$ geodesic surface of radius $r(t)$ and centered at y, $\sqrt{A(r(t))}=r(t)^{n-1}\theta(r(t))$, and $\Delta_{S_{r(t)}}$ is the Laplace-Beltrami operator on $S_{r(t)}$. Replacing the expression of $\sqrt{A}$ in (2.4) yields

$$\Delta_{M_t}=D_r^2+\left[\frac{n-1}{r}+\frac{D_r\theta(r(t))}{\theta(r(t))}\right]D_r+\Delta_{S_{r(t)}}. \tag{2.5}$$

Now we consider the function $G=(4\pi t)^{-n/2}\exp\left(-\frac{r^2}{4t}\right)$, which is in $C^\infty(U_\varepsilon\times(0,T))$, and we set

$$\gamma_k(x,y,t)=G(x,y,t)\sum_{j=0}^{k} t^j u_j(x,y,t), \tag{2.6}$$

where the functions u_j are to be determined. We wish to calculate $L\gamma_k$. To this end we observe that

$$\begin{aligned} L&=\sum_{i,j=1}^{n} D_j(a_{ij}(x,t)D_i)-D_t \\ &=\Delta_{M_t}+\vec{b}(x,t)\cdot\nabla_x-D_t, \end{aligned} \tag{2.7}$$

where $\vec{b}$ is the vector field whose components are

$$b_i(x,t)=-\sum_{j=1}^{n} g^{ij}(x,t)D_j[\ln\sqrt{g(t)(x)}]. \tag{2.8}$$

By (2.5) we obtain

$$\Delta_{M_t}G=\left(\frac{r^2}{4t^2}-\frac{n}{2t}-\frac{1}{2t}\frac{rD_r\theta}{\theta}\right)G. \tag{2.9}$$

Also

$$D_tG=\left(\frac{r^2}{4t^2}-\frac{n}{2t}-\frac{1}{4t}D_t(r^2)\right)G, \tag{2.10}$$

$$\nabla_{M_t}G=-\frac{1}{4t}G\nabla_{M_t}(r^2), \qquad \nabla_xG=-\frac{1}{4t}G\nabla_x(r^2). \tag{2.11}$$

Using (2.7), (2.9), (2.10), and (2.11) we have

$$\begin{aligned} L\gamma_k&=\Delta_{M_t}\gamma_k+\vec{b}\cdot\nabla_x\gamma_k-D_t\gamma_k \\ &=G\Bigg[\left(\frac{r^2}{4t^2}-\frac{n}{2t}-\frac{1}{2t}\frac{rD_r\theta}{\theta}\right)\sum_{j=0}^{k} t^ju_j \\ &\quad+\sum_{j=0}^{k} t^j\Delta_{M_t}u_j+2\sum_{j=0}^{k} t^j\left(-\frac{1}{4t}\right)\langle\nabla_{M_t}(r^2),\nabla_{M_t}u_j\rangle_t \\ &\quad-\frac{1}{4t}\vec{b}\cdot\nabla_x(r^2)\sum_{j=0}^{k} t^ju_j+\sum_{j=0}^{k} t^j\vec{b}\cdot\nabla_xu_j \\ &\quad-\left(\frac{r^2}{4t^2}-\frac{n}{2t}-\frac{1}{4t}D_t(r^2)\right)\sum_{j=0}^{k} t^ju_j-\sum_{j=0}^{k} t^jD_tu_j-\sum_{j=1}^{k} jt^{j-1}u_j\Bigg]. \end{aligned} \tag{2.12}$$

Now we observe that

$$\langle\nabla_{M_t}(r^2),\nabla_{M_t}u_j\rangle_t=2rD_ru_j,$$

therefore (2.12) gives

$$\begin{aligned}
L\gamma_k = G\bigg[&\frac{1}{t}\left(-rD_r u_0 - \frac{rD_r\theta}{2\theta}u_0 + \frac{1}{4}D_t(r^2)u_0 - \frac{1}{4}\vec{b}\cdot\nabla_x(r^2)u_0\right) \\
&+\bigg(-rD_r u_1 - \frac{rD_r\theta}{2\theta}u_1 + \frac{1}{4}D_t(r^2)u_1 \\
&-\frac{1}{4}\vec{b}\cdot\nabla_x(r^2)u_1 - u_1 + \Delta_{M_t}u_0 + \vec{b}\cdot\nabla_x u_0 - D_t u_0\bigg) \\
&+t\bigg(-rD_r u_2 - \frac{rD_r\theta}{2\theta}u_2 + \frac{1}{4}D_t(r^2)u_2 - \frac{1}{4}\vec{b}\cdot\nabla_x(r^2)u_2 \\
&-2u_2 + \Delta_{M_t}u_1 + \vec{b}\cdot\nabla_x u_1 - D_t u_1\bigg) + \dots \\
&+t^{k-1}\bigg(-rD_r u_k - \frac{rD_r\theta}{2\theta}u_k + \frac{1}{4}D_t(r^2)u_k - \frac{1}{4}\vec{b}\cdot\nabla_x(r^2)u_k \\
&-ku_k + \Delta_{M_t}u_{k-1} + \vec{b}\cdot\nabla_x u_{k-1} - D_t u_{k-1}\bigg) \\
&+t^k(\Delta_{M_t}u_k + \vec{b}\cdot\nabla_x u_k - D_t u_k)\bigg].
\end{aligned} \tag{2.13}$$

If we set

$$\begin{aligned}
\Phi_j = &-rD_r u_j - \frac{rD_r\theta}{2\theta}u_j + \frac{1}{4}D_t(r^2)u_j - \frac{1}{4}\vec{b}\cdot\nabla_x(r^2)u_j \\
&-ju_j + \Delta_{M_t}u_{j-1} + \vec{b}\cdot\nabla_x u_{j-1} - D_t u_{j-1},
\end{aligned} \tag{2.14}$$

for $j=0,1,\dots,k$, with $u_{-1}\equiv 0$, and

$$\Phi_{k+1} = \Delta_{M_t}u_k + \vec{b}\cdot\nabla_x u_k - D_t u_k, \tag{2.15}$$

then we can rewrite

$$L\gamma_k = G\sum_{j=0}^{k+1} t^{j-1}\Phi_j. \tag{2.16}$$

At this point we would like to determine the $k+1$ functions $u_0, u_1, \dots, u_k$ in such a way that

$$L\gamma_k = t^k G\Phi_{k+1} \quad \text{in} \quad U_\varepsilon \times (0,T). \tag{2.17}$$

By (2.14), (2.17) will be true if we can solve the $k+1$ equations

$$rD_r u_j = (q-j)u_j + p_j, \quad j=0,1,\dots,k, \tag{2.18}$$

where

$$q = -\frac{rD_r\theta}{2\theta} + \frac{1}{4}D_t(r^2) - \frac{1}{4}\vec{b}\cdot\nabla_x(r^2), \tag{2.19}$$

and

$$p_0 \equiv 0, \quad p_j = \Delta_{M_t} u_{j-1} + \vec{b} \cdot \nabla_x u_{j-1} - D_t u_{j-1}, \quad j = 1, \ldots, k. \tag{2.20}$$

We remark that solving (2.18) amounts to show that $\Phi_j \equiv 0, j = 0, 1, \ldots, k$. From the definition (2.19) of $q = q(r, \theta, t)$ it is clear that

$$q(0, \theta, t) = 0 \quad \text{uniformly in } \theta, t. \tag{2.21}$$

This fact plays a crucial role in the following considerations. We now

Claim. *There exist* $k+1$ $C^\infty(U_\varepsilon \times [0, T])$ *functions,* $u_0, u_1, \ldots, u_k$, *which solve the Eqs.* (2.18). *Moreover,* u_0 *can be chosen so that*

$$u_0(0, \theta, t) \equiv 1, \quad \textit{uniformly in } \theta, t. \tag{2.22}$$

In the proof of the claim we follow closely the inductive argument given in [BGM]. We start with u_0. For $j = 0$ (2.18) becomes [see (2.20)]

$$r D_r u_0 = q u_0. \tag{2.23}$$

A solution to (2.23) is provided formally by

$$u_0(r, \theta, t) = u_0(\theta, t) \exp\left(\int_0^r q(\varrho, \theta, t) \frac{d\varrho}{\varrho} \right).$$

We emphasize that the function q defined by (2.19) belongs to $C^\infty(U_\varepsilon \times [0, T])$ since by (2.5) and the fact that $r^2 \in C^\infty(U_\varepsilon \times [0, T])$ it follows that $\frac{r D_r \theta}{\theta} \in C^\infty(U_\varepsilon \times [0, T])$. Moreover, because of (2.21) it is integrable near $r = 0$ with respect to the measure $\frac{d\varrho}{\varrho}$. Since the choice of the initial value $u_0(\theta, t)$ is up to us, we take

$$u_0(\theta, t) = u_0(0, \theta, t) \equiv 1. \tag{2.24}$$

In what follows for a fixed $t \geqq 0$, and for $0 \leqq s \leqq r(t) = d(x, y, t)$, we denote by x_s the point on the unique geodesic in M_t joining x to y and having distance s from y. If $T_y(M_t)$ denotes the tangent space to M_t in y and $\exp_y : T_y(M_t) \to M_t$ is the exponential map, then for $0 \leqq \alpha \leqq 1$ we have

$$x_{\alpha r} = x_{\alpha r(t)} = \exp_y(\alpha \exp_y^{-1}(x)). \tag{2.25}$$

With these remarks in mind and taking (2.24) into account we have

$$\begin{aligned} u_0(x, y, t) &= \exp\left(\int_0^r q(x_\varrho, y, t) \frac{d\varrho}{\varrho} \right) = \exp\left(\int_0^1 q(x_{\varrho r}, y, t) \frac{d\varrho}{\varrho} \right) \\ &= \exp\left(\int_0^1 q(\exp_y(\varrho \exp_y^{-1}(x)), y, t) \frac{d\varrho}{\varrho} \right), \end{aligned} \tag{2.26}$$

where in the last equality we have used (2.25). Because of (2.21) the map

$$(x, y, t, \varrho) \mapsto \frac{1}{\varrho} q(\exp_y(\varrho \exp_y^{-1}(x)), y, t)$$

is C^∞ in $U_\varepsilon \times [0,T] \times [0,1]$. The claim is thus proved in the case $k=0$. We can then start the induction. Now let $j=k$ and consider (2.18). We assume we have determined k functions $u_0, u_1, \dots, u_{k-1}$ in $C^\infty(U_\varepsilon \times [0,T])$ which solve (2.18). We look for a solution u_k of (2.18) of the form

$$u_k(r,\theta,t) = C_k(r,\theta,t) r^{-k} \theta^{-1/2}. \tag{2.27}$$

By the method of variation of constants C_k must satisfy the equation

$$rD_r C_k = \psi C_k + p_k r^k \theta^{1/2},$$

where we have set $\psi = q + \frac{rD_r\theta}{2\theta}$, so that by (2.19) $\psi(0,\theta,t)=0$ uniformly in θ, t, and $\psi \in C^\infty(U_\varepsilon \times [0,T])$. Formally, a solution C_k is given by

$$C_k(x,y,t) = \exp\left(\int_0^r \psi(x_\tau, y, t)\frac{d\tau}{\tau}\right) \int_0^r p_k(x_\sigma, y, t)\theta^{1/2}(x_\sigma, y, t)\sigma^k$$
$$\times \exp\left(-\int_0^\sigma \psi(x_s, y, t)\frac{ds}{s}\right)\frac{d\sigma}{\sigma}. \tag{2.28}$$

Performing obvious changes of variables in the integrals involved in (2.28) and using (2.25) we can rewrite

$$\begin{aligned} C_k(x,y,t) &= r^k \exp\left(\int_0^1 \psi(x_{\tau r}, y, t)\frac{d\tau}{\tau}\right)\int_0^1 p_k(x_{\sigma r}, y, t)\theta^{1/2}(x_{\sigma r}, y, t))\sigma^k \\ &\quad \times \exp\left(-\int_0^\sigma \psi(x_{sr}, y, t)\frac{ds}{s}\right)\frac{d\sigma}{\sigma} \\ &= r^k \exp\left(\int_0^1 \psi(\exp_y(\tau \exp_y^{-1}(x)), y, t)\frac{d\tau}{\tau}\right)\int_0^1 p_k(\exp_y(\sigma \exp_y^{-1}(x)), y, t) \\ &\quad \times \theta^{1/2}(\exp_y(\sigma \exp_y^{-1}(x)), y, t)\sigma^k \exp\left(-\int_0^\sigma \psi(\exp_y(s\exp_y^{-1}(x)), y, t)\frac{ds}{s}\right)\frac{d\sigma}{\sigma} \\ &= r^k \tilde{C}_k(x,y,t). \end{aligned}$$

From the definition of $\tilde{C}_k$, the fact that $\psi \in C^\infty(U_\varepsilon \times [0,T])$, and $\psi(0,\theta,t) \equiv 0$, it is immediate to recognize that

$$\tilde{C}_k(x,y,t) \in C^\infty(U_\varepsilon \times [0,T]).$$

Therefore, from (2.27) we obtain

$$u_k(x,y,t) = \theta^{-1/2}(x,y,t)\tilde{C}_k(x,y,t),$$

which shows that $u_k \in C^\infty(U_\varepsilon \times [0,T])$. This completes the proof of the claim, from which (2.17) follows.

Now we pick $\chi = C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, $0 \leqq \chi \leqq 1$, with $\chi \equiv 1$ on $U_{\varepsilon/3}$, $\chi \equiv 0$ outside $U_{(2/3)\varepsilon}$. We set

$$H_k(x,y,t) = \chi(x,y)\gamma_k(x,y,t), \tag{2.29}$$

where γ_k is defined by (2.6). If we extend γ_k with zero outside $U_\varepsilon \times (0,T)$, we have

$$H_k \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n \times (0,T)).$$

A computation and (2.17) give

$$LH_k = t^k \chi G \Phi_{k+1} + \gamma_k \operatorname{div}(A \nabla_x \chi) + 2(A \nabla_x \chi) \cdot \nabla_x \gamma_k. \tag{2.30}$$

Let $\pi_1 : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be the map $\pi_1(x, y) = x$. For $\varphi \in C_0^\infty(\pi_1(U_{\varepsilon/3}))$ we set

$$\Gamma_k \varphi(x, t) = \int_{\mathbf{R}^n} [\Gamma(x, y, t) - H_k(x, y, t)] \varphi(y) dy. \tag{2.31}$$

Since Γ is the fundamental solution of L we have $L\left(\int_{\mathbf{R}^n} \Gamma(\cdot, y, \cdot)\varphi(y) dy\right) = 0$. Therefore, (2.30) yields

$$\begin{aligned} L(\Gamma_k \varphi)(x, t) &= -L(H_k \varphi)(x, t) = -\int_{\mathbf{R}^n} LH_k(x, y, t)\varphi(y) dy \\ &= -t^k \int_{\mathbf{R}^n} \chi(x, y) G(x, y, t) \Phi_{k+1}(x, y, t) \varphi(y) dy \\ &\quad - \int_{\mathbf{R}^n} \gamma_k(x, y, t) \operatorname{div}(A \nabla_x \chi)(x, y, t) \varphi(y) dy \\ &\quad - 2 \int_{\mathbf{R}^n} [(A \nabla_x \chi) \cdot \nabla_x \gamma_k](x, y, t) \varphi(y) dy \\ &= f_{k,1}(x, t) + f_{k,2}(x, t) + f_{k,3}(x, t) = f_k(x, t). \end{aligned} \tag{2.32}$$

Using the fact that there exist two positive constants α, β such that for every $(x, y, t) \in U_{(2/3)\varepsilon} \times [0, T]$

$$\alpha |x - y|^2 \leqq d^2(x, y, t) \leqq \beta |x - y|^2, \tag{2.33}$$

it is not difficult to see that the following holds: there exist $C > 0$ and $\gamma > 0$ such that

$$\begin{cases} |f_k(x, t)| \leqq C \exp(\gamma |x|^2), & \text{for every } (x, t) \in \mathbf{R}^n \times (0, T); \\ |\Gamma_k \varphi(x, t)| \leqq C \exp(\gamma |x|^2), & \text{for every } (x, t) \in \mathbf{R}^n \times (0, T); \\ \lim\limits_{t \to 0^+} \Gamma_k \varphi(x, t) = 0, & \text{for every } x \in \mathbf{R}^n. \end{cases}$$

This implies, by uniqueness of the solution of Cauchy problem (see [Fr]) for T small enough

$$\begin{aligned} \Gamma_k \varphi(x, t) &= -\int_0^t \int_{\mathbf{R}^n} \Gamma(x, t; \xi, \tau) f_k(\xi, \tau) d\xi d\tau \\ &= -\sum_{i=1}^3 \int_0^t \int_{\mathbf{R}^n} \Gamma(x, t; \xi, \tau) f_{k,i}(\xi, \tau) d\xi d\tau = I_k + II_k + III_k. \end{aligned} \tag{2.34}$$

Now

$$\begin{aligned} I_k &= \int_{\mathbf{R}^n} \varphi(y) dy \left(\int_0^t \int_{\mathbf{R}^n} \Gamma(x, t; \xi, \tau) G(\xi, y, \tau) \tau^k \chi(\xi, y) \Phi_{k+1}(\xi, y, \tau) d\xi d\tau \right) \\ &= \int_{\mathbf{R}^n} \varphi(y) A_{k,1}(x, y, t) dy, \end{aligned} \tag{2.35}$$

the exchange of order of integration in I_k being possible in virtue of Fubini's theorem and the fact that there exists a $\delta > 0$ such that

$$\begin{aligned} A_{k,1}(x, y, t) &= \int_0^t \int_{\mathbf{R}^n} \Gamma(x, t; \xi, \tau) G(\xi, y, \tau) \tau^k \chi(\xi, y) \Phi_{k+1}(\xi, y, \tau) d\xi d\tau \\ &= t^{k+1-n/2} \exp\left(-\delta \frac{|x - y|^2}{4t} \right) w_k(x, y, t), \end{aligned} \tag{2.36}$$

where $w_k(x,y,t)=0(1)$ uniformly for $(x,y,t)\in \bar{U}_{\varepsilon/3}\times[0,T]$. To see this recall that $\operatorname{supp}\chi\subset \bar{U}_{(2\varepsilon/3)}$ and that $\Phi_{k+1}\in C^\infty(U_\varepsilon\times[0,T])$, therefore

$$|A_{k,1}(x,y,t)|\leqq \sup_{\bar{U}_{(2/3)\varepsilon}\times[0,T]}|\Phi_{k+1}|\int_0^t\int_{\mathbf{R}^n}\Gamma(x,t;\xi,\tau)G(\xi,y,\tau)\tau^k d\xi d\tau$$

(by [Fr, Th. 11, p. 24] and (2.33))

$$\leqq C\int_0^t\int_{\mathbf{R}^n}(t-\tau)^{-n/2}\exp\left(-\frac{\delta|x-\xi|^2}{4(t-\tau)}\right)$$

$$\times\tau^{-(n/2-k)}\exp\left(-\frac{\delta|\xi-y|^2}{4\tau}\right)d\xi d\tau$$

$$=C't^{k+1-n/2}\exp\left(-\frac{\delta|x-y|^2}{4t}\right).$$

In the last equality we have used Lemma 3 on p. 15 of [Fr]. By similar arguments it can be recognized that II_k and III_k can be written, respectively, as

$$II_k=\int_{\mathbf{R}^n}\varphi(y)A_{k,2}(x,y,t)dy \quad\text{and}\quad III_k=\int_{\mathbf{R}^n}\varphi(y)A_{k,3}(x,y,t)dy,$$

with $A_{k,i}$, $i=1,2$, satisfying estimates similar to (2.36). Setting $A_k=\sum_{i=1}^3 A_{k,i}$ from our work above and the fact that φ is an arbitrary function in $C_0^\infty(\pi_1(U_{(2/3)\varepsilon}))$, we conclude that

$$\Gamma(x,y,t)=H_k(x,y,t)+A_k(x,y,t),\qquad (x,y,t)\in U_{(2/3)\varepsilon}\times(0,T),\tag{2.37}$$

with A_k satisfying the estimate

$$A_k(x,y,t)=0\left(t^{k+1-n/2}\exp\left(-\frac{\delta|x-y|^2}{4t}\right)\right),\tag{2.38}$$

for $(x,y,t)\in U_{\varepsilon/3}\times(0,T)$ and a certain $\delta>0$. Recalling now that $H_k=\chi\gamma_k$ and that $\chi\equiv1$ on $\bar{U}_{\varepsilon/3}$ we conclude from (2.37) that

$$\Gamma(x,y,t)=G(x,y,t)\sum_{j=0}^k t^j u_j(x,y,t)+A_k(x,y,t)$$

for $(x,y,t)\in U_{\varepsilon/3}\times(0,T)$, with A_k having the asymptotic behavior given by (2.38). This completes the proof of (2.1). Analogous arguments prove that the derivatives of Γ have similar asymptotic expansions.

3. A Class of Well-behaved Mean Value Formulas

In this section we generalize some mean value formulas relative to the operator L in (1.1) first found in [FG] and [GL]. The advantage of these formulas with respect to those found in [FG] and [GL] consists in the fact that the kernel appearing in them is not only bounded, but possesses a degree of regularity which can be made arbitrarily large. Our starting point is the observation that if $u\in C^2(\mathbf{R}^{n+1})$ and we define a new function in $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}$ by setting for $(x,t) \in \mathbf{R}^{n+1}$

$$\hat{u}(x,y,t) = u(x,t), \qquad y \in \mathbf{R}^m,$$

then with L as in (1.1) we have

$$(L + \Delta_y)\hat{u}(x,y,t) = Lu(x,t). \tag{3.1}$$

Therefore, we can apply to the function $\hat{u}$ the representation formulas established in [FG] or [GL]. Before stating the results we need to introduce some notation. If Γ is the fundamental solution of L, $z=(x,t)$, $\zeta=(\xi,\tau)$, and $r>0$ we set (cf. [GL])

$$E(z;\zeta) = \frac{A(\zeta)(\nabla_\xi \Gamma(z;\zeta)) \cdot \nabla_\xi \Gamma(z;\zeta)}{\Gamma^2(z;\zeta)}, \qquad E_r(z;\zeta) = (4\pi r)^{-n/2} E(z;\zeta). \tag{3.2}$$

Next we define for a fixed $m \in \mathbf{N}$

$$\Phi(z;\zeta) = (4\pi(t-\tau))^{-m/2}\Gamma(z;\zeta), \tag{3.3}$$

$$R_r^2(z;\zeta) = 4(t-\tau)\ln\left[(4\pi r)^{\frac{n+m}{2}}\Phi(z;\zeta)\right]. \tag{3.4}$$

Finally, we recall the definition (1.20) of the *modified parabolic ball* centered at z and of radius r. Using (3.3) we rewrite (1.20) as

$$\Omega_r^m(z) = \left\{\zeta \in \mathbf{R}^{n+1} \mid \Phi(z;\zeta) > (4\pi r)^{-\frac{n+m}{2}}\right\}. \tag{3.5}$$

Theorem 3.1. *Let $u \in C^2(\mathbf{R}^{n+1})$ and let $z \in \mathbf{R}^{n+1}$. Then for every $r>0$ we have*

$$\begin{aligned} &\omega_m (4\pi r)^{-\frac{n+m}{2}} \int_{\Omega_r^m(z)} u(\zeta) R_r^m(z;\zeta)\left[E(z;\zeta) + \frac{m}{m+2}\frac{R_r^2(z;\zeta)}{4(t-\tau)^2}\right] d\zeta \\ &= u(z) + \frac{n+m}{m+2} w_m (4\pi)^{-\frac{n+m}{2}} \int_0^r l^{-\left(\frac{n+m}{2}+1\right)} \int_{\Omega_l^m(z)} Lu(\zeta)\frac{R_l^{m+2}(z;\zeta)}{4(t-\tau)}\, d\zeta\, dl. \end{aligned} \tag{3.6}$$

In (3.6) ω_m *denotes the measure of the unit ball in* $\mathbf{R}^m$.

Proof. Our starting point is formula (1.37) of Theorem 1.6 in [GL] which we now recall.

$$\frac{d}{dr} u_r(z) = \frac{n}{2}(4\pi)^{-n/2} r^{-n/2-1} \int_{\Omega_r(z)} Lu(\zeta)\ln[(4\pi r)^{n/2}\Gamma(z;\zeta)]\, d\zeta, \tag{3.7}$$

where $\Omega_r(z)$ is the parabolic ball defined in (1.4) and we have set [see (3.2)]

$$u_r(z) = \int_{\Omega_r(z)} u(\zeta) E_r(z;\zeta)\, d\zeta. \tag{3.8}$$

By integration $\left[\text{keeping in mind that } \lim_{r \to 0^+} u_r(z) = u(z)\right]$, we obtain from (3.7)

$$u_r(z) = u(z) + \frac{n}{2}(4\pi)^{-n/2}\int_0^r l^{-(n/2+1)} \int_{\Omega_l(z)} Lu(\zeta)\ln[(4\pi l)^{n/2}\Gamma(z;\zeta)]\, d\zeta\, dl. \tag{3.9}$$

We now fix $m \in \mathbf{N}$ and for $y \in \mathbf{R}^m$ we denote by

$$\hat{L} = L + \Delta_y \tag{3.10}$$

the parabolic operator acting on the (x, y, t)-variables in $\mathbf{R}^{n+m+1}$. If $\hat{\Gamma}(x, y, t; \xi, \eta, \tau)$ is the fundamental solution of $\hat{L}$ in (3.10) one easily verifies that

$$\hat{\Gamma}(x, y, t; \xi, \eta, \tau) = \Gamma(x, t; \xi, \tau) K(y-\eta; t-\tau), \tag{3.11}$$

where

$$K(y-\eta; t-\tau) = \begin{cases} (4\pi(t-\tau))^{-m/2} \exp\left(-\dfrac{|y-\eta|^2}{4(t-\tau)}\right), & t > \tau \\ 0, & t \leqq \tau. \end{cases} \tag{3.12}$$

Let now $\hat{z}$ denote the point (x, y, t) in $\mathbf{R}^{n+m+1}$. If $\hat{u}(\hat{z}) = u(z)$, then by (3.1) $\hat{L}\hat{u}(\hat{z}) = Lu(z)$. Therefore, if we apply the $(n+m+1)$-dimensional version of (3.9) to $\hat{u}$ and $\hat{L}$, we obtain

$$(\hat{u})_r(\hat{z}) = u(z) + \frac{n+m}{2} (4\pi)^{-\frac{n+m}{2}} \int_0^r l^{-\left(\frac{n+m}{2}+1\right)}$$
$$\times \int_{\hat{\Gamma}(\hat{z};\hat{\zeta}) > (4\pi l)^{-\frac{n+m}{2}}} Lu(\zeta) \ln\left[(4\pi l)^{\frac{n+m}{2}} \hat{\Gamma}(\hat{z}; \hat{\zeta})\right] d\hat{\zeta}\, dl, \tag{3.13}$$

where the notation in (3.13) means that the inner integral is performed over the set in $\mathbf{R}^{n+m+1}$

$$\left\{\hat{\zeta} \in \mathbf{R}^{n+m+1} \,|\, \hat{\Gamma}(\hat{z}; \hat{\zeta}) > (4\pi l)^{-\frac{n+m}{2}}\right\}.$$

Because of (3.11), keeping in mind (3.3), (3.4), we have

$$\int_{\hat{\Gamma}(\hat{z};\hat{\zeta}) > (4\pi l)^{-\frac{n+m}{2}}} Lu(\zeta) \ln\left[(4\pi l)^{\frac{n+m}{2}} \hat{\Gamma}(\hat{z}; \hat{\zeta})\right] d\hat{\zeta}$$
$$= \int_{\Omega_l^m(z)} Lu(\zeta) \left(\int_{|y-\eta| < R_l(z;\zeta)} \ln\left[(4\pi l)^{\frac{n+m}{2}} \hat{\Gamma}(\hat{z}; \hat{\zeta})\right] d\eta \right) d\zeta$$
$$= \int_{\Omega_l^m(z)} Lu(\zeta) \left(\int_{|y-\eta| < R_l(z;\zeta)} \frac{1}{4(t-\tau)} [R_l^2(z;\zeta) - |y-\eta|^2] d\eta \right) d\zeta$$
$$= \frac{\omega_m}{2(m+2)(t-\tau)} \int_{\Omega_l^m(z)} Lu(\zeta) R_l^{m+2}(z;\zeta) d\zeta. \tag{3.14}$$

From (3.8) and (3.2) we have

$$(\hat{u})_r(\hat{z}) = (4\pi r)^{-\frac{n+m}{2}} \int_{\hat{\Gamma}(\hat{z};\hat{\zeta}) > (4\pi r)^{-\frac{n+m}{2}}} u(\zeta) \hat{E}(\hat{z}; \hat{\zeta}) d\hat{\zeta}. \tag{3.15}$$

Using again (3.11) we obtain

$$\hat{E}(\hat{z}; \hat{\zeta}) = E(z; \zeta) + \frac{|y-\eta|^2}{4(t-\tau)^2}. \tag{3.16}$$

Replacing (3.16) in (3.15) and proceeding along the same lines as above, we end up with

$$(\hat{u})_r(\hat{z}) = \omega_m (4\pi r)^{-\frac{n+m}{2}} \int_{\Omega_r^m(z)} u(\zeta) R_r^m(z;\zeta) \left[E(z;\zeta) + \frac{m}{m+2} \frac{R_r^2(z;\zeta)}{4(t-\tau)^2} \right] d\zeta. \tag{3.17}$$

Inserting (3.14), (3.17) into (3.13) finally gives (3.6).

Remark 3.1. We emphasize that if $m=0$ in Theorem 3.1, then (3.6) reduces to (1.31) of Theorem 1.5 in [GL] if we agree to let $\omega_0=1$.

Remark 3.2. The idea of climbing up in the dimension by adding variables is not new. Kupcov had already employed it in [Ku] to obtain a mean-value formula with a well-behaved kernel for solutions of $Hu=\Delta u-D_t u=0$.

From now on, to avoid clumsy notation we set for $m\in\mathbf{N}$ and $z,\zeta\in\mathbf{R}^{n+1}$

$$E_r^{(m)}(z;\zeta)=(4\pi r)^{-\frac{n+m}{2}}\omega_m R_r^m(z;\zeta)\left[E(z;\zeta)+\frac{m}{m+2}\frac{R_r^2(z;\zeta)}{4(t-\tau)^2}\right]. \tag{3.18}$$

If we denote by $u_r^{(m)}(z)$ the average $(\hat{u})_r(\hat{z})$ appearing in (3.17) [observe that (3.17) says that $(\hat{u})_r(\hat{z})$ is in fact a function of z alone], then (3.17) can be rewritten as

$$u_r^{(m)}(z)=\int_{\Omega_r^m(z)} u(\zeta)E_r^{(m)}(z;\zeta)d\zeta\,. \tag{3.19}$$

Again, one should note that if $m=0$ (3.19) reduces to (3.8).

We close this section by establishing an average formula for solutions of $Lu=0$ which will turn out to be very useful when we will study the smoothing of superparabolic functions in Sect. 6. Such a formula is easily obtained by superposition from (3.19), by suitably adapting the idea in the proof of H. Weyl's lemma on harmonic functions (c.f., e.g., [F], p. 92). In what follows we choose and fix a function $\varphi\in C_0^\infty(\mathbf{R}^+)$ such that $\varphi\geqq 0$, $\operatorname{supp}\varphi\subset(1,2)$ and $\int_0^{+\infty}\varphi(r)dr=1$. For $m\in\mathbf{N}$ and $z\in\mathbf{R}^{n+1}$ we define

$$J_r^{(m)}(u)(z)=\int_0^{+\infty} u_l^{(m)}(z)\varphi\left(\frac{l}{r}\right)\frac{dl}{r} \tag{3.20}$$

where $u\in L^\infty_{\mathrm{loc}}(\mathbf{R}^{n+1})$ and $u_r^{(m)}$ is as in (3.19). Substituting for $u_r^{(m)}$ its expression given by (3.19) and exchanging the order of integration, we obtain from (3.20)

$$J_r^{(m)}(u)(z)=\int_{\mathbf{R}^{n+1}} u(\zeta)M_r^{(m)}(z;\zeta)d\zeta\,, \tag{3.21}$$

where we have set for $z=(x,t)$, $\zeta=(\xi,\tau)$

$$M_r^{(m)}(z;\zeta)=\int_{[4\pi\Phi(z;\zeta)^{\frac{2}{n+m}}]^{-1}}^{+\infty} E_l^{(m)}(z;\zeta)\varphi\left(\frac{l}{r}\right)\frac{dl}{r} \tag{3.22}$$

if $t>\tau$, whereas $M_r^{(m)}(z;\zeta)=0$ for $t\leqq\tau$. Since φ is compactly supported in $\mathbf{R}^+$, $M_r^{(m)}(z;\cdot)$ is supported in a parabolic ball $\Omega_{r_0}^m(z)$ centered at z [see (3.5)]. We will show in Sect. 6 that as a function of z the kernel $M_r^{(m)}$ can be made arbitrarily regular by choosing m large enough.

4. An Elementary Proof of Harnack's Inequality

An immediate consequence of Theorem 3.1 is that if u is a (smooth) solution of $Lu=0$ in $\mathbf{R}^{n+1}$, then for every $z\in\mathbf{R}^{n+1}$ and every $r>0$ we have [see (3.19)]

$$u(z)=u_r^{(m)}(z)=\int_{\Omega_r^m(z)} u(\zeta)E_r^{(m)}(z;\zeta)d\zeta\,, \tag{4.1}$$

where $m \in \mathbf{N}$ is arbitrarily fixed. The advantage of formula (4.1) with respect to (1.18) is that the kernel $E_r^{(m)}(z;\cdot)$ is bounded from above on the parabolic ball $\Omega_r^{(m)}(z)$ by an appropriate power of r, provided that m is large enough. When $m=0$, $E_r^{(0)}(z;\zeta)=E(z;\zeta)$ [see (3.2)] and (4.1) reduces to (1.18). As observed in Sect. 1 $E(z;\cdot)$ is not bounded on $\Omega_r(z)$.

In this section to illustrate the good feature of formula (4.1), we present an elementary proof of Harnack's inequality for parabolic equations which does not use the parabolic BMO machinery developed by Moser in his classical paper [M]. Our approach is direct and imitates the proof of Harnack's inequality for harmonic functions. In what follows we will use the intrinsic notation introduced in Sect. 2. In virtue of Theorem 2.1 for any $\theta>1$ we can find $r_0=r_0(\theta, L)>0$, L as in (1.1), such that for any $r \leqq r_0$, $z \in \mathbf{R}^{n+1}$ and $\zeta \in \Omega_r^{(m)}(z)$, $m \in \mathbf{N}$, we have

$$\theta^{-1} G(z;\zeta) \leqq \Gamma(z;\zeta) \leqq \theta G(z;\zeta). \tag{4.2}$$

where $G(z;\zeta)$ is the generalized Gaussian introduced in Theorem 2.1. Using (4.2) it is easy to check that: *for every $\varepsilon>0$ and a fixed $m \in \mathbf{N}$ there exists $\delta=\delta(L, m, \varepsilon)>0$ such that given $z_0=(x_0, t_0) \in \mathbf{R}^{n+1}$, $r \leqq \frac{r_0}{2}$, and $z=(x,t) \in \Omega_r^m(z_0)$, if $t_0-t \geqq \varepsilon r$ then*

$$\Omega_{\delta r}^m(z) \subset \Omega_{2r}^m(z_0). \tag{4.3}$$

We will use (4.3) in the proof of Theorem 4.1 below.

Theorem 4.1 (Harnack's inequality). *Let $D \subset \mathbf{R}^{n+1}$ be an open set and let $u \geqq 0$ be a solution of $Lu=0$ in D. Given $z_0 \in D$ let $r \leqq \frac{r_0}{4}$ be such that $\Omega_{4r}^m(z_0) \subset D$. If $\varepsilon>0$ let $z \in \Omega_r^m(z_0)$ be such that $t_0-t \geqq \varepsilon r$. Then there exists a constant $C=C(L, m, \varepsilon)>0$ such that*

$$u(z) \leqq C u(z_0). \tag{4.4}$$

Remark 4.1. In the statement above and throughout the discussion below, m is a fixed integer with $m>2$.

Proof of Theorem 4.1. By Theorem 3.1 we have

$$u(z_0)=\int_{\Omega_{3r}^m(z_0)} u(\zeta) E_{3r}^{(m)}(z_0;\zeta)\,d\zeta,$$

$E_{3r}^{(m)}$ being defined by (3.18). By the positivity of u and the observation (4.3) we obtain the following inequalities

$$\begin{aligned} u(z_0) &\geqq \int_{\Omega_{\delta r}^m(z)} u(\zeta) E_{3r}^{(m)}(z_0;\zeta)\,d\zeta \\ &= \int_{\Omega_{\delta r}^m(z)} u(\zeta) \frac{E_{3r}^{(m)}(z_0;\zeta)}{E_{\delta r}^{(m)}(z;\zeta)} E_{\delta r}^{(m)}(z;\zeta)\,d\zeta. \end{aligned} \tag{4.5}$$

In order to get (4.4) all we have to show is the existence of two numbers $C_i=C_i(L, m, \varepsilon)>0$, $i=1,2$, such that

$$\inf_{\zeta \in \Omega_{\delta r}^m(z)} E_{3r}^{(m)}(z_0:\zeta) \geqq C_1 r^{-(n/2+1)}, \tag{4.6}$$

$$\sup_{\zeta \in \Omega_{\delta r}^m(z)} E_{\delta r}^{(m)}(z;\zeta) \leqq C_2 r^{-(n/2+1)}. \tag{4.7}$$

Recalling (3.18) we have for $\zeta\in\Omega^m_{\delta r}(z)$

$$E^{(m)}_{3r}(z_0;\zeta)\geqq C_m r^{-\frac{n+m}{2}}\frac{R^{m+2}_{3r}(z_0;\zeta)}{(t_0-\tau)^2}, \tag{4.8}$$

for a certain $C_m>0$. Now using the definition of $R_{3r}(z_0;\zeta)$ [see (3.4)] and the fact that

$$(12\pi r)^{\frac{n+m}{2}}\Phi(z_0;\zeta)\geqq(\tfrac{3}{2})^{\frac{n+m}{2}}\quad\text{on } \Omega^m_{2r}(z_0),$$

we obtain for $\zeta\in\Omega^m_{\delta r}(z)$

$$E^{(m)}_{3r}(z_0;\zeta)\geqq C'_m r^{-\frac{n+m}{2}}(t_0-\tau)^{m/2-1}. \tag{4.9}$$

Finally, recalling that for $\zeta\in\Omega^m_{\delta r}(z)$ we have $(t_0-\tau)\geqq(t_0-t)\geqq\varepsilon r$, we obtain (4.6) from (4.9).

Now we look at (4.7). We claim that as a consequence of the results in Theorem 2.1 the following estimate for the kernel E_r holds

$$\frac{A(\zeta)(\nabla\Gamma_\xi(z;\zeta))\cdot\nabla\Gamma_\xi(z;\zeta)}{\Gamma^2(z;\zeta)}\leqq C\left(\frac{d^2(x,\xi,\tau)}{(t-\tau)^2}+1\right), \tag{4.10}$$

uniformly for $z=(x,t)\in\mathbf{R}^{n+1}$, $\zeta=(\xi,\tau)\in\Omega^m_r(z)$, and $r\leqq r_0$, where $C>0$ depends only on L. We outline the proof of (4.10). For complete details one should see the proof of Lemma 2.1 in [GL]. Let us fix a $k\in\mathbf{N}$ sufficiently large, for instance $k>\dfrac{m+n}{2}+10^3$. Then from Theorem 2.1, if $z=(x,t)\in\mathbf{R}^{n+1}$ is fixed and $\zeta=(\xi,\tau)\in\Omega^m_r$, $r\leqq r_0$, we can write

$$\Gamma(z;\zeta)=\gamma_k(z;\zeta)+w_k(z;\zeta), \tag{4.11}$$

$$D_{\xi_j}\Gamma(z;\zeta)=D_{\xi_j}\gamma_k(z;\zeta)+w_{k,j}(z;\zeta),\quad j=1,\dots,n, \tag{4.12}$$

where γ_k is defined as in (2.6) and $w_k(z;\cdot)$, $w_{k,j}(z;\cdot)$, $j=1,\dots,n$, are $(n+1)$ functions in $\Omega^m_r(z)$ such that $w_k(z;\cdot)=0(r)$, $w_{k,j}(z;\cdot)=0(r)$, $j=1,\dots,n$, as $r\to 0$. Now let us observe that in the notation of Sect. 2 (4.10) becomes

$$\frac{|\nabla_{M_\tau}\Gamma(z;\zeta)|^2_\tau}{\Gamma^2(z;\zeta)}\leqq C\left(\frac{d^2(x,\xi,\tau)}{(t-\tau)^2}+1\right). \tag{4.13}$$

By (4.12) we have

$$|\nabla_{M_\tau}\Gamma|^2_\tau=|\nabla_{M_\tau}\gamma_k|^2_\tau+2\langle\nabla_{M_\tau}\gamma_k,A(\tau)\vec{w}_k\rangle_\tau+|A(\tau)\vec{w}_k|^2_\tau, \tag{4.14}$$

where we have set $\vec{w}_k=(w_{k,1},w_{k,2},\dots,w_{k,n})$. Replacing in (4.14) the expression for γ_k, using the fact that the function u_0 in the expansion (2.1) is 1 uniformly in x,ξ at the initial time $t=\tau$, and (4.2), we finally obtain (4.13), hence (4.10). For more details one can look at the proof of Lemma 2.1 in [GL] or at Sect. 6 below. Next we observe that because of (2.33) by (3.2) and (4.10) we get

$$E_r(z;\zeta)\leqq Cr^{-n/2}\left(\frac{|x-\xi|^2}{(t-\tau)^2}+1\right), \tag{4.15}$$

for every $z \in \mathbf{R}^{n+1}$, $\zeta \in \Omega_r^m(z)$, and $r \leqq r_0$. Also (3.3), (3.4), and (4.2) yield for a suitable $C > 0$

$$R_r^2(z;\zeta) \leqq C(t-\tau)\ln\left(\frac{Cr}{t-\tau}\right), \tag{4.16}$$

uniformly in $z \in \mathbf{R}^{n+1}$, $\zeta \in \Omega_r^m(z)$, and $r \leqq r_0$. We now take δr instead of r. Replacing then (4.15), (4.16) in (3.18) we finally obtain

$$E_{\delta r}^{(m)}(z;\zeta) \leqq Cr^{-\frac{n+m}{2}}(t-\tau)^{m/2}\left[\ln\left(\frac{Cr}{t-\tau}\right)\right]^{m/2}\left[1+\frac{1}{t-\tau}\ln\left(\frac{Cr}{t-\tau}\right)\right].$$

The right-hand side of the last inequality is bounded by $Cr^{-(n/2+1)}$. This proves (4.7), and therefore the theorem.

5. Averaging of Superparabolic Functions

In this section we study the averaging operators $u \mapsto u_r^{(m)}$ introduced in Sect. 3 [see (3.19)] and their connection with L-superparabolic functions. In [FG] it was proved that if $u \in C^\infty(\mathbf{R}^{n+1})$ is a solution of $Lu = 0$ in $\mathbf{R}^{n+1}$, then for every $z \in \mathbf{R}^{n+1}$ $u(z) \equiv u_r(z)$, for every $r > 0$. In virtue of Theorem 3.1 a similar identity holds for the averages $u_r^{(m)}$. Vice-versa, by the maximum principle it can be easily seen that if $u \in C(\mathbf{R}^{n+1})$ and $u(z) = u_r^{(m)}(z)$ for every $z \in \mathbf{R}^{n+1}$ and every $r > 0$, then $u \in C^\infty(\mathbf{R}^{n+1})$ and $Lu = 0$. In what follows we will use the notion of L-superparabolic function given in Sect. 1.

Proposition 5.1. *Let $u : \mathbf{R}^{n+1} \to \bar{R}$ be a l.s.c. function, and let $m \in \mathrm{N} \cup \{0\}$. The following statements are equivalent:*

(i) *u is L-superparabolic in $\mathbf{R}^{n+1}$;*

(ii) *there exists $r_0 = r_0(L) > 0$ such that for every $z \in \mathbf{R}^{n+1}$ and $r \leqq r_0$*

$$u(z) \geqq u_r^{(m)}(z). \tag{5.1}$$

To prove Proposition 5.1 we will need the following.

Lemma 5.1. *There exists $r_0 = r_0(L) > 0$ such that if $z \in \mathbf{R}^{n+1}$, $r \leqq r_0$, and $\Omega_r(z)$ is the L-parabolic ball* (1.4), *then every point of $\Psi_r(z) \setminus \{z\}$ is L-regular.*

Proof. Because of the assumption (1.3) on L and the asymptotic estimates (4.11) and (4.12) we can find $r_0 = r_0(L) > 0$ small enough such that for $z \in \mathbf{R}^{n+1}$, $r \leqq r_0$ and $\zeta \in \Psi_r(z) \setminus \{z\}$ we have

$$|\nabla_\xi \Gamma(z;\zeta)| \neq 0 \quad \text{or} \quad D_\tau \Gamma(z;\zeta) < 0.$$

This is enough to reach the conclusion.

Proof of Proposition 5.1. That (ii) implies (i) is a consequence of the fact that a function for which (5.1) holds satisfies the strong minimum principle. To show that (i) implies (ii), we first argue in the case $m = 0$. In what follows r_0 is fixed as in Lemma 5.1. For $r \leqq r_0$, let $\varphi \in C(\Psi_r(z))$ with $\varphi \leqq u$ on $\Psi_r(z)$. We

Claim.

$$u(z) \geqq \int_{\Psi_r(z)} \varphi(\zeta) Q_r(z;\zeta)\, dH_n(\zeta),$$

where $Q_r(z;\zeta)$ is the kernel in (1.14). Suppose the claim is true. Taking the supremum over all continuous φ's on $\Psi_r(z)$ such that $\varphi \leqq u$ on $\Psi_r(z)$ we obtain

$$u(z) \geqq \int_{\Psi_r(z)} u(\zeta) Q_r(z;\zeta) dH_n(\zeta). \tag{5.2}$$

Proceeding now as in the proof of Theorem 2 in [FG] we obtain from (5.2) $u(z) \leqq u_r(z)$ for any $r \leqq r_0$, which proves that (i) implies (ii) in the case $m=0$. Let now $m \in \mathbf{N}$. We use the notation of Sect. 3. By thinking of u as a function in $\mathbf{R}^{n+m+1}$, i.e., setting $\hat{u}(x,y,t)=u(x,t)$, then $\hat{u}$ is $\hat{L}$-superparabolic with $\hat{L}$ as in (3.10). Therefore, from the discussion of the case $m=0$ we have

$$\hat{u}(\hat{z}) \geqq (\hat{u})_r(\hat{z}), \tag{5.3}$$

for every $\hat{z} \in \mathbf{R}^{n+m+1}$ and every $r \leqq r_0$. We remark that r_0 does not depend on m, and therefore it can be taken to be the same as in Lemma 5.1. (5.3) is another way to write (5.1), if one observes that $(\hat{u})_r(\hat{z}) = u_r^{(m)}(z)$, see the proof of Theorem 3.1.

We are thus left with the proof of the claim. To this end we place a tiny $(n+1)$-dimensional right circular cone on the top of $\Psi_r(z)$ so to cut out the irregular point $z=(x,t)$. The axis of the cone is along the t-axis and the vertex at the point $N_\varepsilon=(x,t+\varepsilon)$, $\varepsilon>0$. More precisely, let

$$B_\varepsilon(z)=\{\zeta \in \mathbf{R}^{n+1} \mid |\xi-x|^2+(\tau-t)^2 \leqq \varepsilon^2\}$$

be the closed $(n+1)$-ball of radius ε centered at z. Let $\Sigma_{r,\varepsilon}=B_\varepsilon(z) \cap \Psi_r(z)$ and $\partial\Sigma_{r,\varepsilon}(z) = \partial B_\varepsilon(z) \cap \Psi_r(z)$. From N_ε we project a cone onto $\partial\Sigma_{r,\varepsilon}(z)$. We denote this cone by $C_{r,\varepsilon}(z)$ and define

$$\tilde{\Psi}_{r,\varepsilon}(z) = [\Psi_r(z) \setminus \Sigma_{r,\varepsilon}(z)] \cup C_{r,\varepsilon}(z).$$

Now for φ chosen as in the claim we let $\tilde{\varphi} \in C(\tilde{\Psi}_{r,\varepsilon}(z))$ be such that $\tilde{\varphi} \equiv \varphi$ on $\Psi_r(z) \setminus \Sigma_{r,\varepsilon}(z)$. As $\tilde{\Psi}_{r,\varepsilon}(z)$ is the boundary of an L-regular bounded open set in $\mathbf{R}^{n+1}$, we let $H_{\tilde{\varphi}}$ denote the solution of the Dirichlet problem for it with boundary datum $\tilde{\varphi}$. Since u is L-superparabolic and $LH_{\tilde{\varphi}}=0$, using (1.8) we obtain

$$\begin{aligned} u(z) \geqq H_{\tilde{\varphi}}(z) &= \int_{\Psi_r(z)} H_{\tilde{\varphi}}(\zeta) Q_r(z;\zeta) dH_n(\zeta) \\ &= \int_{\Sigma_{r,\varepsilon}(z)} H_{\tilde{\varphi}}(\zeta) Q_r(z;\zeta) dH_n(\zeta) + \int_{\Psi_r(z)\setminus\Sigma_{r,\varepsilon}(z)} \varphi(\zeta) Q_r(z;\zeta) dH_n(\zeta). \end{aligned} \tag{5.4}$$

As $\varepsilon \to 0^+$ the first integral in the right-hand side of (5.4) goes to zero, whereas the second integral tends to the right-hand side of (5.2). This proves the claim, and therefore Proposition 5.1.

Proposition 5.2. *Let u be a L-superparabolic function on $\mathbf{R}^{n+1}$. Then if r_0 is as in Lemma 5.1 we have*

(i) $u_\varrho(z) \leqq u_r(z)$, *for every* $z \in \mathbf{R}^{n+1}$ *and every* $r < \varrho \leqq r_0$;
(ii) $u_r(z) \nearrow u(z)$ *as* $r \to 0^+$, *for every* $z \in \mathbf{R}^{n+1}$;
(iii) u_r *is L-superparabolic on* $\mathbf{R}^{n+1}$ *for every* $r \leqq r_0$.

Proof. We begin by remarking that we can assume that u is the L-potential of a compactly supported nonnegative measure μ on $\mathbf{R}^{n+1}$, i.e., $u=\Gamma_\mu$ where

$$\Gamma_\mu(z) = \int_{\mathbf{R}^{n+1}} \Gamma(z;\zeta) d\mu(\zeta),$$

and Γ is the fundamental solution of L in (1.1). In fact, if u is L-superparabolic on $\mathbf{R}^{n+1}$, then $u \in L^1_{\mathrm{loc}}(\mathbf{R}^{n+1})$ and $Lu \leqq 0$ in the sense of $\mathscr{D}'(\mathbf{R}^{n+1})$. Therefore, there exists a nonnegative measure ν on $\mathbf{R}^{n+1}$ such that $Lu = -\nu$. Now, if D is an arbitrary bounded open set in $\mathbf{R}^{n+1}$ and if we set $\mu = \nu|_{\bar{D}}$, then $L(u - \Gamma_\mu) = 0$ in D, and therefore $u = \Gamma_\mu + h$ in D, where $Lh = 0$ in D. From this representation of u, and the fact that h coincides with its parabolic average at every point $z \in D$, we conclude that it is sufficient to prove (i), (ii), and (iii) above in the case in which $u = \Gamma_\mu$.

We begin with (i). Let $z = (x, t) \in \mathbf{R}^{n+1}$ and consider the parabolic spheres centered at z $\Psi_r(z)$, $\Psi_\varrho(z)$. We want to prove that for $r < \varrho \leqq r_0$

$$\int_{\Psi_\varrho(z)} u(\zeta) Q_\varrho(z;\zeta) dH_n(\zeta) \leqq \int_{\Psi_r(z)} u(\zeta) Q_r(z;\zeta) dH_n(\zeta), \tag{5.5}$$

where Q_r has the same meaning as in (5.2). In the sequel we will denote by $u_r^\sigma(z)$ and $u_\varrho^\sigma(z)$ the surface averages appearing respectively in the left-hand side and in the right-hand side of (5.5). Since by a result of H. Bauer every L-superparabolic function is the pointwise limit of a monotone sequence of continuous L-superparabolic functions (see [Ba, Satz 2.5.8]), we may assume without loss of generality that u is continuous. Now for $\varepsilon > 0$ fixed we perform the same cutting and pasting as in the proof of Proposition 5.1. Letting U denote the L-parabolic extension of u to the set $\tilde{\Omega}_{\varrho,\varepsilon}(z)$, for which $\tilde{\Psi}_{\varrho,\varepsilon}(z) = \partial\tilde{\Omega}_{\varrho,\varepsilon}(z)$, we finally obtain

$$\begin{aligned} u_r^\sigma(z) &\geqq \int_{\Psi_r(z)} U(\zeta) Q_r(z;\zeta) dH_n(\zeta) = U(z) \\ &= \int_{\Psi_\varrho(z)} U(\zeta) Q_\varrho(z;\zeta) dH_n(\zeta) = \int_{\Sigma_{\varrho,\varepsilon}(z)} U(\zeta) Q_\varrho(z;\zeta) dH_n(\zeta) \\ &\quad + \int_{\Psi_\varrho(z)\setminus\Sigma_{\varrho,\varepsilon}(z)} u(\zeta) Q_\varrho(z;\zeta) dH_n(\zeta). \end{aligned} \tag{5.6}$$

Letting $\varepsilon \to 0^+$ we obtain (5.5) from (5.6). To complete the proof of (i) we will show that $u_\varrho^\sigma \leqq u_r^\sigma$ implies $u_\varrho \leqq u_r$. From the results in [FG] we obtain for every $r > 0$

$$\begin{aligned} u_r(z) &= \frac{n}{2} r^{-n/2} \int_0^r l^{n/2-1} \left(\int_{\Psi_l(z)} u(\zeta) Q_l(z;\zeta) dH_n(\zeta) \right) dl \\ &= \frac{n}{2} r^{-n/2} \int_0^r l^{n/2-1} u_l^\sigma(z) dl. \end{aligned} \tag{5.7}$$

Differentiating (5.7) with respect to r we have

$$\frac{d}{dr} u_r(z) = \frac{n}{2} r^{-1} [u_r^\sigma(z) - u_r(z)]. \tag{5.8}$$

On the other hand, since by (5.5) $r \mapsto u_r^\sigma(z)$ is decreasing, we have

$$\begin{aligned} u_r &= \frac{n}{2} r^{-n/2} \int_0^r l^{n/2-1} u_l^\sigma(z) dl \geqq \frac{n}{2} r^{-n/2} u_r^\sigma(z) \int_0^r l^{n/2-1} dl \\ &= u_r^\sigma(z). \end{aligned} \tag{5.9}$$

Using (5.9) in (5.8) we conclude that

$$\frac{d}{dr} u_r(z) \leqq 0.$$

This proves (i). The proof of (ii) is a standard consequence of the lower semicontinuity of u, and we omit it. The proof of (iii) is more delicate. We first establish the following.

Lemma 5.2. *With r_0 as in Lemma 5.1 for every $\varrho, r \leqq r_0$ and every $\zeta, z \in \mathbf{R}^{n+1}$ we have*

$$((\Gamma(\cdot;\zeta))_r)_\varrho(z) \leqq (\Gamma(\cdot;\zeta))_r(z). \tag{5.10}$$

In particular, (5.10) *implies that for every $\zeta \in \mathbf{R}^{n+1}$ the function $z \mapsto (\Gamma(\cdot;\zeta))_r(z)$ is L-superparabolic for every $r \leqq r_0$.*

Proof. For $\zeta \in \mathbf{R}^{n+1}$ fixed we set $w = \Gamma(\cdot;\zeta)$. By w_r^σ we will denote the parabolic surface average of w. The proof of (5.10) is based on the following.

Claim. For every $r \leqq r_0$ and every $z \in \mathbf{R}^{n+1}$ such that $\Gamma(z;\zeta) \neq (4\pi r)^{-n/2}$ we have

$$w_r^\sigma(z) = \min\{\Gamma(z;\zeta), (4\pi r)^{-n/2}\}.$$

Let us take the claim for granted and use it to prove the lemma. First, we observe that the function

$$v_r = \min\{\Gamma(\cdot;\zeta), (4\pi r)^{-n/2}\},$$

being the minimum of two L-superparabolic functions, is L-superparabolic in $\mathbf{R}^{n+1}$. Then by Proposition 5.1

$$(v_r)_\varrho \leqq v_r \tag{5.11}$$

for every $\varrho \leqq r_0$. On the other hand the claim gives

$$w_r^\sigma(z) = v_r(z) \quad \text{if} \quad \Gamma(z;\zeta) \neq (4\pi r)^{-n/2}. \tag{5.12}$$

Now using (5.7) for w_r we have from (3.8) for every $z_0 \in \mathbf{R}^{n+1}$, $\varrho, r \leqq r_0$

$$\begin{aligned}
(w_r)_\varrho(z_0) &= \int_{\Omega_\varrho(z_0)} w_r(z) E_\varrho(z_0;\zeta)dz = \int_{\Omega_\varrho(z_0)} \left[\frac{n}{2} r^{-n/2} \int_0^r l^{n/2-1} w_l^\sigma(z)dl\right] E_\varrho(z_0;z)dz \\
&= [\text{by } (5.12)] \int_{\Omega_\varrho(z_0)} \left[\frac{n}{2} r^{-n/2} \int_0^r l^{n/2-1} v_l(z)dl\right] E_\varrho(z_0;z)dz \\
&= \frac{n}{2} r^{-n/2} \int_0^r l^{n/2-1} \left[\int_{\Omega_\varrho(z_0)} v_l(z) E_\varrho(z_0;z)dz\right] dl \\
&= \frac{n}{2} r^{-n/2} \int_0^r l^{n/2-1} (v_l)_\varrho(z_0)dl \\
&\leqq [\text{by } (5.11)] \frac{n}{2} r^{-n/2} \int_0^r l^{n/2-1} v_l(z_0)dl \\
&= [\text{by } (5.12)] \frac{n}{2} r^{-n/2} \int_0^r l^{n/2-1} w_l^\sigma(z_0)dl = w_r(z_0).
\end{aligned}$$

We are thus left with the proof of the claim. We use (1.31) in Theorem 1.5 of [GL]. Applied to w the latter says that for every $r \leqq r_0$

$$w_r^\sigma(z_0) = w(z_0) + \int_{\Omega_r(z_0)} Lw(z)\,[\Gamma(z_0;z) - (4\pi r)^{-n/2}]dz, \tag{5.13}$$

provided that $\zeta \notin \overline{\Omega_r(z_0)}$. But $\zeta \notin \overline{\Omega_r(z_0)}$ implies $Lw(z) = L(\Gamma(\cdot\,;\zeta))(z) = 0$ for every $z \in \Omega_r(z_0)$. On the other hand $\zeta \notin \overline{\Omega_r(z_0)}$ (iff $\Gamma(z_0;\zeta) < (4\pi r)^{-n/2}$ and $\zeta \neq z_0$. (5.13) then gives for $\Gamma(z_0;\zeta) < (4\pi r)^{-n/2}$ and $\zeta \neq z_0$

$$w_r^\sigma(z_0) = w(z_0) = \Gamma(z_0;\zeta) = v_r(z_0).$$

But $z_0 = \zeta$ trivially implies $w_r^\sigma(z_0) = 0 = v_r(z_0)$, since in this case $w \equiv 0$ on $\Psi_r(z_0)$. To complete the proof of the claim we need only to consider the case in which z_0 is such that $\Gamma(z_0;\zeta) > (4\pi r)^{-n/2}$. Recalling that $Lw(z) = -\delta_\zeta(z)$, whereas $z \mapsto \Gamma(z;\zeta)$ is C^∞ in a neighborhood of z_0 that does not include ζ, again from (1.31) of Theorem 1.5 in [GL] we obtain

$$\begin{aligned} w_r^\sigma(z_0) &= w(z_0) + \int_{\mathbf{R}^{n+1}} Lw(z)\chi_{\Omega_r(z_0)}(z)\,[\Gamma(z_0;z) - (4\pi r)^{-n/2}]dz \\ &= w(z_0) - \langle \delta_\zeta(z), \chi_{\Omega_r(z_0)}(z)\,[\Gamma(z_0;z) - (4\pi r)^{-n/2}]\rangle \\ &= w(z_0) - \chi_{\Omega_r(z_0)}(\zeta)\Gamma(z_0;\zeta) + (4\pi r)^{-n/2} = (4\pi r)^{-n/2}. \end{aligned}$$

In the last equality we have used the fact that, since $\Gamma(z_0;\zeta) > (4\pi r)^{-n/2}$ iff $\zeta \in \Omega_r(z_0)$, $\chi_{\Omega_r(z_0)}(\zeta) = 1$. The proof of the claim is thus completed and, with it, the proof of Lemma 5.2.

We then return to the

Proof of Proposition 5.2 (continued). We are left with proving (iii) in the case in which $u = \Gamma_\mu$, where μ is a nonnegative compactly supported measure on $\mathbf{R}^{n+1}$. We observe first that if $\chi_{\Omega_r(z_0)}$ is the characteristic function of the parabolic ball $\Omega_r(z_0)$, we can write

$$\begin{aligned} u_r(z_0) &= \int_{\Omega_r(z_0)} u(z)E_r(z_0;z)dz = \int_{\mathbf{R}^{n+1}} \chi_{\Omega_r(z_0)}(z)u(z)E_r(z_0;z)dz \\ &= \int_{\mathbf{R}^{n+1}} \chi_{\Omega_r(z_0)}(z)\left[\int_{\mathbf{R}^{n+1}} \Gamma(z;\zeta)d\mu(\zeta)\right]E_r(z_0;z)dz \\ &= \int_{\mathbf{R}^{n+1}}\left[\int_{\mathbf{R}^{n+1}} \chi_{\Omega_r(z_0)}(z)\Gamma(z;\zeta)E_r(z_0;z)dz\right]d\mu(\zeta) \\ &= \int_{\mathbf{R}^{n+1}} (\Gamma(\cdot\,;\zeta))_r(z_0)d\mu(\zeta). \end{aligned}$$

The exchange of order of integration above is justified by Tonelli's theorem. The above identity and a similar exchange of order of integration give

$$\begin{aligned} (u_r)_\varrho(z_0) &= \int_{\mathbf{R}^{n+1}} \chi_{\Omega_\varrho(z_0)}(z)u_r(z)E_\varrho(z_0;z)dz \\ &= \int_{\mathbf{R}^{n+1}} \chi_{\Omega_\varrho(z_0)}(z)\int_{\mathbf{R}^{n+1}} [(\Gamma(\cdot\,;\zeta))_r(z)d\mu(\zeta)]\,E_\varrho(z_0;z)dz \\ &= \int_{\mathbf{R}^{n+1}}\left[\int_{\Omega_\varrho(z_0)} (\Gamma(\cdot\,;\zeta))_r(z)\,E_\varrho(z_0;z)dz\right]d\mu(\zeta) \\ &= \int_{\mathbf{R}^{n+1}} ((\Gamma(\cdot\,;\zeta))_r)_\varrho(z_0)d\mu(\zeta) \\ &\leqq \int_{\mathbf{R}^{n+1}} (\Gamma(\cdot\,;\zeta))_r(z_0)d\mu(\zeta) = u_r(z_0). \end{aligned} \tag{5.17}$$

In the last inequality above we have used (5.10). (5.17) shows that (ii) of Proposition 5.1 holds for u_r (with $m = 0$), and therefore u_r is L-superparabolic. This completes the proof of Proposition 5.2.

Corollary 5.1. *Let u be L-superparabolic on $\mathbf{R}^{n+1}$. Then the conclusions* (i)–(iii) *of Proposition 5.2 hold unchanged if we replace u_r by $u_r^{(m)}$ for any $m \in \mathbf{N} \cup \{0\}$.*

Proof. It follows by the same arguments of the proof of Proposition 5.2 once we observe that $u_r^{(m)}(z)$ coincides with $\hat{u}_r(\hat{z})$ and that $\hat{u}$ is $\hat{L}$-superparabolic in $\mathbf{R}^{n+m+1}$; see Sect. 3.

Before stating the next corollary of Proposition 5.2, we recall the Weyl type formulas (3.20), (3.21).

Corollary 5.2. *Let u be L-superparabolic on $\mathbf{R}^{n+1}$. Then for every $m \in \mathbf{N} \cup \{0\}$ we have:*

(i) $J_\varrho^{(m)} u \leqq J_r^{(m)} u$ *for every* $r < \varrho \leqq \frac{r_0}{2}$.

(ii) $J_r^{(m)} u \nearrow u$ *as* $r \to 0^+$.

(iii) $J_r^{(m)} u$ *is L-superparabolic on* $\mathbf{R}^{n+1}$ *for every* $r \leqq \frac{r_0}{2}$.

Proof. By the definition (3.20) of the operators $J_\varrho^{(m)}$ we obtain for every $z \in \mathbf{R}^{n+1}$

$$\begin{aligned} J_\varrho^{(m)}(u)(z) &= \int_0^{+\infty} u_l^{(m)}(z)\varphi\left(\frac{l}{\varrho}\right)\frac{dl}{\varrho} = \int_0^{+\infty} u_{\varrho l/r}^{(m)}(z)\varphi\left(\frac{l}{r}\right)\frac{dl}{r} \\ &\leqq \int_0^{+\infty} u_l^{(m)}(z)\varphi\left(\frac{l}{r}\right)\frac{dl}{r} = J_r^{(m)}(u)(z). \end{aligned}$$

In the inequality above we have used the fact that $\frac{\varrho}{r} > 1$ and Corollary 5.1. This proves (i). Since for every $z \in \mathbf{R}^{n+1}$

$$u(z) - J_r^{(m)}(u)(z) = \int_0^{+\infty} [u(z) - u_l^{(m)}(z)]\varphi\left(\frac{l}{r}\right)\frac{dl}{r},$$

from Proposition 5.1 and Corollary 5.1 we have

$$\begin{aligned} 0 &\leqq u(z) - J_r^{(m)}(u)(z) \leqq \sup_{0 < l \leqq 2r} [u(z) - u_l^{(m)}(z)] \\ &= u(z) - u_{2r}^{(m)}(z) \to 0 \quad \text{as} \quad r \to 0^+. \end{aligned}$$

Hence (ii) holds. We finally look at (iii). It is immediate to check that $J_r^{(m)}(u)$ is l.s.c. To prove that it is L-superparabolic we show that $J_r^{(m)}(u)$ is super mean valued. In fact, for every $z \in \mathbf{R}^{n+1}$ and $r \leqq \frac{r_0}{2}$ we have

$$\begin{aligned} (J_r^{(m)}(u))_\varrho(z) &= \int_{\Omega_\varrho(z)} J_r^{(m)}(u)(\zeta)E_\varrho(z;\zeta)d\zeta \\ &= \int_{\Omega_\varrho(z)} E_\varrho(z;\zeta)\left[\int_0^{+\infty} u_l^{(m)}(\zeta)\varphi\left(\frac{l}{r}\right)\frac{dl}{r}\right]d\zeta \\ &= \int_0^{+\infty}\left[\int_{\Omega_\varrho(z)} u_l^{(m)}(\zeta)E_\varrho(z;\zeta)d\zeta\right]\varphi\left(\frac{l}{r}\right)\frac{dl}{r} \\ &\quad \text{[by (iii) of Corollary 5.1 and Proposition 5.1]} \\ &\leqq \int_0^{+\infty} u_l^{(m)}(z)\varphi\left(\frac{l}{r}\right)\frac{dl}{r} = J_r^{(m)}(u)(z). \end{aligned}$$

This completes the proof of the corollary.

6. Smoothing of L-Superparabolic Functions

In several questions in potential theory a crucial problem is that of regularizing superharmonic functions. For harmonic functions this can be achieved by means of the usual mollification process. The same procedure can be followed for supertemperatures as the heat operator has constant coefficients. However, for an operator like L in (1.1) usual mollification does not allow to approximate L-superparabolic functions by smooth functions which are still L-superparabolic. In this section we show that this approximation problem can be solved by means of the Weyl type averaging operators introduced in (3.20). We emphasize that the operators $u \mapsto u_r^{(m)}$ defined by (3.19) could also be used, although it would be much more complicated to prove their regularizing properties.

Theorem 6.1. *Let u be a L-superparabolic function in $\mathbf{R}^{n+1}$ and let $\nu \in \mathbf{N}$ be fixed. Then there exists a sequence of functions $(u_j)_{j\in\mathbf{N}}$ such that*

(i) $u_j \in C^\nu(\mathbf{R}^{n+1})$, $j \in \mathbf{N}$,

(i)) u_j *is L-superparabolic in* $\mathbf{R}^{n+1}$, $j \in \mathbf{N}$,

(iii) $u_j \leqq u_{j+1}$, $j \in \mathbf{N}$,

(iv) $u_j(z) \to u(z)$ *as* $j \to +\infty$ *for every* $z \in \mathbf{R}^{n+1}$

(v) *if for a given compact $K \subset \mathbf{R}^{n+1}$ $Lu=0$ in $\mathbf{R}^{n+1} \backslash K$, then for every $\delta>0$ there exists $j_0 \in \mathbf{N}$ such that $Lu_j=0$ in $\mathbf{R}^{n+1} \backslash K_\delta$ for every $j \geqq j_0$, where*

$$\cdot K_\delta = \{z \in \mathbf{R}^{n+1} | \operatorname{dist}(z,K) \geqq \delta\}.$$

Proof. Let $(r_j)_{j\in\mathbf{N}}$ *be a sequence of positive numbers such that* $r_j \to 0$ *as* $j \to \infty$ *and* $r_{j+1} \leqq r_j \leqq \frac{r_0}{2}$ for every $j \in \mathbf{N}$ (r_0 is as in Corollary 5.2). For every $j \in \mathbf{N}$ we set

$$u_j = J_{r_j}^{(m)}(u), \tag{6.1}$$

where m is a positive integer to be fixed later on and $J_r^{(m)}$ is defined as in (3.21). By Corollary 5.2 it immediately follows that the sequence $(u_j)_{j\in\mathbf{N}}$ verifies (ii), (iii), and (iv) above. We now prove (v). By Theorem 2.1 we can find two positive numbers $C, \bar{r}$, depending only on L, m, and n, such that

$$\Omega_r^{(m)}(z) \subset \{(\xi,\tau) \in \mathbf{R}^{n+1} \,|\, |x-\xi|^2 \leqq Cr,\ 0<t-\tau<Cr\} \tag{6.2}$$

for every $z=(x,t) \in \mathbf{R}^{n+1}$ and $0<r<\bar{r}$. It is therefore clear that given $\delta>0$ there exists $j_0 \in \mathbf{N}$ such that

$$\Omega_l^{(m)}(z) \subset \mathbf{R}^{n+1} \backslash K$$

for every $l \leqq 2r_j$, $j \geqq j_0$, and $z \in \mathbf{R}^{n+1}$ such that $\operatorname{dist}(z,K)>\delta$. For such z's and l's we then have (see Theorem 3.1)

$$u_l^{(m)}(z) = u(z).$$

By (3.20), recalling that $\operatorname{supp}\varphi \subset (1,2)$, we obtain

$$u_j(z) = u(z)$$

for every $j \geqq j_0$ and $z \in \mathbf{R}^{n+1}$ with $\operatorname{dist}(z,K)>\delta$. From this (v) immediately follows.

We are left with proving (i). We will show that:

$$\textit{There exists } m \in \mathbf{N} \textit{ such that } J_r^{(m)}(u) \in C^\nu(\mathbf{R}^{n+1}) \textit{ for every } r,\ 0<r<\frac{r_0}{2}. \tag{6.3}$$

For ease of notation we set

$$\varphi_r(l) = \omega_m (4\pi l)^{-\frac{n+m}{2}} \frac{1}{r} \varphi\left(\frac{l}{r}\right),$$

where φ is the function appearing in (3.20). Then by (3.18) and (3.22), the kernel $M_r^{(m)}(z;\zeta)$ in (3.21) takes the form

$$M_r^{(m)} = \int\limits_{\left[4\pi\Phi^{\frac{2}{n+m}}\right]^{-1}}^{+\infty} R_l^m \left[E + \frac{m}{m+2} \frac{R_l^2}{4(t-\tau)^2}\right] \varphi_r(l) dl. \tag{6.4}$$

In (6.4) and in what follows we simply write $M_r^{(m)}$, R_l, and E, instead of $M_r^{(m)}(z;\zeta)$, $R_l(z;\zeta)$, $E(z;\zeta)$. Moreover, if $z=(x,t)$ and $\zeta=(\xi,\tau)$, the reader should keep in mind that $M_r^{(m)}$ is given by (6.4) if $t>\tau$, whereas $M_r^{(m)} \equiv 0$ if $t \leqq \tau$. Let us now take $m=2h$, for $h \in \mathbf{N}$. From (6.4) and (3.4) we obtain for $t>\tau$

$$\begin{aligned} M_r^{(2h)} = & \int\limits_{\left[4\pi\Phi^{\frac{2}{n+2h}}\right]^{-1}} (4(t-\tau))^h [\ln((4\pi l)^{n/2+h}\Phi)]^h \\ & \times \left[E + \frac{h}{h+1} \frac{1}{t-\tau} \ln((4\pi l)^{n/2+h}\Phi)\right] \varphi_r(l) dl \\ = & \sum_{k=0}^{h} c_{h,k} (t-\tau)^h (\ln \Phi)^{h-k} E \int\limits_{\left[4\pi\Phi^{\frac{2}{n+2h}}\right]^{-1}}^{+\infty} (\ln(4\pi l))^k \varphi_r(l) dl \\ & + \sum_{k=0}^{h+1} c'_{h,k} (t-\tau)^{h-1} (\ln\Phi)^{h+1-k} \int\limits_{\left[4\pi\Phi^{\frac{2}{n+2h}}\right]^{-1}}^{+\infty} (\ln(4\pi l))^k \varphi_r(l) dl, \end{aligned} \tag{6.5}$$

where $c_{h,k}$ and $c'_{h,k}$ are suitable constants. For $k=0,1,\dots,h+1$, we set

$$\psi_k(s) = \begin{cases} (\ln s)^{h-k} \int\limits_{\left[4\pi s^{\frac{2}{n+2h}}\right]^{-1}}^{+\infty} (\ln(4\pi l))^k \varphi_r(l) dl, & s>0, \\ 0, & s \leqq 0. \end{cases} \tag{6.6}$$

Since $\operatorname{supp}\varphi_r \subset (r, 2r)$, $\psi_k \in C^\infty(\mathbf{R})$ and $\operatorname{supp}\psi_k \subset ((8\pi r)^{-n/2-h}, +\infty)$. If we replace (6.6) in (6.5), letting $c_{h,h+1}=0$, we obtain

$$M_r^{(2h)} = (t-\tau)^{h-1} \sum_{k=0}^{h+1} \psi_k(\Phi) \left[c_{h,k}(t-\tau)E + c'_{h,k}(\ln\Phi)\right], \tag{6.7}$$

which we think valid over all of $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ if we agree to set $\psi_k(s)\ln s = 0$ for $s \leqq 0$. Using (6.7) we are going to prove the following:

Claim. *For any fixed* $r \in \left(0, \frac{r_0}{2}\right)$, *for any* $\nu \in \mathbf{N}$ *and any multi-index* $\alpha = (\alpha_1, \dots, \alpha_{n+1})$, $\alpha_i \in \mathbf{N} \cup \{0\}$ *and* $\alpha_1 + \dots + \alpha_{n+1} \leqq \nu$, *there exist* $h = h(\nu) \in \mathbf{N}$ *and a constant* $C = C(r, \alpha, h) > 0$ *such that*

$$|D_z^\alpha M_r^{(2h)}(z;\zeta)| \leqq C \tag{6.8}$$

for every $z, \zeta \in \mathbf{R}^{n+1}$, *with* $z \neq \zeta$. *In* (6.8)

$$D_z^\alpha = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \dots D_{x_n}^{\alpha_n} D_t^{\alpha_{n+1}}.$$

It is clear that from the claim, and (3.21), (6.3) follows, and therefore (i). To see this we differentiate under the integral sign in (3.21), use (6.8), observe that $u \in L^1_{\mathrm{loc}}(\mathbf{R}^{n+1})$ since u is L-superparabolic, and finally recall that for every $z \in \mathbf{R}^{n+1}$ fixed, the support of $M_r^{(m)}(z;\cdot)$ is contained in the parabolic ball $\Omega_{r_0}^{(m)}(z)$ and the latter by (6.2) is contained in the cylinder

$$\{(\xi,\tau)\in\mathbf{R}^{n+1} \mid |x-\xi|^2 \leqq Cr_0, 0<t-\tau\leqq Cr_0\}.$$

We are therefore left with proving the claim. For every $\zeta \in \mathbf{R}^{n+1}$ fixed the support of the function $M_r^{(2h)}(\cdot,\zeta)$ is contained in

$$\Lambda_{r_0}(\zeta)=\{z\in\mathbf{R}^{n+1} \mid \Phi(z;\zeta)>(4\pi r_0)^{-(n/2+h)}\}$$

[we remark that $\Lambda_{r_0}(\zeta)$ is a modified parabolic ball relatively to the operator L^*, adjoint of L]. Hence, it will be enough to prove (6.8) for every $\zeta \in \mathbf{R}^{n+1}$ and $z \in \Lambda_{r_0}(\zeta)$. From Theorem 2.1 [see also (4.11) and (4.12)] we deduce the following estimate. For every multi-index $\alpha=(\alpha_1,\ldots,\alpha_{n+1})$ and for every $s\in\mathbf{N}$ there exist $q=q(\alpha,s)\in\mathbf{N}$ and a constant $C=C(r_0,\alpha,s)>0$ such that

$$|(t-\tau)^q D_z^\alpha \Phi(z;\zeta)| \leqq C(t-\tau)^{-h}[G(z;\zeta)+(t-\tau)^s], \tag{6.9}$$

for every $\zeta \in \mathbf{R}^{n+1}$ and $z \in \Lambda_{r_0}(\zeta)$. In (6.9) $G(z;\zeta)$ is the generalized Gaussian introduced in Sect. 2. If $s \geqq 1$, then on $\Lambda_{r_0}(\zeta)$ $G(z;\zeta) \geqq C(t-\tau)^s$, by suitably modifying the constant C in (6.9). On the other hand, as for (4.2) we have $G(z;\zeta) \leqq C\Gamma(z;\zeta)$ for every $\zeta \in \mathbf{R}^{n+1}$ and $z \in \Lambda_{r_0}(\zeta)$. From these considerations and (6.9) we obtain

$$|(t-\tau)^q D_z^\alpha \Phi(z;\zeta)| \leqq C\Phi(z;\zeta) \tag{6.10}$$

for every $\zeta \in \mathbf{R}^{n+1}$ and $z \in \Lambda_{r_0}(\zeta)$. Observing now that the function $(t-\tau)^{n/2+h}\Phi(z;\zeta)$ is bounded on $\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}$ we obtain from (6.10): for every multi-index $\alpha=(\alpha_1,\ldots,\alpha_{n+1})$ there exist $q=q(\alpha)\in\mathbf{N}$ and a constant $C=C(r_0,\alpha,h)>0$ such that

$$|(t-\tau)^q D_z^\alpha \Phi(z;\zeta)| \leqq C \tag{6.11}$$

for every $\zeta \in \mathbf{R}^{n+1}$, $z \in \Lambda_{r_0}(\zeta)$. We then prove that an estimate similar to (6.11) holds for the functions $\psi_k(\Phi)$ defined through (6.6). To this purpose we observe that $D^\alpha \psi_k(\Phi)$ is a finite sum of terms of the type

$$\operatorname{const} \psi_k^{(p)}(\Phi) D^{\beta_{(1)}}\Phi \ldots D^{\beta_{(p)}}\Phi,$$

where $p\in\mathbf{N}$ and $p \leqq \alpha_1+\ldots+\alpha_{n+1}$, $\beta_{(1)},\ldots,\beta_{(p)}$ are multi-index whose length does not exceed the length of α, and $\psi_k^{(p)}$ is the p^{th} order derivative of ψ_k. Now from (6.6) one easily obtains the following estimate

$$|\psi_k^{(p)}(s)| \leqq \operatorname{const}(\ln s)^h, \quad \text{for every } s\in((8\pi r)^{-n/2-h}, +\infty),$$

whereas by (4.2)

$$|\ln \Phi(z;\zeta)| \leqq \ln[C_2(t-\tau)^{-n/2-h}], \quad \text{for} \quad \zeta\in\mathbf{R}^{n+1} \text{ and } z\in\Lambda_{r_0}(\zeta).$$

Using (6.11) we finally have: for every multi-index α there exist $q=q(\alpha)\in\mathbf{N}$ and $C=C(\alpha,r,h)>0$ such that

$$|(t-\tau)^q D_z^\alpha \psi_k(\Phi(z;\zeta))| \leqq C, \quad \zeta\in\mathbf{R}^{n+1} \text{ and } z\in\Lambda_{r_0}(\zeta). \tag{6.12}$$

In the same way one proves an analogous estimate for the functions $\ln\Phi$ and E. Concerning the derivatives of E, it is enough to observe that by (3.2) we can write

$$E(z;\zeta)=A(\zeta)\,[\nabla_\xi(\ln\Phi(z;\zeta))]\cdot\nabla_\xi(\ln\Phi(z;\zeta)).$$

From (6.7) and (6.12), and analogous estimates for $\ln\Phi$ and E we finally obtain: for every $\nu\in\mathbf{N}$ and for every multi-index $\alpha=(\alpha_1,\ldots,\alpha_{n+1})$, with $\alpha_1+\ldots+\alpha_{n+1}\leqq\nu$, there exist $h=h(\nu)\in\mathbf{N}$ and a constant $C=C(\alpha,r,h)>0$ such that

$$|D_z^\alpha M_r^{(2h)}(z;\zeta)|\leqq C,\qquad \zeta\in\mathbf{R}^{n+1},\ z\in\Lambda_{r_0}(\zeta).$$

This proves the claim, thus completing the proof of Theorem 6.1.

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Received December 28, 1987; in revised form April 25, 1988