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On the Principle of Linearized Stability for Variational Inequalities

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Introduction

In this paper we study the stability of stationary solutions of parabolic variational inequalities in Hilbert spaces. It is well known that in the case of semilinear parabolic equations the spectrum of "linearized" operator determines the Lyapunov stability of an equilibrium (e.g. [4]). In the case of inequalities this problem is more complicated: the stability of an equilibrium may "substantially" change if we linearize the operator in the inequality (see Example 1). Nevertheless, it is still possible to find some conditions which are independent of the nonlinearity and which are sufficient for the stability or the instability of the equilibrium, respectively. Moreover, these conditions are under some additional assumptions also necessary.

To point out the main results, let us consider the following problem, to which our theory can be applied: let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, let L be a second-order elliptic operator of the form $Lu := -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$ with $a_{ij} \in L^\infty(\Omega)$, $a_{ij}(x)\xi_i\xi_j \geqq \alpha|\xi|^2$, $\alpha > 0$, and let $F(u) := f(x, u, \nabla u)$, where $f = f(x,u,p)$: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is measurable in x and C^1 in u and p, $f(\cdot, 0, 0) \in L^2(\Omega)$, $\frac{\partial f}{\partial p}$ is bounded and $\left|\frac{\partial f}{\partial u}(x,u,p)\right| \leqq a(x) + C(|u|^\gamma + |p|^{2/N})$ for some $a \in L^N(\Omega)$, $\gamma \leqq 2/(N-2)$ (if $N > 2$) and $C > 0$. Let $\tilde{K}$ be a closed convex set in the Sobolev space $H_0^1(\Omega)$ and let $\tilde{u} \in \tilde{K}$ be a stationary solution of the inequality

$$
\begin{aligned}
& u(t) \in \tilde{K} \\
& \left(\frac{du}{dt} + Lu + F(u), v - u\right) \geqq 0 \quad \forall v \in \tilde{K} \quad \text{a.e. in } (0,T) \qquad (0) \\
& u(0) = u_0 ,
\end{aligned}
$$

i.e.

$$
(L\tilde{u} + F(\tilde{u}), v - \tilde{u}) = \int_\Omega \left(a_{ij}(x)\frac{\partial \tilde{u}}{\partial x_j}\frac{\partial (v - \tilde{u})}{\partial x_i} + f(x, \tilde{u}, \nabla \tilde{u})(v - \tilde{u})\right) dx \geqq 0 \quad \forall v \in \tilde{K} .
$$

Put

$$Au := Lu + F'(\tilde{u})u$$
$$K := \tilde{K} - \tilde{u} = \{v - \tilde{u};\ v \in \tilde{K}\}$$
$$F_0 := L\tilde{u} + F(\tilde{u})$$
$$\lambda_I := \liminf_{\substack{u \in K \\ \|u\|_{H^1} \to 0}} \frac{(Au + F_0, u)}{(u, u)} .$$

Then the condition $\lambda_I > 0$ is sufficient for the asymptotic stability of the stationary solution $u_0 = \tilde{u}$ of the inequality (0) in the topology of $H_0^1(\Omega)$ (more precisely see Theorem 1).

In order to see that the condition $\lambda_I < 0$ may be sufficient for the instability result, suppose, moreover, that K is a cone with its vertex at zero, $F_0 = 0$ and A is symmetric (i.e. $a_{ij} = a_{ji}$ and f is independent of p) and denote by $\sigma(A)$ the spectrum of the operator A and by $\sigma_K(A)$ the set of all (real) eigenvalues of the inequality

$$u \in K\colon (Au - \lambda u, v - u) \geqq 0 \quad \forall v \in K , \tag{1}$$

i.e. $\sigma_K(A) := \{\lambda \in \mathbb{R};$ the inequality (1) has a nontrivial solution$\}$.

Then we have

$$\lambda_I = \min_{\substack{u \in K \\ \|u\|_{H^1} = 1}} \frac{(Au, u)}{(u, u)} = \min \sigma_K(A) \geqq \min \sigma(A) .$$

Let u_I be any nontrivial solution of (1) with $\lambda = \lambda_I$ and let $\delta > 0$. Then the function $u(t) = \tilde{u} + \delta e^{-\lambda_I t} u_I$ is a solution of the linearized inequality

$$u(t) \in \tilde{K}$$
$$\left(\frac{du}{dt} + Lu + F(\tilde{u}) + F'(\tilde{u})(u - \tilde{u}), v - u\right) \geqq 0 \quad \forall v \in \tilde{K} \quad \text{a.e. in } (0, T)$$
$$u(0) = \tilde{u} + \delta u_I ,$$

which implies that the condition $\lambda_I < 0$ guarantees the instability of the solution $u_0 = \tilde{u}$ for the linearized inequality. If, moreover,

$$\lambda_I = \min_{\substack{u \in K - u_I \\ u \neq 0}} \frac{(Au, u)}{(u, u)} ,$$

then Theorem 2 implies the instability result also for the nonlinear inequality (0) (in the topology of $L^2(\Omega)$).

An application of these results to a more concrete problem is given in Example 3. Example 2 shows that the condition $\operatorname{Re}(\sigma_K(A) \cup \sigma(A)) > 0$ is not sufficient for the stability in the nonsymmetric case and in Example 1 it is shown that the condition $\exists \lambda \in \sigma_K(A), \lambda < 0$, is, in general, not sufficient for the instability result.

The proofs of Theorems 1 and 2, which are the main results of this paper, are based on the existence and regularity results of Brézis [1, 2].

Main Results

We shall suppose that V and H are real Hilbert spaces with norms $\|\cdot\|$ and $|\cdot|$, respectively, and $V\subset H\subset V'$, where the inclusions are dense and compact. By $(\cdot,\cdot)$ we denote the duality between V' and V and also the scalar product in H. We shall study the inequality

$$\begin{aligned} &u(t)\in K \\ &\left(\frac{du}{dt}+Au+N(u)+F_0, v-u\right)\geqq 0 \quad \forall v\in K \quad \text{a.e. in } (0,T) \\ &u(0)=u_0 , \end{aligned} \tag{2}$$

where

K is a closed convex set in V, $0\in K$,
$A: V\to V'$ is a continuous linear map, $A=A_1+A_2$,
  $A_1: V\to V'$ is symmetric and coercive
    (i.e. $(A_1u,v)=(A_1v,u)$ and $(A_1v,v)\geqq\alpha\|v\|^2$ for some $\alpha>0$ and any $u,v\in V$),
  $A_2: V\to H$ is continuous (i.e. $|A_2v|\leqq C_1\|v\|$ for some $C_1>0$ and any $v\in V$),
$N: V\to V'$ is a nonlinear map of the form $N=N_1+N_2$,
  N_1 is a gradient of a C^1 convex functional $\Phi: V\to\mathbb{R}$, $\Phi(0)=0$, $\Phi'(0)=0$, $\Phi''(0)=0$,
  $N_2: V\to H$ is a continuous map satisfying
    $|N_2(u)-N_2(v)|\leqq C_2\|u-v\|$ for any $u,v\in B$ and some $C_2>0$,
    $(N_2(u),u)\geqq -\varphi(\|u\|)\,\|u\|^2$ for any $u\in B$,
  where $\varphi(t)\to 0$ as $t\to 0+$ and B is a given neighbourhood of zero in V,
$F_0\in V'$, $(F_0,v)\geqq 0$ for any $v\in K$,
$u_0\in K$.

By a (strong) *solution* of (2) we mean a function $u\in C([0,T],K)$ such that $u:(0,T)\to H$ is differentiable a.e. and fulfils (2). The main result of this paper is the following

Theorem 1. *Let*

$$\lambda_I := \liminf_{u\in K,\ \|u\|\to 0} \frac{(Au+F_0,u)}{|u|^2} > 0 .$$

Then the solution $u_0=0$ of (2) is asymptotically stable in the topology of V, i.e. for every $\varepsilon>0$ there exists $\delta>0$ such that for any $u_0\in K$ with $\|u_0\|<\delta$ there exists an unique solution u of (2) on $[0,+\infty)$ and fulfils $\|u(t)\|<\varepsilon$ for any $t\geqq 0$. Moreover, for any $\eta>0$ we have

$$\frac{du}{dt}\in L^2((0,\infty),H)\cap L^2((\eta,\infty),V)$$

and for any $\lambda<\lambda_I$, $\Lambda\geqq 0$ there exist constants $C=C(\lambda,\Lambda)>0$ and $\delta=\delta(\lambda,\varepsilon)>0$ such that $\|u_0\|<\delta$ implies

$$|u(t)| \leqq e^{-\lambda t}|u_0| \quad \textit{for any} \quad t \geqq 0$$

$$\|u(t)\| \leqq Ce^{-\lambda t}(\|u_0\| + \sqrt{(F_0, u_0)}e^{-\Lambda t}) \quad \textit{for any} \quad t \geqq 0$$

$$\left|\frac{du}{dt}(t)\right| \leqq C\left(1 + \frac{1}{\sqrt{t}}\right)e^{-\lambda t}(\|u_0\| + \sqrt{(F_0, u_0)}e^{-\Lambda t}) \quad \textit{for a.e.} \ t \geqq 0 \ .$$

If there exists $g_0 \in H$ *such that* $(A_1 u_0 + N_1(u_0) + F_0 - g_0, v - u_0) \geqq 0$ *for any* $v \in K$, *then*

$$\frac{du}{dt} \in L^\infty((0, \infty), H) \cap L^2((0, \infty), V) \ .$$

If, moreover, $N_1 \equiv 0$, $F_0 \in V$ *and* $(I + \mu A_1)^{-1}(K) \subset K$ *for any* $\mu > 0$, *then* $A_1 u(t) \in H$ *for any* $t \geqq 0$ *and*

$$\begin{aligned} |A_1 u(t)| &\leqq Ce^{-\lambda t}(\|u_0\| + |A_1 u_0| \max(1 - t, 0) + \sqrt{(F_0, u_0)}e^{-\Lambda t}) + \sqrt{2(A_1 F_0, u(t))^-} \\ &\leqq C(e^{-\lambda t}|A_1 u_0| + e^{-\lambda t/2}\|F_0\|) \quad \textit{for any} \quad t \geqq 0 \ , \end{aligned}$$

where $x^- := \max(-x, 0)$.

Proof. By C we shall denote various constants, which may depend on the given constants $\alpha, C_1, C_2, \|A_1\|_{L(V,V')}, \lambda, \Lambda$ and the function φ, but do *not* depend on u_0 and F_0. Without loss of generality we may suppose that $B = V$ (otherwise we redefine N_2 outside B) and φ is strictly increasing and continuous.

Let $u_0 \in K$, $\|u_0\| < \delta < 1$. Put

$$\Phi_1(v) := \Phi(v) + \tfrac{1}{2}(A_1 v, v) + |(F_0, v)| \ ,$$

$$\tilde{A}_1(v) := A_1 v + N_1(v) + F_0 \ ,$$

$$\tilde{A}(v) := Av + N(v) + F_0 \ .$$

Then $\tilde{A} : V \to V'$ is pseudomonotone, $(\tilde{A}(v), v) \geqq \frac{\alpha}{2}\|v\|^2 - C|v|^2$ for any $v \in K$, thus the results of Brézis [1, Corollaire II.1, Remarque II.5] imply the existence of a weak solution of (2), i.e. for any $T > 0$ there exists $\hat{u} \in L^2((0, T), K) \cap C([0, T], H)$ such that $\hat{u}(0) = u_0$ and

$$\int_0^T \left(\frac{dv}{dt} + \tilde{A}(\hat{u}), v - \hat{u}\right)e^{-2Ct}dt \geqq -\frac{1}{2}|v(0) - u_0|^2 \quad \forall v \in L^2((0, T), K) \ ,$$

$$\frac{dv}{dt} \in L^2((0, T), V') \ .$$

Put $f(t) := -A_2\hat{u}(t) - N_2(\hat{u}(t)$. Then $f \in L^2((0, T), H)$ and again the results of Brézis [1, Théorème II.8, the proof of Corollaire II.2] and [2, Proposition 5, Lemme 6] imply that there exists a unique $u \in C([0, T], H)$ such that

$$\begin{gathered} u(0) = u_0 \\ \left(\frac{du}{dt} + \tilde{A}_1(u) - f, v - u\right) \geqq 0 \quad \forall v \in K \quad \text{a.e. in } (0, T) \ . \end{gathered} \tag{3}$$

Moreover $u:[0,T]\to H$ is absolutely continuous and differentiable a.e.,

$$\begin{aligned}&\frac{du}{dt}\in L^2((0,T),H),\quad \left(\int_0^T\left|\frac{du}{dt}\right|^2 dt\right)^{1/2}\leqq\left(\int_0^T |f|^2 dt\right)^{1/2}+\sqrt{\Phi_1(u_0)}\\ &u\in C([0,T],K)\\ &t\mapsto\Phi_1(u(t))\quad\text{is absolutely continuous and}\quad\left|\frac{du}{dt}\right|^2+\frac{d}{dt}\Phi_1(u(t))=\left(f,\frac{du}{dt}\right)\quad\text{a.e.}\end{aligned}\tag{4}$$

Since both u and $\hat{u}$ are weak solutions of the inequality (3) (cf. [1, Remarque II.11], the uniqueness result of [1, Théorème II.3, Remarque II.5] implies $u=\hat{u}$.

By putting $v=0$ in (3) we get

$$\left(\frac{du}{dt}+\tilde{A}(u),u\right)\leqq 0\qquad\text{a.e. in }(0,T)\ .\tag{5}$$

Let $\lambda\in(0,\lambda_I)$ be fixed. Then there exists $\varepsilon_1>0$ such that

$$(Au+F_0,u)\geqq\frac{\lambda+\lambda_I}{2}|u|^2\qquad\text{for any }u\in K,\ \|u\|\leqq\varepsilon_1\ ,$$

$$(Au,u)\geqq\frac{\alpha}{2}\|u\|^2-C|u|^2\qquad\text{for any }u\in V\ ,$$

$$(N(u),u)\geqq-\varphi(\|u\|)\|u\|^2\qquad\text{for any }u\in V\ .$$

By choosing $\eta:=(\lambda_I-\lambda)/(4\max(\lambda_I,C))$ we obtain

$$(Au+F_0,u)\geqq\frac{\eta\alpha}{2}\|u\|^2+\lambda|u|^2+\eta(F_0,u)\qquad\text{for any }u\in K,\ \|u\|\leqq\varepsilon_1\ .$$

By using (5) and putting $\beta:=\eta\alpha/4$ we get

$$\frac{1}{2}\frac{d}{dt}|u|^2+\lambda|u|^2+\beta\|u\|^2+\eta(F_0,u)\leqq 0\qquad\text{a.e. in }(0,T_\beta)\ ,\tag{6}$$

where $T_\beta:=\sup\{t;\|u(\tau)\|\leqq\min(\varphi^{-1}(\beta),\varepsilon_1)$ for any $\tau\in[0,t]\}>0$ if δ is small enough. The inequality (6) implies

$$|u(t)|\leqq|u_0|e^{-\lambda t}\tag{6a}$$

$$\int_{t_1}^{t_2}\|u\|^2dt\leqq C(|u(t_1)|^2-|u(t_2)|^2)\leqq C|u_0|^2e^{-2\lambda t_1}\tag{6b}$$

$$\int_{t_1}^{t_2}(F_0,u)dt\leqq C(|u(t_1)|^2-|u(t_2)|^2)\leqq C|u_0|^2e^{-2\lambda t_1}\tag{6c}$$

for any $t,t_1,t_2\in[0,T_\beta]$. Now it follows from (4) and (6b) that

$$\alpha\|u(t)\|^2 \leqq \Phi_1(u(t)) = \Phi_1(u_0) + \int_0^t \left(\left(f, \frac{du}{dt}\right) - \left|\frac{du}{dt}\right|^2\right) dt$$
$$\leqq \Phi_1(u_0) + \frac{1}{2}\int_0^t |f|^2 dt \leqq C\left(\|u_0\|^2 + (F_0, u_0) + \int_0^t \|u\|^2 dt\right)$$
$$\leqq C(\|u_0\|^2 + (F_0, u_0) + |u_0|^2) < \alpha(\min(\varphi^{-1}(\beta/2), \varepsilon_1/2))^2 \tag{6d}$$

for any $t \in [0, T_\beta]$ provided δ is sufficiently small. Hence $T_\beta = T$.

By putting $v := u(t+h)$ in (3) we obtain

$$\left(\frac{du}{dt}(t) + Au(t) + N(u(t)) + F_0, \frac{u(t+h) - u(t)}{h}\right) \geqq 0 \quad \text{a.e. in } (0, T-h) .$$

Similarly,

$$\left(\frac{du}{dt}(t+h) + Au(t+h) + N(u(t+h)) + F_0, \frac{u(t) - u(t+h)}{h}\right) \geqq 0$$
$$\text{a.e. in } (0, T-h) .$$

By adding the last two inequalities we get

$$\left(\left(\frac{du}{dt}\right)_h (t) + Au_h(t) + (N \circ u)_h(t), u_h(t)\right) \leqq 0 \qquad \text{a.e. in } (0, T-h) ,$$

where $w_h(t) := (w(t+h) - w(t))/h$. We have

$$(Au_h, u_h) \geqq \frac{\alpha}{2}\|u_h\|^2 - C|u_h|^2$$

$$((N \circ u)_h, u_h) \geqq -C_2\|u_h\| \cdot |u_h| \geqq -\frac{\alpha}{4}\|u_h\|^2 - C|u_h|^2 ,$$

therefore,

$$\frac{1}{2}\frac{d}{dt}|u_h|^2 + \frac{\alpha}{4}\|u_h\|^2 - C|u_h|^2 \leqq 0 \qquad \text{a.e. in } (0, T-h) . \tag{7}$$

Consequently,

$$|u_h(t_2)| \leqq |u_h(t_1)| e^{C(t_2 - t_1)} \tag{7a}$$

$$\int_{t_1}^{t_2} \|u_h\|^2 dt \leqq C\left(|u_h(t_1)|^2 - |u_h(t_2)|^2 + \int_{t_1}^{t_2} |u_h|^2 dt\right) \tag{7b}$$

for any $t_1, t_2 \in [0, T-h]$.

The last inequality, together with (4), implies that $u : (0, T) \to V$ is (locally) absolutely continuous, $\frac{du}{dt} \in L^2_{loc}((0, T), V)$ and

$$\int_{t_1}^{t_2} \left\|\frac{du}{dt}\right\|^2 dt \leqq C\left(\left|\frac{du}{dt}(t_1)\right|^2 + \int_{t_1}^{t_2} \left|\frac{du}{dt}\right|^2 dt\right) \tag{7c}$$

for a.e. $t_1 \in (0, T)$ and any $t_2 \in (t_1, T)$.

Choose $\Lambda \geqq 0$, $C_* \geqq 1$ and suppose there exists $t_2^* \in (0, T)$ such that

$$\|u(t_2^*)\| \geqq 10\, C_* \exp(-\lambda t_2^*)(\|u_0\| + \sqrt{(F_0, u_0)} \exp(-\Lambda t_2^*)) \ .$$

We may suppose

$$t_2^* = \min\{t \in (0, T);\ \|u(t)\| \geqq 10\, C_* \exp(-\lambda t)(\|u_0\| + \sqrt{(F_0, u_0)} \exp(-\Lambda t))\}$$

and we put

$$t_1^* := \min\{t \in (0, t_2^*);\ \|u(\tau)\| \geqq C_* \exp(-\lambda\tau)(\|u_0\| + \sqrt{(F_0, u_0)} \exp(-\Lambda\tau)) \quad \text{for any } \tau \in [t, t_2^*]\} \ .$$

Since (6d) implies $\|u(t)\| \leqq C(\|u_0\| + \sqrt{(F_0, u_0)})$ for every t, we have $t_1^* \geqq 3$ (if C_* is sufficiently large). By using (6b) we get

$$C|u_0|^2 e^{-2\lambda t_1^*} \geqq \int_{t_1^*}^{t_2^*} \|u\|^2 dt \geqq \int_{t_1^*}^{t_2^*} C_*^2 e^{-2\lambda t} \|u_0\|^2 dt = \frac{C_*^2 \|u_0\|^2}{2\lambda} (e^{-2\lambda t_1^*} - e^{-2\lambda t_2^*}) \tag{8}$$

hence $e^{-2\lambda t_1^*} < (1+\zeta) e^{-2\lambda t_2^*}$, where $\zeta \leqq C/(C_*^2 - C)$. Consequently, for any $\gamma > 0$ fixed we have

$$t_2^* - t_1^* \leqq \min\left(1, \frac{1}{\Lambda}, \frac{1}{\lambda}, \gamma\right) , \tag{9}$$

if $C_* = C_*(\gamma, \Lambda, \lambda)$ is sufficiently large. Now

$$\|u(t_1^*)\| = C_* e^{-\lambda t_1^*}(\|u_0\| + e^{-\Lambda t_1^*} \sqrt{(F_0, u_0)}) \leqq 9\, C_* e^{-\lambda t_2^*}(\|u_0\| + e^{-\Lambda t_2^*} \sqrt{(F_0, u_0)}) ,$$

so that

$$(C_* e^{-\lambda t_2^*}(\|u_0\| + e^{-\Lambda t_2^*} \sqrt{(F_0, u_0)}))^2 \leqq \|u(t_2^*) - u(t_1^*)\|^2 \leqq (t_2^* - t_1^*) \int_{t_1^*}^{t_2^*} \left\|\frac{du}{dt}\right\|^2 dt \ . \tag{10}$$

Choose $t_0^* \in [t_1^* - 2, t_1^* - 1]$ such that

$$(F_0, u(t_0^*)) \leqq \int_{t_1^*-2}^{t_1^*-1} (F_0, u) dt \tag{11a}$$

and choose $t_1 \in [t_0^*, t_1^*]$ such that

$$\left|\frac{du}{dt}(t_1)\right|^2 \leqq \int_{t_0^*}^{t_1^*} \left|\frac{du}{dt}\right|^2 dt \ . \tag{11b}$$

Then (10, 9, 7c, 11b, 4, 6b, 11a, 6c, 9) imply

$$\begin{aligned}(C_* e^{-\lambda t_2^*}(\|u_0\| + e^{-\Lambda t_2^*} \sqrt{(F_0, u_0)}))^2 &\leqq \gamma \int_{t_1}^{t_2^*} \left\|\frac{du}{dt}\right\|^2 dt \leqq C\gamma \int_{t_0^*}^{t_2^*} \left|\frac{du}{dt}\right|^2 dt \\ &\leqq C\gamma \left(\int_{t_0^*}^{t_2^*} |f|^2 dt + \Phi_1(u(t_0^*))\right) \\ &\leqq C\gamma \left(\int_{t_0^*}^{t_2^*} \|u\|^2 dt + \|u(t_0^*)\|^2 + (F_0, u(t_0^*))\right)\end{aligned}$$

$$\leqq C\gamma(|u_0|^2 e^{-2\lambda t_0^*} + (10 C_* e^{-\lambda t_0^*}(\|u_0\| + e^{-\Lambda t_0^*}\sqrt{(F_0, u_0)}))^2$$
$$+ |u_0|^2 e^{-2\lambda(t_1^* - 2)})$$
$$\leqq C\gamma(C_* e^{-\lambda t_2^*}(\|u_0\| + e^{-\Lambda t_2^*}\sqrt{(F_0, u_0)}))^2 \ ,$$

where the constant C does not depend on γ and C_*. Hence choosing $\gamma < 1/C$ we get

$$\|u(t)\| \leqq 10 C_* e^{-\lambda t}(\|u_0\| + e^{-\Lambda t}\sqrt{(F_0, u_0)}) \tag{12}$$

for any t (the constant C_* does not depend on T).

In order to prove the estimate for $\left|\frac{du}{dt}\right|$ let us prove the following inequality

$$(F_0, u(t)) \leqq C e^{-2\lambda t}(\|u_0\|^2 + e^{-2\Lambda t}(F_0, u_0)) \ . \tag{13}$$

By using (4) we get

$$(F_0, u(t)) - \Phi_1(u(t_0)) \leqq \Phi_1(u(t)) - \Phi_1(u(t_0))$$
$$\leqq \int_{t_0}^{t} |f|^2 dt \leqq C \int_{t_0}^{t} \|u\|^2 dt \leqq C|u_0|^2 e^{-2\lambda t_0}$$

for any $t_0 < t$, thus

$$(F_0, u(t)) \leqq (F_0, u(t_0)) + C e^{-2\lambda t_0}(\|u_0\|^2 + e^{-2\Lambda t_0}(F_0, u_0)) \ . \tag{14}$$

Now the inequality (13) is obvious for $t \leqq 1$. If $t > 1$, then there exists $t_0 \in (t-1, t)$ such that

$$(F_0, u(t_0)) \leqq \int_{t-1}^{t} (F, u(t)) dt \leqq C|u_0|^2 e^{-2\lambda t}$$

(we have used (6c)), which, together with (14), proves (13).

By using (4, 6b, 12, 13) we obtain further

$$\int_{t_1}^{t_2} \left|\frac{du}{dt}\right|^2 dt \leqq C\left(\int_{t_1}^{t_2} \|u\|^2 dt + \|u(t_1)\|^2 + (F_0, u(t_1))\right)$$
$$\leqq C e^{-2\lambda t_1}(\|u_0\|^2 + e^{-2\Lambda t_1}(F_0, u_0)) \ . \tag{15}$$

Choose $t > 0$ such that $\frac{du}{dt}(t)$ exists and put $t_0 = \max(t/2, t-1)$. Then there exists $t_1 \in [t_0, t]$ such that

$$\left|\frac{du}{dt}(t_1)\right|^2 \leqq \frac{1}{t - t_0}\int_{t_0}^{t} \left|\frac{du}{dt}\right|^2 dt \leqq \frac{C e^{-2\lambda t_0}}{t - t_0}(\|u_0\|^2 + e^{-2\Lambda t_0}(F_0, u_0))$$
$$\leqq \frac{C e^{-2\lambda t}}{\min(1, t)}(\|u_0\|^2 + e^{-2\Lambda t}(F_0, u_0))$$

by (15). According to (7a) we get

$$\left|\frac{du}{dt}(t)\right| \leqq e^{C(t - t_1)}\left|\frac{du}{dt}(t_1)\right| \leqq C\left(1 + \frac{1}{\sqrt{t}}\right) e^{-\lambda t}(\|u_0\| + e^{-\Lambda t}\sqrt{(F_0, u_0)}) \ . \tag{16}$$

Now let $g_0 \in H$, $(A_1 u_0 + N_1(u_0) + F_0 - g_0, v - u_0) \geqq 0$ for any $v \in K$. Then it follows from [2] (Proposition 5 used for the functional $\Phi(u) = \chi_K(u) + \Phi_1(u) - \Phi_1(u_0) + (g_0, u_0 - u)$, where χ_K is the indicatrix function of the set K)

$$\int_0^T \left|\frac{du}{dt}\right|^2 dt \leqq 4 \int_0^T |f + g_0|^2 dt \leqq C\left(T|g_0|^2 + \int_0^T \|u\|^2 dt\right),$$

which, together with (7a, 12), implies

$$\left|\frac{du}{dt}\right| \leqq C(|g_0| + \|u_0\| + \sqrt{(F_0, u_0)}) \quad \text{a.e. in } (0, \infty) . \tag{17}$$

By using (7c, 16, 17) we get $\frac{du}{dt} \in L^2((0, \infty), V)$.

Finally, let $N_1 \equiv 0$, $F_0 \in V$ and $(I + \mu A_1)^{-1}(K) \subset K$ for any $\mu > 0$. Since $\frac{du}{dt} \in L^2((0, \infty), V)$, we have $\frac{df}{dt} \in L^2((0, \infty), H)$. The results of [1, Lemme II.4] imply that $A_1 u(t) \in H$ for any $t \geqq 0$ and the function $u: [0, \infty) \to H$ is differentiable from the right everywhere. Moreover,

$$\left(\frac{d^+u}{dt}(t) + A_1 u(t), A_1 u(t)\right) \leqq (f(t) - F_0, A_1 u(t)) \quad \text{for any} \quad t \geqq 0 . \tag{18}$$

It follows from (7a), (16) and (17) that

$$\left|\frac{d^+u}{dt}(t)\right| \leqq Ce^{-\lambda t}(\|u_0\| + |g_0| \max(1 - t, 0) + \sqrt{(F_0, u_0)}\, e^{-At})$$

By using (18) we get

$$\begin{aligned}
|A_1 u(t)|^2 &\leqq C\left(|f(t)|^2 + \left|\frac{d^+u}{dt}(t)\right|^2\right) - 2(A_1 F_0, u(t)) \\
&\leqq C\left(\|u(t)\|^2 + \left|\frac{d^+u}{dt}(t)\right|^2\right) - 2(A_1 F_0, u(t)) \\
&\leqq Ce^{-2\lambda t}(\|u_0\|^2 + |A_1 u_0|^2 \max(1 - t, 0) + (F_0, u_0)e^{-2At}) \\
&\quad -2(A_1 F_0, u(t)) \\
&\leqq C(e^{-\lambda t}|A_1 u_0| + e^{-\lambda t/2}\|F_0\|)^2 .
\end{aligned}$$

It remains to prove the uniqueness of our solution. Let u_1, u_2 be two (weak) solutions of the inequality (2). It follows from the preceeding considerations that u_1, u_2 are also strong solutions, $\|u_i(t)\| < \varepsilon$ for any $t \geqq 0$, $i = 1, 2$,

$$\left(\frac{du_i}{dt} + Au_i + N(u_i) + F_0, v - u_i\right) \geqq 0 \quad \text{for any } v \in K \quad \text{a.e. in } (0, T) .$$

Choosing $v = u_{3-i}$ in this inequality and adding the resulting inequalities we get

$$\left(\frac{d(u_1 - u_2)}{dt} + A(u_1 - u_2) + N(u_1) - N(u_2), u_1 - u_2\right) \leqq 0 \quad \text{a.e. in } (0, T) ,$$

which implies (cf. the deriving of (7))

$$\frac{1}{2}\frac{d}{dt}|u_1-u_2|^2 \leqq C|u_1-u_2|^2 \quad \text{a.e. in } (0,T) \ .$$

Now $u_1(0)=u_2(0)$ implies $u_1 \equiv u_2$. □

Theorem 2. *Let, in addition to our general assumptions, K be a cone with its vertex at zero, $F_0=0$, $|N_2(v)| \leqq C\|v\|^2$ for any $v \in B$. Let $u_1 \in K$, $|u_1|=1$, be an eigenvector of the inequality (1) with an eigenvalue $\lambda_1<0$ (i.e. $(Au_1-\lambda_1 u_1, v-u_1) \geqq 0$ for any $v \in K$) and let*

$$(Ay,y) \geqq \lambda_1 |y|^2 \quad \textit{for any } y \in K-u_1 := \{v-u_1 \,; v \in K\} \ . \tag{A1}$$

Then the zero solution of (2) is unstable in the toplology of H; more precisely, there exists $\varepsilon>0$ such that for any $\delta>0$ the solution u of (2) with $u_0=\delta u_1$ exists on $[0,T_\delta]$ and $|u(T_\delta)| \geqq \varepsilon$.

Proof. Similarly as in the proof of Theorem 1 we get the existence of a strong solution of (2) (with N_2 redefined outside B). Moreover, there exist ε_0, $C>0$ such that $\{u \in V; \|u\| \leqq \varepsilon_0\} \subset B$ and

$$\left.\begin{aligned} &\int_0^t \|u\|^2 dt \leqq C\left(\int_0^t |u|^2 dt + |u_0|^2\right) \\ &\|u(t)\|^2 \leqq C\left(\int_0^t |u|^2 dt + \|u_0\|^2\right) \end{aligned}\right\} \text{for any } t \leqq T_{\max} \ , \tag{19}$$

where $T_{\max} := \inf\{t; \|u(t)\| \geqq \varepsilon_0\} > 0$ if $\|u_0\|$ is small enough (cf. the derivation of (6, 6d)).

Choose $\varepsilon<\varepsilon_0$, $\delta>0$ and let $u_0=\delta u_1$. By putting $v=0$ and $v=2u$ in (2) we get

$$\left(\frac{du}{dt}+Au+N(u), u\right)=0 \ ,$$

so that

$$\frac{d}{dt}|u| \leqq -\lambda_1 |u| + C\|u\|^2 \quad \text{a.e. in } \{t \in (0,T_{\max}); u(t) \neq 0\} \ . \tag{20}$$

Choose $\beta \in (0,1)$ fixed and suppose $|u(s)| \leqq (1+\beta)|w(s)|$ for $s \leqq t \leqq T_{\max}$, where $w(s)=\delta u_1 e^{-\lambda_1 s}$ is the solution of the linearized inequality. By using (20, 19) we obtain

$$\begin{aligned} |u(t)| &\leqq e^{-\lambda_1 t}|u_0| + \int_0^t e^{-\lambda_1(t-s)} C\|u(s)\|^2 ds \\ &= \delta e^{-\lambda_1 t} + C\int_0^t \|u(s)\|^2 ds - \lambda_1 C \int_0^t \left(\int_0^s \|u(\tau)\|^2 d\tau\right) e^{-\lambda_1(t-s)} ds \\ &\leqq \delta e^{-\lambda_1 t} + C\left(\int_0^t |u(s)|^2 ds + |u_0|^2\right) + C\int_0^t e^{-\lambda_1(t-s)}\left(\int_0^s |u(\tau)|^2 d\tau + |u_0|^2\right) ds \\ &\leqq |w(t)| + C|w(t)|^2 < \left(1+\frac{\beta}{2}\right)|w(t)| \end{aligned}$$

whenever $|w(t)| \leqq \beta/C_0$, i.e. $t \leqq T_\delta := -\frac{1}{\lambda_1} \log \frac{\beta}{C_0 \delta}$ (where C_0 is some fixed constant). Therefore,

$$|u(t)| \leqq (1+\beta)|w(t)| \quad \text{for any } t \leqq \min(T_\delta, T_{\max}) \ . \tag{21}$$

By putting $y := u - w$ we get

$$\left(\frac{du}{dt} + Au + N(u), y\right) \leqq 0 \ ,$$

$$\left(\frac{dw}{dt} + Aw, y\right) \geqq 0 \ ,$$

hence

$$\left(\frac{dy}{dt} + Ay + N(u), y\right) \leqq 0 \ ,$$

which implies

$$\frac{d}{dt} |y| \leqq -\lambda_1 |y| + C\|u\|^2 \quad \text{a.e. in } \{t \in (0, T_{\max}); y(t) \neq 0\} \ ,$$

$$|y(t)| \leqq \int_0^t e^{-\lambda_1(t-s)} C\|u(s)\|^2 ds \leqq \beta |w(t)| \quad \text{for any } t \leqq \min(T_\delta, T_{\max}) \ , \tag{22}$$

$$|u(t)| \geqq (1-\beta)|w(t)| \quad \text{for any } t \leqq \min(T_\delta, T_{\max}) \ .$$

We have by (19) and (21)

$$\|u(t)\| \leqq C(\|u_0\| + \delta e^{-\lambda_1 t}) \quad \text{for} \quad t \leqq \min(T_\delta, T_{\max}) \ ,$$

thus

$$\|u(t)\| \leqq C(\|u_0\| + \delta e^{-\lambda_1 T_\delta}) = C\left(\|u_0\| + \frac{\beta}{C_0}\right) < \frac{\varepsilon_0}{2}$$

if $\beta < C_0\left(\frac{\varepsilon_0}{2C} - \|u_0\|\right)$ and $t \leqq \min(T_\delta, T_{\max})$, which implies $T_\delta \leqq T_{\max}$. By (22)

$$|u(T_\delta)| \geqq (1-\beta)|w(T_\delta)| \geqq \frac{(1-\beta)\beta}{C_0} \ ,$$

which proves our assertion. □

Remark 1. Theorem 2 holds also if we replace the assumption $|N_2(v)| \leqq C\|v\|^2$ by the assumption $\|N_2(v)\|_{V'} \leqq C\|v\| \cdot |v|$.

Examples

Example 1. Let

$$V = H = \mathbb{R}^2, \ K = \left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2; \ u_2 \geqq u_1 \geqq 0\right\} \ ,$$

$$A = \begin{pmatrix} -3 & 0 \\ 5 & -1 \end{pmatrix}, \quad N\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -(u_2)^2 \\ 0 \end{pmatrix}, \quad F_0 = 0.$$

Then $\sigma(A)=\{-3,-1\}$, $\sigma_K(A)=\{-1,1/2\}$, nevertheless the zero solution is stable.

Proof. Let us choose $\varepsilon>0$ ($\varepsilon \ll 1$). We shall show that the solution $u(t)$ of (0) with $u_0=\begin{pmatrix}0\\ \delta\end{pmatrix}$, δ sufficiently small, satisfies $\|u(t)\|\leqq\varepsilon$ for any $t\geqq 0$, which, together with the geometry of K and of the trajectory of $u(t)$, implies our assertion (see Fig. 1).

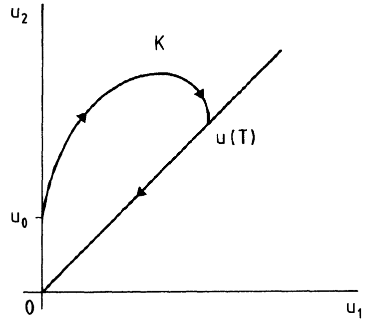

Fig. 1

To prove this let us study the solution $v(t)$ of the corresponding equation

$$\begin{aligned} \dot v_1 &= 3v_1+v_2^2 \\ \dot v_2 &= -5v_1+v_2 \\ v_1(0)&=0\ , \quad v_2(0)=\delta\ , \end{aligned} \tag{23}$$

where $\begin{pmatrix}\dot v_1\\ \dot v_2\end{pmatrix}=\frac{dv}{dt}$.

Since $\dot v_1(t)>0$, $v_1(t)>0$ for any $t>0$, we have $v\equiv u$ on $[0,T]$, where $T=\inf\{t; v_1(t)\geqq v_2(t)\}$. Thus it is sufficient to prove $T<\infty$ and $\|v(t)\|\leqq\varepsilon$ for $t\leqq T$, since $u(t)$ decays exponentially to zero for $t>T$.

Since $v_1\geqq 0$, we have $\dot v_2\leqq v_2$, hence

$$v_2(t)\leqq\delta e^t\ . \tag{24}$$

Now

$$v_1(t)=\int_0^t e^{3(t-s)}v_2^2(s)ds\leqq e^{3t}\delta^2\int_0^t e^{-s}ds\leqq e^{3t}\delta^2\leqq\frac{\delta}{5}$$

if $t\leqq 1$ and δ is sufficiently small. Consequently, $\dot v_2(t)\geqq 0$ and $v_2(t)\geqq\delta$ for $t\leqq 1$, which implies

$$v_1(1)\geqq\int_0^1 v_2^2(s)ds\geqq\delta^2\ .$$

Therefore,

$$v_1(t)\geqq\delta^2e^{3(t-1)} \quad \text{for any} \quad t\geqq 1\ . \tag{25}$$

Now (24) and (25) give us the following estimate for T

$$T\leqq T^*\ , \qquad \text{where} \quad \delta^2e^{3(T^*-1)}=\delta e^{T^*}\ .$$

Finally,

$$v_1(t) \leqq v_2(t) \leqq \delta e^t \leqq \delta e^T \leqq \delta e^{T^*} = e^{3/2}\sqrt{\delta}$$

for any $t \leqq T$, which proves our assertion. □

Remark 2. In [3, 5] there are given some general assumptions, under which stationary solutions of certain reaction-diffusion systems loose their "linearized stability" when we add unilateral conditions to the system. In an abstract setting we have $\operatorname{Re}\sigma(A) > 0$, $\sigma_K(A) \cap \{\lambda; \lambda < 0\} \neq \emptyset$ and, moreover, the results of [3, 5] imply that there exists $\lambda \in \sigma_K(A)$, $\lambda < 0$, which is also a bifurcation point of the stationary "nonlinear" inequality

$$(Au + N(u) - \lambda u, v - u) \geqq 0 \qquad \forall v \in K \tag{26}$$

(for any "suitable" N). Example 1 is *not* counterexample for the linearization principle in this case, since in this example

(i) we do not have $\operatorname{Re}\sigma(A) > 0$

(ii) $\lambda = -1$ is not a bifurcation point for the inequality (26).

Example 2. Let $V = H = \mathbb{R}^3$, $K = \{u \in \mathbb{R}^3; u_3 = 0\}$, $F_0 = 0$, $N \equiv 0$,

$$A = \begin{pmatrix} -1 & -2 & 16 \\ 2 & -1 & 0 \\ -2 & 0 & 9 \end{pmatrix}.$$

Then $\sigma(A) = \{1, 3 \pm 2i\}$ and $\sigma_K(A) = \emptyset$, since any $\lambda \in \sigma_K(A)$ is an eigenvalue of the operator $B := PA/K$ (where $P: \mathbb{R}^3 \to K$ is the orthogonal projection) and $\sigma(B) = \{-1 \pm 2i\}$. The inequality (2) is equivalent to the equation

$$\frac{du}{dt} + Bu = 0 ,$$

hence the zero solution is unstable.

Example 3. Let $\Omega = (0, \pi) \subset \mathbb{R}^1$, $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $F_0 = 0$, $Au = -u'' + \lambda u$, $K = \{u \in V; u(\pi/2) \leqq 0, \quad u(2\pi/3) \geqq 0\}$, $N(u) = f(u)$, where $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(0) = f'(0) = 0$.

Then $u_0 = 0$ is a stable solution of the equation

$$\frac{du}{dt} + Au + N(u) = 0$$

$$u(0) = u_0$$

provided $\lambda > -1$ and it is unstable if $\lambda < -1$. Similarly, $u_0 = 0$ is a stable solution of the inequality (2) if $\lambda > -9/4$ and it is unstable if $\lambda < -9/4$.

The stability result follows from Theorem 1, the instability result from the proof of Theorem 2. Note that the assumption (A1) is not satisfied, nevertheless, by putting $u_0 = \delta u_1$, where $u_1(x) = -\sin(3x/2)$ for $x \leqq 2\pi/3$ and $u_1(x) = 0$ for $x \geqq 2\pi/3$, and by using the notation from the proof of Theorem 2 we get $(Ay, y) \geqq \lambda_1 |y|^2$ for $y = u - w$, since $u(t)(2\pi/3) = 0$ for $t \leqq T_\delta$.

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