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An Index Theory and Existence of Multiple Brake Orbits for Star-Shaped Hamiltonian Systems

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1. Introduction

Let $H:\mathbf{R}^{2N}\to\mathbf{R}$ be a continuously differentiable function and denote

$$J=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix},$$

where I is the $N\times N$ identity matrix. Consider the Hamiltonian system of differential equations

$$\dot{x}=JH'(x). \tag{HS}$$

If $x(t)$ satisfies (HS), then $\frac{d}{dt}H(x(t))=0$, so each solution of (HS) must necessarily lie on some energy surface $\{x\in\mathbf{R}^{2N}:H(x(t))=\text{const}\}$.

In this paper we will be concerned with the existence of periodic solutions of (HS) (of a priori unknown period) on a given compact hypersurface $H^{-1}(1)$. Such solutions are called periodic orbits. The problem of existence of at least one periodic orbit on $H^{-1}(1)$ has been studied by several authors, see e.g. Seifert [16], Weinstein [22], Rabinowitz [12], Viterbo [20], and Hofer and Zehnder [8].

Let $x=(p,q)\in\mathbf{R}^N\times\mathbf{R}^N$. A special kind of periodic orbits, henceforth called *brake orbits* (cf. [22]), are those for which the q-component of the solution of (HS) oscillates back and forth between two restpoints. More precisely, the corresponding $p(t)$ and $q(t)$ are T-periodic functions for some $T>0$, p odd and q even about 0 and $\frac{T}{2}$. Recently Rabinowitz [14] has shown that if H is even in p [i.e., $H(-p,q)=H(p,q)$] and $H^{-1}(1)$ bounds a compact star-shaped neighbourhood of $0\in\mathbf{R}^{2N}$ such that $x\cdot H'(x)\neq 0$ $\forall x\in H^{-1}(1)$, then (HS) possesses a brake orbit (see also [15] for a more general result).

In [4] Ekeland and Lasry have shown that (HS) has at least N periodic orbits if $H^{-1}(1)$ bounds a convex region and satisfies a certain geometric condition. This

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result has been later generalized by Berestycki et al. [2] to the case of $H^{-1}(1)$ bounding a star-shaped region. Results of the same type, for Hamiltonians having certain symmetry properties, have been obtained by Girardi [5] and van Groesen [19]. The proofs in these papers were carried out by reducing the problem to the one of finding critical points of an appropriate S^1- or $\mathbf{Z}_2$-symmetric functional. Multiple critical points were then obtained by invoking minimax arguments and topological index theories for symmetric sets.

When looking for brake orbits of (HS) it is natural to work in the space of periodic functions $(p(t), q(t))$ such that p is odd and q even in t. Although the functional is no longer symmetric in this space, it has a useful partial $\mathbf{Z}_2$-symmetry as we will see later. Our goal is to show that if $H(-p,q)=H(p,q)$ and $H^{-1}(1)$ bounds a star-shaped region and satisfies a geometric condition similar to that in [2], then (HS) possesses at least N brake orbits on $H^{-1}(1)$. The main tool in the proof is an index theory which we construct in Sect. 2. It is a variant of the relative index introduced by Berestycki et al. [2] and further developed by Tarantello [18] (see also Benci [1] for a related concept of pseudoindex). A special feature of our index is a strong dimension property (see Proposition 2.8 and Remark 2.9). In Sect. 3 we establish an abstract minimax principle which we employ in Sect. 4 to the problem of existence of N brake orbits.

An important role in the proof of dimension property is played by the generalized Borsuk theorem (Lemma 2.10). Its different versions are known in the literature (see e.g. Michalek [10], Nirenberg [11], Wang [21] and the references there). Our version is a simple adaptation of that in [10, 11, 21] to the case of partial $\mathbf{Z}_2$-symmetry. For the reader's convenience we include an appendix containing a proof of Lemma 2.10.

2. An Index Theory

In this section we develop an index theory similar to that in [2, 18]. Although we restrict our attention to the symmetry group $\mathbf{Z}_2$, it is clear that at least for separable spaces one obtains a corresponding theory for other symmetry groups (in particular, for S^1 and $\mathbf{Z}_p$ with p a prime integer).

Let E be a real Hilbert space and T a unitary representation of $\mathbf{Z}_2$ in E. That is, $T_0=I_E$ (the identity mapping on E) and T_1 is a linear isometry such that $T_1=T_1^{-1}$. A subset $A\subset E$ is said to be *T-invariant* (or simply *invariant*) if $T_1A\subset A$. Let

$$E^G=\{x\in E: T_1x=x\}$$

be the fixed point set of T. To T there corresponds an orthogonal decomposition $E=E^G\oplus F$. It is easy to see that F is invariant and

$$T_1(x+y)=x-y \ \forall x\in E^G,\ y\in F.$$

Let

$$\Sigma=\{A\subset E: A \text{ is closed and invariant}\}.$$

For $A\in\Sigma$ we define *the index of* A, denoted $\gamma(A)$, to be the smallest integer k such that there exists a continuous mapping $f: A\to\mathbf{R}^k-\{0\}$ satisfying $f(T_1x)=-f(x)$.

If there is no such k, then $\gamma(A)=\infty$. For the empty set $\emptyset$ we define $\gamma(\emptyset)=0$. Observe that if $A\cap E^G\neq\emptyset$, then $\gamma(A)=\infty$ (because $f(x)=f(T_1x)=-f(x)\ \forall x\in A\cap E^G$). It is easy to verify that γ satisfies the usual properties of index (which may be found e.g. in [13, 17]). In particular, denoting $N_\delta(A)=\{x\in E: d(x,A)\leqq\delta\}$, where $d(x,A)$ is the distance from x to A, we have

2.1. Proposition (Continuity Property). *If $A\in\Sigma$ is compact, then $\gamma(A)=\gamma(N_\delta(A))$ for some $\delta>0$.*

Let Y be a closed subspace of E. Henceforth P_Y will denote the orthogonal projection from E to Y and Y^G will be the set $Y\cap E^G$. A function $\xi: A\to\mathbf{R}$ is said to be *T-invariant* (or *invariant*) if $\xi(T_1x)=\xi(x)\ \forall x\in E$. A mapping $f: A\to E$ is *T-equivariant* (or *equivariant*) if $f(T_1(x))=T_1f(x)\ \forall x\in E$, and f is *compact* if the image of each bounded subset of A is contained in a compact set.

Let now $E=Y\oplus X$, where X, Y are orthogonal to each other and invariant. For $A\in\Sigma$, let $\mathscr{F}_k(A)$ be the set of all continuous mappings $f=(f_1,f_2): A\to Y\times\mathbf{R}^k-\{(0,0)\}$ satisfying the following conditions:

(i) *f is equivariant in the sense that $f_1(T_1x)=T_1f_1(x)$, $f_2(T_1x)=-f_2(x)$,*

(ii) *$f_1=P_Y-K$, where K is compact and $K(A)$ is bounded in Y,*

(iii) *$f_1(x)=x\ \forall x\in A\cap Y^G$ (and $f_2(x)=0$ by equivariance).*

Let $A\in\Sigma$. We define *the index of A relative to X*, denoted $\gamma_r(A,X)$, or shortly $\gamma_r(A)$ when no ambiguity can arise, to be the smallest integer k such that $\mathscr{F}_k(A)\neq\emptyset$. If there is no such k, we set $\gamma_r(A)=\infty$, and we define $\gamma_r(\emptyset)=0$.

It should be noted that the main difference between our index and that in [2, 18] lies in the requirement that K in (ii) above be of bounded range. In the following propositions we collect some basic properties of γ_r.

2.2. Proposition (Mapping Property). *Let $A,B\in\Sigma$ and let $g: A\to B$ be a continuous mapping such that $g(x)=e^{-\xi(x)L}x-K(x)$, where*

(i) *$L: E\to E$ is linear, equivariant, selfadjoint and $LY\subset Y$,*

(ii) *$\xi: A\to\mathbf{R}$ is invariant and $\xi(A)$ is bounded,*

(iii) *$K: A\to E$ is equivariant, compact and $K(A)$ is bounded.*

If $\xi|_{A\cap Y^G}=0$ and $K|_{A\cap Y^G}=0$ (i.e., if $g|_{A\cap Y^G}=I_{A\cap Y^G}$), then $\gamma_r(A)\leqq\gamma_r(B)$.

Proof. The conclusion is trivial if $\gamma_r(B)=\infty$. Assume that $\gamma_r(B)=k<\infty$. Then there exists an $f\in\mathscr{F}_k(B)$, $f=(f_1,f_2)$, $f_1=P_Y-C$. Note that $P_YL=LP_Y$ by selfadjointness of L. Define $\varphi: A\to Y\times\mathbf{R}^k$ by setting

$$\varphi(x)=(\varphi_1(x),\varphi_2(x))=(e^{\xi(x)L}f_1g(x), f_2g(x)).$$

Then φ is equivariant, $\varphi(x)\neq(0,0)\ \forall x\in A$ and

$$\begin{aligned}\varphi_1(x)&=e^{\xi(x)L}(P_Y-C)g(x)=e^{\xi(x)L}(P_Y-C)(e^{-\xi(x)L}x-K(x))\\&=P_Yx-e^{\xi(x)L}(P_YK(x)+C(g(x)))\equiv P_Yx-N(x).\end{aligned}$$

It is easy to see that N is compact and $N(A)$ bounded. If $x\in A\cap Y^G$, then $g(x)=x\in B\cap Y^G$, so $\varphi_1(x)=x$. Hence $\varphi\in\mathscr{F}_k(A)$ and $\gamma_r(A)\leqq k=\gamma_r(B)$. □

2.3. Proposition (Monotonicity). *If $A,B\in\Sigma$ and $A\subset B$, then $\gamma_r(A)\leqq\gamma_r(B)$.*

Proof. Take $g(x)=x$ in the preceding proposition. □

2.4. Proposition (Subadditivity). *If $A, B \in \Sigma$, then $\gamma_r(A \cup B) \leqq \gamma_r(A) + \gamma(B)$.*

Proof. It suffices to consider A, B with $\gamma_r(A) = k < \infty$, $\gamma(B) = m < \infty$. Suppose $f = (P_Y - K, f_2) \in \mathscr{F}_k(A)$ and $g: B \to \mathbf{R}^m - \{0\}$ satisfies $g(T_1 x) = -g(x)$. Since the convex hull of a compact set is compact, there exists a compact extension $\tilde{K}: E \to Y$ of K such that $\tilde{K}(E)$ is bounded. We may assume that $\tilde{K}$ is equivariant (if it is not, replace it by $\frac{1}{2}\tilde{K} + \frac{1}{2} T_1 \tilde{K} T_1$). Similarly, there exist extensions $\tilde{f}_2: E \to \mathbf{R}^k$, $\tilde{g}: E \to \mathbf{R}^m$ of f_2 and g such that $\tilde{f}_2(T_1 x) = -\tilde{f}_2(x)$, $\tilde{g}(T_1 x) = -\tilde{g}(x)$. Let now

$$h(x) = (P_Y x - \tilde{K}(x), \tilde{f}_2(x), \tilde{g}(x)) \in Y \times \mathbf{R}^k \times \mathbf{R}^m.$$

Then $h: A \cup B \to Y \times \mathbf{R}^{k+m}$, h is equivariant, $h = (P_Y - \tilde{K}, h_2)$ and $h(x) \neq (0,0)$ $\forall x \in A \cup B$ because $(P_Y x - \tilde{K}(x), f_2(x)) = (P_Y x - K(x), f_2(x)) \neq (0,0)$ on A and $\tilde{g}(x) = g(x) \neq 0$ on B. Since $\gamma(B) < \infty$, $B \cap Y^G = \emptyset$. Therefore $(A \cup B) \cap Y^G = A \cap Y^G$, and for $x \in A \cap Y^G$, $P_Y x - \tilde{K}(x) = x$. Hence $h \in \mathscr{F}_{k+m}(A \cup B)$ and $\gamma_r(A \cup B) \leqq k + m$. □

2.5. Proposition. *If $A, B \in \Sigma$ and $\gamma(B) < \infty$, then $\gamma_r(\overline{A - B}) \geqq \gamma_r(A) - \gamma(B)$.*

Proof. It follows from Propositions 2.3 and 2.4 that $\gamma_r(A) \leqq \gamma_r(\overline{A - B} \cup B) \leqq \gamma_r(\overline{A - B}) + \gamma(B)$. □

2.6. Proposition (Intersection Property). *Let $A \in \Sigma$. Suppose that $X = X_0 \oplus X_1$, where X_0, X_1 are orthogonal to each other, invariant and $X_0 \cap E^G = \{0\}$. If $\dim X_0 = k < \infty$ and $\gamma_r(A) > k$, then $A \cap X_1 \neq \emptyset$.*

Proof. Suppose that $A \cap X_1 = \emptyset$ and let $f(x) = P_Y x + P_{X_0} x$. Then $f: A \to Y \oplus X_0 - \{0\}$. Since X_0 is invariant and $X_0 \cap E^G = \{0\}$, $T_1 x_0 = -x_0$ $\forall x_0 \in X_0$ (because $x_0 + T_1 x_0 \in X_0 \cap E^G$). So $P_{X_0} T_1 x = T_1 P_{X_0} x = -P_{X_0} x$. It follows that X_0 can be identified with $\mathbf{R}^k$ and f with the equivariant mapping $x \mapsto (P_Y x, f_2(x))$ from A to $Y \times \mathbf{R}^k$ ($f_2(x) \in \mathbf{R}^k$ corresponds to $P_{X_0} x \in X_0$). Hence $\mathscr{F}_k(A) \neq \emptyset$ and $\gamma_r(A) \leqq k$, a contradiction. □

In order to state the next property of γ_r we will need the following geometric condition.

2.7. Definition. A set A is said to satisfy *condition* ($\mathscr{G}$) if for each finite dimensional subspace E_0 of E and each $r > 0$ there exists an $R > 0$ such that if $\|x\| \leqq r$, then $A \cap (x + E_0) \subset B_R$.

Here $x + E_0 = \{x + y \in E : y \in E_0\}$ and $B_R = \{x \in E : \|x\| < R\}$.

2.8. Proposition (Dimension Property). *Let $X_0 \subset X$ be an invariant subspace with $\dim X_0 = k$ and $X_0 \cap E^G = \{0\}$. Let U be an open invariant neighbourhood of the origin in $Y \oplus X_0$. If $\bar{U}$ satisfies condition ($\mathscr{G}$), then $\gamma_r(\partial U) = k$, where ∂U is the boundary of U in $Y \oplus X_0$.*

2.9. Remarks. (i) For the index in [2, 18] the conclusion of Proposition 2.8 remains valid if U is bounded but fails in general (see Bögle [3]). In Sect. 4 we will employ the above proposition to sets which satisfy condition ($\mathscr{G}$) and are unbounded. It was the need of a dimension property for unbounded sets that has motivated us to look for an index theory different from already existing ones.

(ii) As we have just observed, our index satisfies a stronger dimension property than that in [2, 18]. On the other hand, our mapping property is weaker (because the mapping K in Proposition 2.2 has bounded range). This has the disadvantage that in order to establish the minimax principle of the next section it seems necessary to modify the standard deformation lemma. Such a modification will be carried out for a class of functionals satisfying a compactness hypothesis which is somewhat stronger than the usual Palais-Smale condition.

In the proof of Proposition 2.8 we will use the following generalized Borsuk theorem (cf. Appendix).

2.10. Lemma. *Let W be an open bounded neighbourhood of $0 \in \mathbf{R}^m \times \mathbf{R}^n$ such that if $(x, y) \in W$, then $(x, -y) \in W$. Let $f = (g, h): \bar{W} \to \mathbf{R}^m \times \mathbf{R}^n$ be a continuous mapping with $g(x, -y) = g(x, y)$, $h(x, -y) = -h(x, y)$ $\forall (x, y) \in \bar{W}$, $f|_{\partial W} \neq 0$ and $f(x, 0) = (x, 0)$ $\forall (x, 0) \in \partial W$. Then the Brouwer degree* $\deg(f, W, 0)$ *is an odd integer.*

Proof of Proposition 2.8. By Proposition 2.6, $\gamma_r(\partial U) \leqq k$ (because ∂U does not intersect the orthogonal complement of X_0 in X). Suppose $\gamma_r(\partial U) < k$. Let $f = (P_Y - K, f_2): \partial U \to Y \times \mathbf{R}^{k-1}$, where $f \in \mathscr{F}_{k-1}(\partial U)$. By replacing $f_2(x)$ with $f_2(x)/(1 + |f_2(x)|)$ if necessary we may assume that $f_2(\partial U)$ is bounded. Since X_0 is invariant and $X_0 \cap E^G = \{0\}$, we may also assume (as in the proof of Proposition 2.6) that $f = P_Y - K + f_2: \partial U \to Y \oplus X_0'$, where X_0' is a $k-1$ dimensional subspace of X_0. Let $\tilde{K}: \bar{U} \to Y$, $\tilde{f}_2: \bar{U} \to X_0'$ be equivariant extensions of K and f_2 such that $\tilde{K}$ is compact and $\tilde{K}(\bar{U})$, $\tilde{f}_2(\bar{U})$ are bounded (cf. the proof of Proposition 2.4). Since $K|_{\partial U \cap Y^G} = 0$, we may assume that $\tilde{K}|_{\bar{U} \cap Y^G} = 0$. Let r be a number such that $\tilde{K}(\bar{U}) \subset B_r$. By condition ($\mathscr{G}$), there is an $R > 0$ such that

$$\bar{U} \cap (y + X_0) \subset B_R \text{ whenever } \|y\| \leqq r. \tag{1}$$

Let $x \in \bar{U}$. Suppose $\|P_Y x\| \leqq r$. Since $x = P_Y x + P_{X_0} x \subset P_Y x + X_0$, $x \in \bar{U} \cap (P_Y x + X_0)$ $\subset B_R$ according to (1). It follows that if $x \in \bar{U} \cap \partial B_R$, then $\|P_Y x\| > r$, and therefore $P_Y x - \tilde{K}(x) \neq 0$. Hence $\tilde{f}(x) = P_Y x - \tilde{K}(x) + \tilde{f}_2(x) \neq 0$ whenever $x \in \partial(U \cap B_R)$ (recall that $\tilde{f} = f \neq 0$ on ∂U). Furthermore, $\tilde{f} = I - C$, where $C = \tilde{K} + P_{X_0} - \tilde{f}_2$ is a compact mapping. It follows that the Leray-Schauder degree (see e.g. [9]) $\deg(\tilde{f}, U \cap B_R, 0)$ is well defined. Since $\tilde{f}(U \cap B_R) \subset Y \oplus X_0'$, $\tilde{f}(x) + te \neq 0$ for any $x \in \overline{U \cap B_R}$, $t > 0$ and $e \in X_0 - X_0'$. So by the homotopy invariance,

$$\deg(\tilde{f}, U \cap B_R, 0) = 0. \tag{2}$$

It follows from the properties of $\tilde{K}$ and $\tilde{f}_2$ that the mapping C is equivariant and $C|_{\bar{U} \cap Y^G} = 0$. Given $\varepsilon > 0$, there exists a mapping C_ε such that $\|C(x) - C_\varepsilon(x)\|$ $< \varepsilon$ $\forall x \in \overline{U \cap B_R}$ and $C_\varepsilon(U \cap B_R)$ is contained in a finite dimensional space [9, Theorem 4.2.2]. By a slight modification of the proof in [9] we will obtain C_ε which has some additional properties. Since the set $\overline{C(U \cap B_R)}$ is compact, it may be covered by open balls

$$B_\varepsilon(v_i) = \{z \in Y \oplus X_0 : \|z - v_i\| < \varepsilon\}, \quad 0 \leqq i \leqq 2N.$$

Moreover, we may assume that $v_0 = 0$ and $v_{i+N} = -v_i$ for $i = 1, \ldots, N$. Let $m_i(x)$ $= \max\{0, \varepsilon - \|C(x) - v_i\|\}$ and

$$\theta_i(x) = m_i(x) \Big/ \sum_{j=0}^{2N} m_j(x), \quad 0 \leqq i \leqq 2N.$$

Define

$$D_\varepsilon(x) = \sum_{i=0}^{2N} \theta_i(x) v_i$$

and $C_\varepsilon(x) = \frac{1}{2}D_\varepsilon(x) + \frac{1}{2}T_1 D_\varepsilon(T_1 x)$. It is easy to see (cf. [9]) that C_ε has the properties mentioned above, it is equivariant and $C_\varepsilon|_{\overline{U \cap B_R} \cap Y^G} = 0$. (Indeed, if $x \in \overline{U \cap B_R} \cap Y^G$, then $C(x) = C(T_1 x) = 0$, so $m_i(x) = m_{i+N}(x)$ for $1 \leqq i \leqq N$. Therefore $D_\varepsilon(x) = D_\varepsilon(T_1 x) = 0$. It follows that $C_\varepsilon(x) = 0$.) Choose a finite dimensional invariant subspace Y_0 of Y such that $Y_0 \oplus X_0$ contains the range of C_ε. If ε is sufficiently small,

$$\deg(\tilde{f}, U \cap B_R, 0) = \deg(I - C_\varepsilon, U \cap B_R \cap (Y_0 \oplus X_0), 0). \tag{3}$$

By Lemma 2.10, the right-hand side of (3) is an odd integer, a contradiction to (2). Hence $\gamma_r(\partial U)$ cannot be less than k. □

3. A Minimax Principle

Let E be a real Hilbert space with an inner product $\langle\ ,\ \rangle$ and T a unitary representation of Z_2 in E. Let $L: E \to E$ be an equivariant and selfadjoint bounded linear operator. Define $\Phi(x) = \frac{1}{2}\langle Lx, x\rangle$. Then Φ is an invariant functional and $\Phi'(x) = L(x)$ (here we assume via the Riesz representation theorem that $\Phi'(x) \in E$). Let ψ be an invariant functional on E such that $\psi^{-1}(1) \neq \emptyset$. Set $M = \psi^{-1}(1)$. Suppose further that $\psi \in C^{1,1}(E, \mathbf{R})$ and ψ' is a compact mapping which is bounded away from 0 on bounded subsets of M. Then M is an invariant $C^{1,1}$-manifold. For $x \in M$ denote

$$\lambda(x) = \frac{\langle Lx, \psi'(x)\rangle}{\|\psi'(x)\|^2}$$

and observe that $\langle Lx - \lambda(x)\psi'(x), \psi'(x)\rangle = 0$. So $Lx - \lambda(x)\psi'(x)$ is an element of the tangent space $T_x(M)$. In particular, $x \in M$ is a critical point of $\Phi|_M$ if and only if $Lx = \lambda(x)\psi'(x)$. We will need the following compactness hypothesis which is stronger than the usual Palais-Smale condition:

(C^*) *If* $(x_n) \subset M$ *is a sequence such that* $\Phi(x_n) \to c \in \mathbf{R}$ *and* $\dfrac{Lx_n - \lambda(x_n)\psi'(x_n)}{(\|x_n\| + 1)^{1/2}} \to 0$, *then* (x_n) *has a convergent subsequence.*

Let

$$\Phi_c = \{x \in M : \Phi(x) \leqq c\} \quad \text{and} \quad K_c = \{x \in M : \Phi(x) = c, Lx = \lambda(x)\psi'(x)\}.$$

In the proof of the minimax principle we will employ the following deformation lemma.

3.1. Lemma. *Suppose that* Φ, ψ *and* M *are as above and* $\Phi|_M$ *satisfies* (C^*). *Given* $c \in \mathbf{R}$, $\bar{\varepsilon} > 0$ *and a neighbourhood* U *of* K_c *in* M, *there exist* $\varepsilon \in (0, \bar{\varepsilon})$ *and a mapping* $\eta: [0,1] \times M \to M$ *such that:*

(i) $\eta(t, \cdot)$ *is a homeomorphism* $\forall t \in [0,1]$,

(ii) $\eta(0, x) = x$ $\forall x \in M$,

(iii) $\eta(1,x)=x \ \forall x\in M, \Phi(x)\notin(c-\bar{\varepsilon},c+\bar{\varepsilon})$,

(iv) $\|\eta(1,x)-x\|\leqq 2\|L\| \ \forall x\in M$ *($\|L\|$ is the norm of the operator L),*

(v) $\eta(1,\Phi_{c+\varepsilon}-U)\subset\Phi_{c-\varepsilon}$,

(vi) $\eta(1,x)=e^{-\theta(x)L}x-K(x)$, *where* $\theta(x)\in[0,1] \ \forall x\in M$, *$K$ is compact and $K(M)$ is bounded,*

(vii) $\eta(1,\cdot)$ *is equivariant.*

Proof. Since results of similar type are well known (see e.g. [13, Appendix A]), we will only outline the argument. By (C^*), K_c is a compact set. It follows that if $N_\delta=\{x\in M: d(x,K_c)\leqq\delta\}$, where $d(x,K_c)$ is the distance from x to K_c, then $N_\delta\subset U$ for $\delta>0$ sufficiently small. Thus we may assume that $U=N_\delta$. We claim that there exist $\hat{\varepsilon}\in(0,\bar{\varepsilon})$ and $b>0$ such that

$$\|Lx-\lambda(x)\psi'(x)\|\geqq b(\|x\|+1)^{1/2}\geqq b \ \forall x\in\Phi_{c+\hat{\varepsilon}}-(\Phi_{c-\hat{\varepsilon}}\cup N_{\delta/8}). \tag{4}$$

For if not, we find $b_n\to 0$, $\hat{\varepsilon}_n\to 0$ and $x_n\in\Phi_{c+\hat{\varepsilon}_n}-(\Phi_{c-\hat{\varepsilon}_n}\cup N_{\delta/8})$ such that

$$\|Lx_n-\lambda(x_n)\psi'(x_n)\|\leqq b_n(\|x_n\|+1)^{1/2}.$$

By (C^*), $x_n\to\bar{x}\in K_c$ after passing to a subsequence. This is impossible because $x_n\notin N_{\delta/8}$. So (4) is satisfied for some $\hat{\varepsilon}$ and b. Choose $\varepsilon\in(0,\hat{\varepsilon})$ and let $\chi_1,\chi_2: M\to[0,1]$ be two Lipschitz continuous functions such that $\chi_1(x)=0$ if $\Phi(x)\notin(c-\hat{\varepsilon},c+\hat{\varepsilon})$, $\chi_1(x)=1$ if $\Phi(x)\in[c-\varepsilon,c+\varepsilon]$ and $\chi_2(x)=0$ if $x\in N_{\delta/8}$, $\chi_2(x)=1$ if $x\notin N_{\delta/4}$. Since the sets in the definitions of χ_1 and χ_2 are invariant, we may assume that χ_1, χ_2 are invariant functions.

Let $\chi(x)=\chi_1(x)\chi_2(x)$ and consider the initial value problem

$$\frac{d\eta}{dt}=-\frac{\chi(\eta)}{\|\eta\|+1}(L\eta-\lambda(\eta)\psi'(\eta)), \qquad \eta(0,x)=0, \tag{5}$$

where $x\in M$. Since $\left\|\frac{d\eta}{dt}\right\|\leqq 2\|L\|$ and the vector field in (5) is locally Lipschitz continuous, (5) has a unique solution $\eta(t,x)$ defined for all $t\in\mathbf{R}$. It is therefore clear that (i)–(iv) are satisfied. Since

$$\begin{aligned}\frac{d}{dt}\Phi(\eta(t,x))&=\left\langle L\eta,\frac{d\eta}{dt}\right\rangle=-\frac{\chi(\eta)}{\|\eta\|+1}\langle L\eta, L\eta-\lambda(\eta)\psi'(\eta)\rangle\\&=-\frac{\chi(\eta)}{\|\eta\|+1}\|L\eta-\lambda(\eta)\psi'(\eta)\|^2\leqq 0,\end{aligned}$$

$\Phi(\eta(t,x))$ is nonincreasing as t increases. Furthermore, according to the first inequality in (4), $\frac{d}{dt}\Phi(\eta(t,x))\leqq -b^2$ whenever $\chi(\eta)=1$. Now one can follow the argument in [13, p. 84] (with obvious changes) in order to obtain (v).

Denote $\omega(x)=\frac{\chi(x)}{\|x\|+1}$. As in [13, p. 86] or [14], one sees from (5) that $\eta(1,x)=e^{-\theta(x)L}x-K(x)$, where $\theta(x)=\theta(1,x)$,

$$\theta(t,x)=\int_0^t \omega(\eta(s,x))ds$$

and

$$K(x) = -\int_0^1 [\exp(\theta(t,x)-\theta(1,x))L]\,\omega(\eta(t,x))\,\lambda(\eta(t,x))\,\psi'(\eta(t,x))\,dt\,. \tag{6}$$

Hence $\theta(x) \in [0,1]$. Using the definitions of ω and λ,

$$\|\omega(x)\lambda(x)\psi'(x)\| \leqq \frac{\|Lx\|}{\|x\|+1},$$

so it follows from (6) that $K(M)$ is bounded. Recall that ψ' is bounded away from 0 on bounded subsets of M. Therefore the mapping $x \mapsto \lambda(x)\psi'(x)$, $x \in M$, is compact, and the same argument as in [13] or [14] (using (iv)) shows that K is compact. So (vi) is verified. Finally, (vii) is a direct consequence of the fact that the vector field in (5) is equivariant. □

Now we are ready to state the main result of this section.

3.2. Theorem. *Suppose that Φ, ψ and M satisfy the assumptions at the beginning of this section and $\Phi|_M$ satisfies (C^*). Let $E = Y \oplus X$, where $Y = X^\perp$, X and Y are invariant and $LY \subset Y$. Define*

$$\Gamma_j = \{A \in \Sigma : A \subset M, \gamma_r(A, X) \geqq j\}$$

and

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} \Phi(x), \qquad j = 1, \ldots, N\,.$$

Suppose also that there exist finite numbers a, b such that $c_j \in (a,b)$ for $1 \leqq j \leqq N$ and $Y^G \cap \Phi^{-1}([a,b]) = \emptyset$. Then all c_j are critical values of $\Phi|_M$. Furthermore, if $c_j = \ldots = c_{j+p}$ for some j and some $p \geqq 0$, then $\gamma(K_{c_j}) \geqq p+1$.

Proof. It suffices to prove the second assertion. Let $c_j = \ldots = c_{j+p} \equiv c$, where $p \geqq 0$, and suppose that $\gamma(K_c) \leqq p$. Since K_c is compact (by (C^*)), it follows from Proposition 2.1 that $\gamma(K_c) = \gamma(N_\delta(K_c))$ for $\delta > 0$ sufficiently small. Let $\bar\varepsilon < \min\{b-c, c-a\}$, $U = N_\delta(K_c)$, and let ε, η be as in the conclusion of Lemma 3.1. Choose $A \in \Gamma_{j+p}$ such that $A \subset \Phi_{c+\varepsilon}$. According to Proposition 2.5, $\gamma_r(\overline{A - N_\delta(K_c)}) \geqq \gamma_r(A) - \gamma(N_\delta(K_c)) \geqq j+p-p = j$. So $\overline{A - N_\delta(K_c)} \in \Gamma_j$. Let $B = \eta(1, \overline{A - N_\delta(K_c)})$. Since $a < c - \bar\varepsilon$, $b > c + \bar\varepsilon$ and $Y^G \cap \Phi^{-1}([a,b]) = \emptyset$, $\eta(1,x) = x$ $\forall x \in \overline{A - N_\delta(K_c)} \cap Y^G$ by (iii) of Lemma 3.1. It follows therefore from Proposition 2.2 (using (vi) and (vii) of Lemma 3.1) that $\gamma_r(B) \geqq j$. This is a contradiction because $B \subset \Phi_{c-\varepsilon}$ according to (v) of Lemma 3.1. □

3.3. Remark. (i) Note that it does not follow from the theorem that $\Phi|_M$ has more than one critical point. Indeed, if all c_j coincide and if K_{c_j} consists of a single point in E^G, then $\gamma(K_{c_j}) = \infty$. So the conclusion is not violated.

(ii) Suppose that $E^G \cap K_{c_j} = \emptyset$ for $j = 1, \ldots, N$. Then $\Phi|_M$ has at least N distinct pairs of critical points because either all c_j are distinct or $c_j = \ldots = c_{j+p}$ for some j and some $p > 0$. In the latter case $\gamma(K_{c_j}) \geqq 2$, so K_{c_j} is an infinite set.

4. Existence of Brake Orbits

Let $H \in C^2(\mathbf{R}^{2N}, \mathbf{R})$ and suppose that $H(-p,q)=H(p,q)\ \forall (p,q)\in \mathbf{R}^{2N}$. Suppose also that the set $S=H^{-1}(1)$ is compact, bounds a star-shaped neighbourhood of the origin in $\mathbf{R}^{2N}$ and $x\cdot H'(x)\neq 0$ on S (the dot denotes the inner product in $\mathbf{R}^{2N}$). There exist two positive numbers, r and R, such that

$$r \leqq |x| \leqq R \ \forall x \in S. \tag{7}$$

Let ϱ be the largest number for which

$$T_y(S) \cap \{x \in \mathbf{R}^{2N} : |x| < \varrho\} = \emptyset \ \forall y \in S, \tag{8}$$

where $T_y(S)$ is the tangent hyperplane to S at y and $|x|=(x\cdot x)^{1/2}$.

We will be concerned with the existence of brake orbits for the Hamiltonian system

$$\dot{x} = JH'(x), \tag{HS}$$

which lie on the hypersurface S. Our goal is to prove the following

4.1. Theorem. *Let $H \in C^2(\mathbf{R}^{2N}, \mathbf{R})$ be such that:*

(i) $H(-p,q)=H(p,q)\ \forall (p,q)\in \mathbf{R}^{2N}$,

(ii) *The set $\mathscr{A}=\{x\in\mathbf{R}^{2N}: H(x)\leqq 1\}$ is nonempty, compact, star-shaped with respect to the origin and $S=H^{-1}(1)$ is the boundary of $\mathscr{A}$,*

(iii) $x\cdot H'(x)\neq 0\ \forall x\in S$.

If $R^2<2\varrho^2$, where R and ϱ are as in (7), (8), *then* (*HS*) *has at least N distinct brake orbits on S.*

As the first step towards proving the above result we will find a convenient variational formulation of the problem. In doing this we essentially follow [14]. It is known that changing H outside S does not change the orbits (see e.g. [12, Lemma 1.5] or [13, Proposition 6.47]). We may therefore assume that $H(0)=0$ and $H(x)=\alpha(x)^2$ if $x\neq 0$, where $\alpha(x)$ is the unique positive number such that $\frac{x}{\alpha(x)}\in S$. Then H is homogeneous of degree two, $H\in C^2(\mathbf{R}^{2N}-\{0\}, \mathbf{R}) \cap C^{1,1}(\mathbf{R}^{2N},\mathbf{R})$ and $\frac{|H'(x)|}{|x|}$ is bounded [12, p. 160]. It is also clear that $H(-p,q) = H(p,q)$. Since

$$r \leqq \frac{|x|}{\alpha(x)} \leqq R$$

according to (7), it follows that

$$\frac{|x|^2}{R^2} \leqq H(x) \leqq \frac{|x|^2}{r^2} \ \forall x \in \mathbf{R}^{2N}. \tag{9}$$

By [14, Lemma 2.3], see also [12, p. 161], there is a one-to-one correspondence between brake orbits for (*HS*) on S (of unknown period T) and 2π-periodic brake orbits for

$$\dot{x} = \lambda JH'(x), \qquad \lambda > 0, \tag{10}$$

on S. The number λ in (10) is unknown and related to T.

Now we introduce a suitable function space. Let $H^{1/2}(S^1, \mathbf{R}^{2N})$ be the Sobolev space of 2π-periodic $\mathbf{R}^{2N}$-valued functions

$$x = \sum_{k\in\mathbf{Z}} c_k e^{ikt}, \quad \text{where} \quad c_k \in \mathbf{C}^{2N} \quad \text{and} \quad c_{-k} = \bar{c}_k,$$

such that

$$\sum_{k\in\mathbf{Z}} (1+|k|)\,|c_k|^2 < \infty .$$

Let

$$E = \{x = (p,q) \in H^{1/2}(S^1, \mathbf{R}^{2N}) : p(-t) = -p(t), q(-t) = q(t)\ \forall t\} .$$

For $x = (p,q) \in E$, set $z = p + iq$. Then

$$z = \sum_{k\in\mathbf{Z}} a_k e^{ikt}, \quad \text{where} \quad a_k \in \mathbf{C}^N . \tag{11}$$

Since p is odd and q is even, it is easy to see that $Re\, a_k = 0\ \forall k$. A convenient norm in E, equivalent to the $H^{1/2}$-norm, is given by

$$\|x\|^2 = 2\pi\Big(|a_0|^2 + \sum_{k\neq 0} |k|\,|a_k|^2\Big). \tag{12}$$

Note that

$$\|x\|_{L^2}^2 = \int_0^{2\pi} |x|^2 dt = 2\pi \sum_{k\in\mathbf{Z}} |a_k|^2 \leqq \|x\|^2 . \tag{13}$$

Let $E = E^- \oplus E^0 \oplus E^+$ be the orthogonal decomposition of E into the parts corresponding to $k<0$, $k=0$ and $k>0$ in (11). If $e_1, \ldots, e_N$ is the standard basis in $\mathbf{R}^N$, then E^0 is spanned by $(p,q) = (0, e_j)$, $1 \leqq j \leqq N$, and $E^{\pm}$ by

$$(p,q) = (e_j \sin kt, \mp e_j \cos kt), \qquad 1 \leqq j \leqq N, \qquad 1 \leqq k < \infty .$$

For $x \in E$, let

$$\Phi(x) = \frac{1}{2} \int_0^{2\pi} (-J\dot{x} \cdot x)\, dt$$

and $\Phi'(x) = Lx$ (recall that $\Phi'(x) \in E$ via the Riesz representation theorem). It is easy to see that

$$\Phi(x) = \frac{1}{2} \int_0^{2\pi} (-i\dot{z} \cdot z)\, dt$$

(here $\cdot$ denotes the inner product in $\mathbf{C}^N$), so by (11) and (12), if $x = x^- + x^0 + x^+ \in E^- \oplus E^0 \oplus E^+$, then

$$\Phi(x) = \tfrac{1}{2}\|x^+\|^2 - \tfrac{1}{2}\|x^-\|^2 \quad \text{and} \quad Lx = x^+ - x^- .$$

Let

$$\psi(x) = \frac{1}{2\pi} \int_0^{2\pi} H(x)\, dt \quad \text{and} \quad M = \psi^{-1}(1) .$$

By [14, pp. 607–608], each critical point of $\Phi|_M$ corresponds to a 2π-periodic brake orbit for (10) on S, and therefore to a brake orbit for (HS) on S.

By [14, Proposition 2.10], $\psi \in C^{1,1}(E, \mathbf{R})$ and ψ' is compact. Since ψ is homogeneous of degree 2, $\langle \psi'(x), x\rangle = 2\psi(x) = 2 \ \forall x \in M$. So M is a $C^{1,1}$-manifold and ψ' is bounded away from 0 on bounded subsets of M. By (9),

$$\|x\|_{L^2}^2 \leqq R^2 \int_0^{2\pi} H(x)dt \leqq 2\pi R^2 \quad \textit{whenever} \quad \psi(x) \leqq 1 . \tag{14}$$

In particular, M is bounded in L^2.

4.2. Lemma. *$\Phi|_M$ satisfies condition (C^*).*

Proof. We slightly modify the argument of [2, Lemma 3.7]. Suppose that $(x_n) \subset M$, $\Phi(x_n) \to c$ and

$$z_n = \frac{Lx_n - \lambda(x_n)\psi'(x_n)}{(\|x_n\| + 1)^{1/2}} \to 0 . \tag{15}$$

Since $\Phi(x_n) = \frac{1}{2}\|x_n^+\|^2 - \frac{1}{2}\|x_n^-\|^2$, there are positive constants a_1, a_2, a_3 such that

$$-a_3 + a_1 \|x_n^+\| \leqq \|x_n^-\| \leqq a_2 \|x_n^+\| + a_3 . \tag{16}$$

Recall that $\langle \psi'(x_n), x_n\rangle = 2\psi(x_n) = 2$ and $\langle Lx_n, x_n\rangle = 2\Phi(x_n)$. This and (15) imply

$$\lambda(x_n) = \Phi(x_n) - \tfrac{1}{2}(\|x_n\| + 1)^{1/2} \langle z_n, x_n\rangle .$$

So $|\lambda(x_n)| \leqq a_4 + a_5 \|x_n\|^{3/2}$. Scalar multiplication of (15) by x_n^+ gives

$$\|x_n^+\|^2 = (\|x_n\| + 1)^{1/2} \langle z_n, x_n^+\rangle + \lambda(x_n) \int_0^{2\pi} H'(x_n) \cdot x_n^+ \, dt .$$

Since $\dfrac{|H'(x)|}{|x|}$ is bounded and M is bounded in L^2, the integral above is also bounded. Consequently,

$$\|x_n^+\|^2 \leqq a_6 + a_7 \|x_n\|^{3/2} . \tag{17}$$

Since $\|x_n^0\|_{L^2} = \|x_n^0\|$, (x_n^0) is bounded in E. This, (16) and (17) imply that (x_n) is bounded. After passing to a subsequence, $x_n \to \bar{x}$ weakly in E, strongly in L^2, and $\lambda(x_n) \to \bar{\lambda}$. Since ψ' is compact, it follows from (15) that $Lx_n = x_n^+ - x_n^-$ converges strongly. Therefore $x_n \to \bar{x}$ strongly in E. □

4.3. Lemma. *The set $\mathscr{B} = \{x \in E : \psi(x) \leqq 1\}$ satisfies condition $(\mathscr{G})$ of Definition 2.7.*

Proof. Let E_0 and r be given. Since the E- and the L^2-norm are equivalent on E_0, $\|w\| \leqq C\|w\|_{L^2}$, where C is a constant independent of $w \in E_0$. Let $y \in \mathscr{B} \cap (x + E_0)$, $\|x\| \leqq r$. Then $y = x + w$, $w \in E_0$. Using (13) and (14), it follows that

$$\|w\|_{L^2} \leqq \|y\|_{L^2} + \|x\|_{L^2} \leqq \sqrt{2\pi}\, R + r .$$

Therefore

$$\|y\| \leqq \|x\| + \|w\| \leqq r + C\|w\|_{L^2} \leqq (C+1)r + \sqrt{2\pi}\, RC . \quad \square$$

Proof of Theorem 4.1. Let $T_0 x = x$ and $T_1 x(t) = x(t+\pi)$. Then $T = \{T_0, T_1\}$ is a unitary representation of $\mathbf{Z}_2$ in E. The fixed point set E^G of T consists of those $x \in E$ which are π-periodic [i.e., $a_k = 0$ for all odd k in the Fourier expansion (11)].

We will use Theorem 3.2. Note that L is equivariant and selfadjoint, $L(E^- \oplus E^0) \subset E^- \oplus E^0$ (because $Lx = x^+ - x^-$), and ψ is invariant. Recall that ψ' is compact and bounded away from zero on bounded subsets of M. By Lemma 4.2, $\Phi|_M$ satisfies (C^*). Let $\gamma_r(\cdot) = \gamma_r(\cdot, E^+)$ and

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} \Phi(x), \quad j = 1, \ldots, N,$$

where $\Gamma_j = \{A \in \Sigma : A \subset M, \gamma_r(A) \geqq j\}$. If $A \in \Gamma_j$, then $\gamma_r(A) \geqq 1$, so $A \cap E^+ \neq \emptyset$ according to Proposition 2.6. Let $x \in A \cap E^+ \subset M \cap E^+$. Then, using (13) and (9),

$$\Phi(x) = \frac{1}{2}\|x\|^2 \geqq \frac{1}{2}\|x\|_{L^2}^2 \geqq \frac{1}{2} r^2 \int_0^{2\pi} H(x)\,dt = \pi r^2 .$$

So $c_j \geqq \pi r^2$. Let E_1 be the N-dimensional subspace of E^+ corresponding to $k=1$ in (11) (that is, E_1 is spanned by $(p,q) = (e_j \sin t, -e_j \cos t)$, $1 \leqq j \leqq N$). Let $A = M \cap (E^- \oplus E^0 \oplus E_1)$. Then A is the boundary of the set $\{x \in E^- \oplus E^0 \oplus E_1 : \psi(x) \leqq 1\}$ in $E^- \oplus E^0 \oplus E_1$, and this set satisfies condition ($\mathscr{G}$) according to Lemma 4.3. It follows therefore from Proposition 2.8 with $Y = E^- \oplus E^0$ and $X_0 = E_1$ that $\gamma_r(A) = N$. If $x = x^- + x^0 + x^+ \in A$, then, by (9),

$$\|x^+\|_{L^2}^2 \leqq \|x\|_{L^2}^2 \leqq R^2 \int_0^{2\pi} H(x)\,dt = 2\pi R^2 .$$

Observe that $\|x^+\|_{L^2} = \|x^+\|$ whenever $x^+ \in E_1$ [cf. (12), (13)]. Hence

$$\Phi(x) = \tfrac{1}{2}\|x^+\|^2 - \tfrac{1}{2}\|x^-\|^2 \leqq \tfrac{1}{2}\|x^+\|^2 = \tfrac{1}{2}\|x^+\|_{L^2}^2 \leqq \pi R^2 ,$$

so $c_j \leqq \pi R^2$.

We have shown that $c_j \in [\pi r^2, \pi R^2]$ for $j = 1, \ldots, N$. Since $\Phi(x) = \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2$, $\Phi|_{Y^G} \leqq 0$, where $Y^G = (E^- \oplus E^0)^G$. Now all hypotheses of Theorem 3.2 are verified. Assume for the moment that all $x \in K_{c_j}$, $1 \leqq j \leqq N$, have minimal period 2π. Then $E^G \cap K_{c_j} = \emptyset$ and it follows from Remark 3.3(ii) that $\Phi|_M$ possesses at least N pairs of critical points. If x_1 and x_2 are two distinct points in K_{c_j} corresponding to the same brake orbit for (10), then $x_2(t) = x_1(t+\alpha)$, where $\alpha \in (0, 2\pi)$, and the Fourier expansion (11) shows that $\alpha = \pi$. So $x_2 = T_1 x_1$. Therefore there exist at least N distinct brake orbits.

It remains to show that if $x \in K_{c_j}$, $1 \leqq j \leqq N$, then indeed x has minimal period 2π. Our argument is close to that in [6, Lemma 6]. Suppose x is $\frac{2\pi}{m}$-periodic, $m \geqq 1$. Then, using (10) and the fact that $H'(x) \cdot x = 2H(x)$,

$$\Phi(x) = \frac{1}{2} \int_0^{2\pi} (-J\dot{x} \cdot x)\,dt = \frac{1}{2}\lambda \int_0^{2\pi} H'(x) \cdot x\,dt = \lambda \int_0^{2\pi} H(x)\,dt = 2\pi\lambda . \tag{18}$$

Let $x(t)=\bar{x}+\tilde{x}(t)$, where $\bar{x}\in E^0$ and $\tilde{x}$ has mean value zero. Then, by the Wirtinger inequality $\|\dot{\tilde{x}}\|_{L^2}\leqq\frac{1}{m}\|\tilde{x}\|_{L^2}$ and (10),

$$\Phi(x)=\frac{1}{2}\int_0^{2\pi}(-J\dot{x}\cdot x)dt=\frac{1}{2}\int_0^{2\pi}(-J\dot{\tilde{x}}\cdot\tilde{x})dt\leqq\frac{1}{2}\|\dot{\tilde{x}}\|_{L^2}\|\tilde{x}\|_{L^2}$$

$$\leqq\frac{1}{2m}\|\dot{\tilde{x}}\|_{L^2}^2=\frac{1}{2m}\|\dot{x}\|_{L^2}^2=\frac{1}{2m}\int_0^{2\pi}|\lambda H'(x)|^2dt.$$

Since $|H'(x)|\leqq\frac{2}{\varrho}\;\forall x\in S$ (see e.g. [2, Proof of Theorem 4.10]),

$$\Phi(x)\leqq\frac{4\pi\lambda^2}{m\varrho^2}.$$

Combining this with (18), $\lambda\geqq\frac{1}{2}m\varrho^2$ and $\Phi(x)\geqq\pi m\varrho^2$. Since $c_j\leqq\pi R^2$, it follows from the hypothesis $R^2<2\varrho^2$ that

$$\pi m\varrho^2\leqq\Phi(x)\leqq\pi R^2<2\pi\varrho^2.$$

Hence $m<2$. The minimal period of x is therefore 2π. □

4.4. Remarks. (i) If we remove the assumption that $R^2<2\varrho^2$ in Theorem 4.1, then $\Phi|_M$ will still have a critical value in $[\pi r^2,\pi R^2]$. So there exists at least one brake orbit of (HS) on S, and we recover the main result of [14].

(ii) Using [6, Lemma 6] it is easy to see that the hypothesis $R^2<2\varrho^2$ may be replaced by $R^2<\sqrt{2}r\varrho$.

Appendix

We will prove the following

Generalized Borsuk Theorem. *Let W be an open bounded neighbourhood of $0\in\mathbf{R}^m\times\mathbf{R}^n$ such that if $(x,y)\in W$, then $(x,-y)\in W$. Let $f=(g,h):\bar{W}\to\mathbf{R}^m\times\mathbf{R}^n$ be a continuous mapping with $g(x,-y)=g(x,y)$, $h(x,-y)=-h(x,y)\;\forall(x,y)\in\bar{W}$, $f|_{\partial W}\neq 0$ and $f(x,0)=(x,0)\;\forall(x,0)\in\partial W$. Then the Brouwer degree $\deg(f,W,0)$ is an odd integer.*

Proof. Our argument is essentially an adaptation of Nirenberg's proof in [11] (see also [10] and [21]) to the simpler case of $\mathbf{Z}_2$-symmetry.

Given $\varepsilon>0$, there exists a C^∞ mapping f_ε such that $|f(x,y)-f_\varepsilon(x,y)|\leqq\varepsilon\;\forall(x,y)\in\bar{W}$. We may assume that f_ε is equivariant by replacing it if necessary with $\bar{f}_\varepsilon(x,y)=\frac{1}{2}(g_\varepsilon(x,y),h_\varepsilon(x,y))+\frac{1}{2}(g_\varepsilon(x,-y),-h_\varepsilon(x,-y))$. For $\delta>0$, let $\chi:[0,\infty)\to[0,1]$ be a C^∞ function satisfying $\chi(t)=1$ if $0\leqq t\leqq\frac{\delta}{2}$ and $\chi(t)=0$ if $t\geqq\delta$. Define

$$\tilde{f}(x,y)=\chi(|y|)(x,y)+(1-\chi(|y|))f_\varepsilon(x,y).$$

It is easy to see that if $\delta=\delta(\varepsilon)$ is small enough, $|\tilde{f}(x,y)-f(x,y)|\leqq 2\varepsilon$ $\forall (x,y)\in\partial W$. By the continuity property of degree,

$$\deg(f,W,0)=\deg(\tilde{f},W,0) \tag{A.1}$$

whenever ε is sufficiently small. Let

$$V=\left\{(x,y)\in W:|y|<\frac{\delta}{2}\right\}.$$

Since $\tilde{f}|_{\partial V}\neq 0$ and $\tilde{f}|_V$ is the identity mapping,

$$\deg(\tilde{f},W,0)=\deg(\tilde{f},V,0)+\deg(\tilde{f},W-\bar{V},0)=1+\deg(\tilde{f},W-\bar{V},0). \tag{A.2}$$

We will show that $\deg(\tilde{f},W-\bar{V},0)$ is an even integer. Let $A=(a_{ij})$ be an $m\times n$ and $B=(b_{ij})$ an $n\times n$ real matrix. Let $(x,y)=(x_1,\dots,x_m,y_1,\dots,y_n)$, $\tilde{f}=(\tilde{g}_1,\dots,\tilde{g}_m,\tilde{h}_1,\dots,\tilde{h}_m)$ and

$$\begin{aligned}\bar{g}_i(x,y,A,B)&=\tilde{g}_i(x,y)+\sum_{j=1}^{n}a_{ij}y_j^2, \qquad 1\leqq i\leqq m,\\ \bar{h}_i(x,y,A,B)&=\tilde{h}_i(x,y)+\sum_{j=1}^{n}b_{ij}y_j, \qquad 1\leqq i\leqq n.\end{aligned} \tag{A.3}$$

If A and B are fixed and sufficiently small (in the sense that $|a_{ij}|<\varepsilon_0$, $|b_{ij}|<\varepsilon_0$ for all i,j and an appropriate $\varepsilon_0>0$), then, setting $\bar{f}=\bar{f}_{A,B}=(\bar{g}_1,\dots,\bar{g}_m,\bar{h}_1,\dots,\bar{h}_n)$,

$$\deg(\tilde{f},W-\bar{V},0)=\deg(\bar{f},W-\bar{V},0) \tag{A.4}$$

by the continuity property of degree. Assume for the moment that 0 is a regular value of $\bar{f}|_{W-\bar{V}}$. Then $\bar{f}(x,y)=0$ for finitely many $(x,y)\in W-\bar{V}$ and each such (x,y) gives a contribution of $+1$ or -1 to the degree. Since $y\neq 0$ on $W-\bar{V}$ and $\bar{f}$ is equivariant, the zeros of $\bar{f}|_{W-\bar{V}}$ come in pairs $(x,\pm y)$. So $\deg(\bar{f},W-\bar{V},0)$ is the sum of an even number of terms, each equal to $+1$ or -1, and is therefore an even integer. Using this and (A.1), (A.2), (A.4), it follows that $\deg(f,W,0)$ is an odd integer.

It remains to show that A and B may be chosen in such a way that 0 is a regular value of $\bar{f}$. Let M_A and M_B be the spaces of all sufficiently small $m\times n$ and $n\times n$ real matrices. Let

$$F:(W-\bar{V})\times M_A\times M_B\to\mathbf{R}^m\times\mathbf{R}^n$$

be the mapping given by $F(x,y,A,B)=\bar{f}_{A,B}(x,y)$. We claim that the derivative of F is transversal to the origin (i.e. surjective) whenever $F(x,y,A,B)=0$. To see this, let $z=(\xi,\eta)\in\mathbf{R}^m\times\mathbf{R}^n$. We need to find $\bar{x}$, $\bar{y}$, $\bar{A}$, $\bar{B}$ such that

$$(D_xF)\bar{x}+(D_yF)\bar{y}+(D_AF)\bar{A}+(D_BF)\bar{B}=z, \tag{A.5}$$

where D_xF etc.... are the partial derivatives of F at (x,y,A,B). A simple computation using (A.3) shows that (A.5) with $\bar{x}=0$ and $\bar{y}=0$ is equivalent to the system of equations

$$\begin{aligned}\sum_{j=1}^{n}\bar{a}_{ij}y_j^2&=\xi_i, \qquad 1\leqq i\leqq m,\\ \sum_{j=1}^{n}\bar{b}_{ij}y_j&=\eta_i, \qquad 1\leqq i\leqq n.\end{aligned} \tag{A.6}$$

Since $y \neq 0$ in $W - \bar{V}$, $y_j \neq 0$ for some j. So (A.6) can be solved for $\bar{A}$ and $\bar{B}$. Therefore (A.5) is satisfied with $\bar{x}=0$, $\bar{y}=0$ and $\bar{A}$, $\bar{B}$ just found. This proves the claim. Finally, it follows from the transversality theorem (see e.g. [7, p. 68]) that 0 is a regular value of $\bar{f}_{A,B}$ for almost all $A \in M_A$ and $B \in M_B$. $\square$

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