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Symmetries of Möbius Ladders

Erica Flapan*

Department of Mathematics, Pomona College, Claremont, CA 91711, USA

Introduction

Chemistry has recently motivated the study of graphs embedded in $\mathbb{R}^3$, and of their symmetries as an extension of knot theory. We are interested in the following question: Given a graph G embedded in $\mathbb{R}^3$ or $S^3 = \mathbb{R}^3 \cup \infty$, what can be said about its symmetries just from the topology of the graph itself? More precisely, we shall let $\mathrm{Sym}(G)$ denote the group of homeomorphisms of G, up to isotopy. If G is embedded in a manifold M, then $\mathrm{Sym}(M, G)$ is the group of diffeomorphisms of M which leave G invariant, up to isotopy respecting G. We are interested in the general question of how an element of $\mathrm{Sym}(G)$ can be represented by an element of $\mathrm{Sym}(S^3, G)$, for some embedding of G in S^3. Of course, not every graph G can be embedded in such a way that a given element of $\mathrm{Sym}(G)$ can be represented by some element of $\mathrm{Sym}(S^3, G)$. In Sect. 1, we will provide an example of a graph G and a particular element of $\mathrm{Sym}(G)$ such that, no matter what the embedding of G in S^3, that element cannot be represented by an element of $\mathrm{Sym}(S^3, G)$.

In the case where each element of $\mathrm{Sym}(G)$ can be represented by an element of $\mathrm{Sym}(S^3, G)$, we are interested in which elements of $\mathrm{Sym}(G)$ can be represented by periodic and/or orientation reversing elements of $\mathrm{Sym}(S^3, G)$. Since not all periodic elements of $\mathrm{Sym}(S^3, G)$ restrict to periodic elements of $\mathrm{Sym}(\mathbb{R}^3, G)$, we consider separately the question of which elements of $\mathrm{Sym}(G)$ can be represented by periodic elements of $\mathrm{Sym}(\mathbb{R}^3, G)$. In Sects. 2 and 3, we completely answer these questions for one class of graphs. Understanding the symmetries of this particular class of graphs also has some applications in chemistry, which we explain below.

It is often important in the field of chemistry to determine whether a molecule is distinct from its mirror image. A molecule which can convert itself to its mirror image is said to be *chemically achiral*, whereas one which cannot is *chemically chiral*. The existence of such a molecular deformation depends on a variety of physical conditions, and thus cannot be completely characterized mathematically. Instead, we abstract the molecule as a graph in space, and ask whether this embedded graph can be deformed in space to its mirror image. A graph which can

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be deformed to its mirror image is *topologically achiral*, while one which cannot be deformed to its mirror image is *topologically chiral* [Wa]. (The property of topological achirality for graphs is analogous to the property of amphicheirality for knots.) A molecule whose associated graph is topologically chiral will necessarily be chemically chiral, hence this concept is of some use to chemists.

One particular molecular graph which is of interest is the "molecular Möbius ladder", which was first synthesized by Walba et al. [WRH]. This is a molecule shaped like a ladder with three rungs which was made to join itself end-to-end with one half twist (see Fig. 1). The sides of the ladder represent a molecular chain while the rungs represent double bonds; hence in the associated molecular graph we distinguish between the edges making up the sides and those making up the rungs. The synthesis of this molecule was a significant achievement in chemistry because of its topologically interesting molecular structure. Walba had conjectured that this molecule was chemically chiral [Wa], however chemical achirality could not be completely ruled out until Simon [Si] proved that its associated embedded molecular graph was topologically chiral.

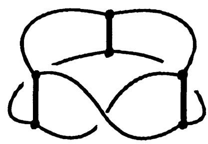

Fig. 1

More generally, let M_n denote the graph illustrated in Fig. 2, with $n \geqq 3$, where the rungs of the ladder are $\alpha_1, \ldots, \alpha_n$ and the sides of the ladder together form the loop K. Observe that the graph M_3 is just the bipartite graph (3, 3) which is one of Kuratowski's non-planar graphs. For all $n > 3$, M_n contains this non-planar graph and hence is itself non-planar.

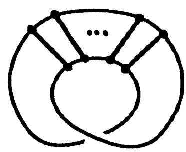

Fig. 2

What Simon showed is that for the embedding of M_n illustrated in Fig. 2, for any $n \geqq 3$, there is no orientation reversing diffeomorphism h of S^3 with $h(M_n) = M_n$ and $h(K) = K$. The chemical motivation for the requirement that $h(K) = K$ is that the loop K represents a molecular chain, which is chemically different from the rungs which represent molecular bonds. We note however, that Simon [Si] has also shown that if $n \geqq 4$ then every automorphism of M_n leaves K setwise invariant. Thus if we restrict our attention to Möbius ladders with at least four rungs, then $h(K) = K$ will follow whenever $h(M_n) = M_n$. So only in the case where $n = 3$ does the hypothesis that $h(K) = K$ make any difference.

Simon's results naturally led to the question of topological chirality for other embeddings of the graph M_n. That is, is it possible to reembed M_n in S^3 in such a way that there is an orientation reversing diffeomorphism h of S^3 with $h(M_n) = M_n$ and $h(K) = K$? For $n \geqq 4$, this is just the question of whether there is any element of $\mathrm{Sym}(M_n)$ which can be represented by an orientation reversing element of

$\mathrm{Sym}(S^3, M_n)$. We answer this particular question by showing that, for any $n \geqq 3$ which is odd, no matter how M_n is embedded there is no such h. On the other hand, for any n which is even there is an embedding of M_n in S^3 and an orientation reversing diffeomorphism h of S^3 with $h(M_n) = M_n$ and $h(K) = K$. In addition, we provide a general analysis of how elements of $\mathrm{Sym}(M_n)$ can be represented by elements of $\mathrm{Sym}(S^3, M_n)$ and $\mathrm{Sym}(\mathbb{R}^3, M_n)$. In particular, the group $\mathrm{Sym}(M_n)$ is generated by those rotations and reflections of K which leave the set of rungs invariant. Each element of $\mathrm{Sym}(M_n)$ can be realized by a periodic orientation preserving element of $\mathrm{Sym}(S^3, M_n)$, as will be illustrated in Figs. 5, 9, and 10. Theorem 4 together with Figs. 6 and 3, analyzes which symmetries of M_n can be realized by periodic orientation preserving elements of $\mathrm{Sym}(\mathbb{R}^3, M_n)$. The question of which elements of $\mathrm{Sym}(M_n)$ can be represented by orientation reversing elements of $\mathrm{Sym}(S^3, M_n)$ is answered by Theorem 2 together with Fig. 4. For periodic orientation reversing elements of $\mathrm{Sym}(S^3, M_n)$ and $\mathrm{Sym}(\mathbb{R}^3, M_n)$ the question is answered by Theorem 3 and Lemma 2. Finally, in Sect. 4, we make some observations about topological chirality for a slightly larger class of graphs which includes the graphs M_n.

1. An Element of Sym(G) which Cannot be Represented by an Element of Sym(S^3, G)

We are interested in whether there is a graph G such that some element of $\mathrm{Sym}(G)$ cannot be realized by any element of $\mathrm{Sym}(S^3, G)$, no matter how G is embedded in S^3. The following theorem shows that K_6 (the complete graph on six vertices) is an example of such a graph.

Theorem 1. *For any embedding of K_6 in S^3, and any labelling of the vertices of K_6 by the numbers one through six, there is no element of* $\mathrm{Sym}(S^3, K_6)$ *which induces the permutation* (1234) *on the vertices of K_6.*

Proof. Choose some labelling of the vertices of K_6 by the numbers one through six. Any unordered set of three such numbers will determine a loop consisting of the three vertices with those numbers together with the edges between them. However, since K_6 only has six vertices, this set also determines a disjoint complementary loop. So we shall let each set of three such numbers represent a pair of disjoint loops. Since each pair of loops can be represented in two different ways by complementary sets of numbers, there are ten pairs of such loops in K_6.

We consider the orbits of these pairs of loops under the permutation (1234). Being careful not to list the same pair in two different ways, we see that the collection of orbits of loop pairs is $\langle 123, 234, 341, 412\rangle$, $\langle 125, 235, 345, 415\rangle$, and $\langle 135, 245\rangle$. The observation we wish to make here is that the set of all loop pairs is partitioned into orbits which each contain an even number of elements.

Now suppose that the graph K_6 is embedded in S^3 in such a way that there is a diffeomorphism $h: S^3 \to S^3$ with $h(K_6) = K_6$ and h induces the permutation (1234) on the vertices of K_6. For each pair of disjoint loops A and B in K_6, let $\omega(A, B)$ denote the mod 2 linking number of A and B in S^3. Since h is a diffeomorphism, $\omega(A, B) = \omega(h(A), h(B))$. Thus all the pairs in a given orbit will have the same mod 2 linking number. Define $\lambda = \sum \omega(A, B)$, where the summation is in $\mathbb{Z}_2$ over all pairs of disjoint loops in K_6. Since every orbit has an even number of pairs in it, $\lambda=0$. However, Conway and Gordon have proved in [CG] that for any embedding of K_6 in S^3, $\lambda=1$. Thus there could not have been such a diffeomorphism h. □

2. Orientation Reversing Symmetries of Möbius Ladders

Any graph which is homeomorphic, as a 1-complex, to the graph in Fig. 2, is a *Möbius ladder* as defined originally by Harary and Guy [HG]. More formally,

Definition. For $n \geqq 3$ we define a *Möbius ladder* M_n to be any graph which is homeomorphic to a $2n$-gon K together with disjoint chords $\alpha_1, \ldots, \alpha_n$ joining opposite pairs of vertices. We will refer to K as the *loop* of M_n and the chords $\alpha_1, \ldots, \alpha_n$ as the *rungs* of M_n.

Lemma 1. *Let M_n be a Möbius ladder which is embedded in S^3 with loop K and rungs $\alpha_1, \ldots, \alpha_n$. Suppose $h: S^3 \to S^3$ is an orientation reversing diffeomorphism with $h(M_n)=M_n$ and $h(K)=K$. Then there are at most two rungs α_i such that $h(\alpha_i)=\alpha_i$.*

Proof. Let X denote the double branched cover of S^3, branched over K. Let $K_1, \ldots, K_n$ be the preimages of $\alpha_1, \ldots, \alpha_n$ respectively. For each i, let k_i denote the simple closed curve consisting of α_i together with some component of $K-\alpha_i$. Let F_i be a Seifert surface for k_i; *and let* S_i denote the preimage of F_i in X. Observe that for each i, K_i is the boundary of S_i.

It is not hard to show that $H_1(X)$ is finite (see [Ro]); and so, by Poincaré duality, $H_2(X)$ is trivial. Thus if S and S' are both surfaces bounded by K_i then the algebraic intersection number of K_j with S must equal the algebraic intersection number of K_j with S'. Hence we can define $\operatorname{lk}(K_i, K_j)$ as the algebraic intersection of K_j with the surface S_i.

Let p be the algebraic intersection number of $\operatorname{Int}(\alpha_j)$ with $F_i - K$. Since α_j meets $F_i \cap K$ at one point, then $\operatorname{lk}(K_i, K_j) = \pm(2p+1)$, depending on orientations. In particular, for all $i \neq j$, we have $\operatorname{lk}(K_i, K_j) \neq 0$. Now suppose $h(\alpha_i)=\alpha_i$ for $i=1,2,3$. Since S^3 has a unique double branched cover over K, any homeomorphism of (S^3, K) will lift to X. So let $g: X \to X$ be one lift of h. Then g is orientation reversing and $g(K_i)=K_i$ for $i=1,2,3$. Give K_1, K_2, K_3 orientations, then suppose that g preserves the orientation of K_1. Since g is orientation reversing and $\operatorname{lk}(K_1, K_2) \neq 0$ and $\operatorname{lk}(K_1, K_3) \neq 0$, it must be that g reverses the orientations of both K_2 and K_3. But this is impossible because $\operatorname{lk}(K_2, K_3) \neq 0$. We obtain a similar contradiction if we suppose that g reverses the orientation of K_1. Thus we could not have had $h(\alpha_i)=\alpha_i$ for $i=1,2,3$. □

Theorem 2. *Let M_n be a Möbius ladder which is embedded in S^3 with loop K, where n is an odd number. Then there is no diffeomorphism $h: S^3 \to S^3$ which is orientation reversing with $h(M_n)=M_n$ and $h(K)=K$.*

Proof. Suppose there were such an h. First we shall consider the case where h reverses the orientation of K. In this case h performs a permutation of order two on the rungs. Since the number of rungs, n, is odd there must be some rung which h maps to itself. By the definition of a Möbius ladder $n \geqq 3$, so we can assume the rungs have been labelled in such a way that $h(\alpha_1)=\alpha_1$, $h(\alpha_2)=\alpha_n$, and $h(\alpha_n)=\alpha_2$. As in Lemma 1, let X be the double branched cover of S^3 with branch set K; and let g be one lift of h and, for each i, let K_i be the preimage of α_i. Then $g(K_1)=K_1$, $g(K_2)=K_n$, and $g(K_n)=K_2$. As in the proof of Lemma 1, the algebraic linking number is well defined and $\mathrm{lk}(K_1,K_2)\neq 0$, $\mathrm{lk}(K_1,K_n)\neq 0$, and $\mathrm{lk}(K_2,K_n)\neq 0$. Since g is orientation reversing and $\mathrm{lk}(K_2,K_n)\neq 0$, we can assume without loss of generality that $g(K_2)=-K_n$ and $g(K_n)=+K_2$. Now suppose that $g(K_1)=+K_1$. Then $g(K_2)=-K_n$ implies that $\mathrm{lk}(K_1,K_2)=\mathrm{lk}(K_1,K_n)$. But $g(K_n)=+K_2$ implies that $\mathrm{lk}(K_1,K_n)=-\mathrm{lk}(K_1,K_2)$. Since these linking numbers are non-zero, this is impossible. Supposing that $g(K_1)=-K_1$ yields a similar contradiction. Thus no such h could exist with $h(K)=-K$.

Now we consider the case where h preserves the orientation of K. We can assume that the rungs $\alpha_1,\ldots,\alpha_n$ have been labelled consecutively. In this case there exists a number p, such that h rotates each α_i to α_{i+p}. Since n is odd, there is some odd number q, such that $h^q(\alpha_i)=\alpha_i$ for all $i=1,\ldots,n$. But h^q is orientation reversing and $n\geqq 3$, hence this contradicts Lemma 1. □

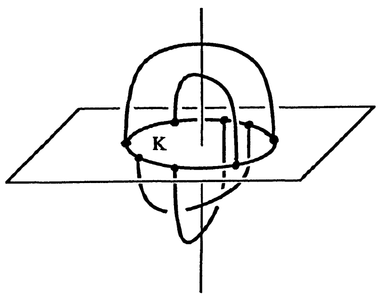

Fig. 3

In Fig. 3, we illustrate an example of an embedded Möbius ladder M_4 with four rungs, which has an orientation reversing diffeomorphism $h\colon S^3\to S^3$ with $h(M_4)=M_4$ and $h(K)=K$. In this example the map h can be seen as the composition of a reflection through the plane containing the loop K followed by a rotation of 90° about an axis perpendicular to that plane. Thus h preserves the orientation of K, and h has order equal to four. For any n which is even we can draw a similar example of a Möbius ladder M_n, with loop K lying in a plane, such that there is an orientation reversing diffeomorphism $h\colon S^3\to S^3$ which is the composition of a reflection though the plane containing K followed by a rotation of 90° about an axis perpendicular to that plane with $h(M_n)=M_n$ and $h(K)=K$. Thus, for any n which is even there is a Möbius ladder M_n and an orientation reversing diffeomorphism $h\colon S^3\to S^3$ with $h(M_n)=M_n$, $h(K)=+K$, and h is of order four.

We will also illustrate a Möbius ladder M_n and an orientation reversing diffeomorphism $h\colon S^3\to S^3$ with $h(M_n)=M_n$ and $h(K)=-K$; however, in order to explain the way we have drawn our example, we first prove the following lemma.

Lemma 2. *Let M_n be a Möbius ladder which is embedded in $\mathbb{R}^3$. Let $h\colon\mathbb{R}^3\to\mathbb{R}^3$ be an orientation reversing diffeomorphism with $h(M_n)=M_n$ and $h(K)=K$. Then h is not of finite order.*

Proof. Suppose h is of finite order. Then $h|K$ is also of finite order and h reverses the orientation of K. Thus, h fixes two antipodal points, x and y, of K. The points x and y separate K into arcs A and B, with $h(A)=B$ and $h(B)=A$. Since h leaves the collection of rungs setwise invariant, it follows from the definition of a Möbius ladder that each rung α_i has one endpoint in A and the other endpoint in B. By Smith Theory [Sm], the fixed point set of an orientation reversing finite order diffeomorphism of $\mathbb{R}^3$ is either one point or a plane. In this case, the fixed point set of h must contain the points x and y, so the fixed point set of h must be a plane P. Now $P \cap K=\{x, y\}$, hence the arc A is contained in one component of $\mathbb{R}^3-P$, and the arc B is contained in the other component of $\mathbb{R}^3-P$. Since P separates $\mathbb{R}^3$, each rung α_i must intersect P. But this means that h fixes a point of each rung α_i, which implies that $h(\alpha_i)=\alpha_i$ for all i. By the definition of Möbius ladder $n \geqq 3$, hence this contradicts Lemma 1. □

Suppose that M_n is an example of a Möbius ladder which is embedded in S^3 (i.e. $\mathbb{R}^3 \cup \infty$) in such a way that there is an orientation reversing diffeomorphism $h: S^3 \to S^3$ with $h(M_n)=M_n$ and $h(K)=-K$, and without loss of generality, $h(\infty)=\infty$. Then by the proof of Lemma 2, one of the fixed points of $h|K$ must actually be the point at infinity. Thus if we want to illustrate an example where $h(K)=-K$ and h can be seen easily as the composition of a reflection and a rotation, then we must draw M_n so that the point at infinity is on K.

In Fig. 4, we illustrate a Möbius ladder M_6, which is embedded in S^3 in such a way that the ends of the loop K meet at the point at infinity. We define an orientation reversing diffeomorphism $h: S^3 \to S^3$ which reflects S^3 through the origin. That is, h is the composition of a reflection through the plane drawn in the Fig. 4, followed by a rotation of 180° about K. Thus, $h(M_6)=M_6$, and $h(K)=-K$, and h has order two. For any n which is even we can embed M_n in S^3 in a similar way, so that there is an orientation reversing diffeomorphism $h: S^3 \to S^3$ with $h(M_n)=M_n$, and $h(K)=-K$, and h has order two. By moving K slightly at the point at infinity we obtain an embedding of M_n in $\mathbb{R}^3$, and an orientation reversing diffeomorphism $g: S^3 \to S^3$ with $g(M_n)=M_n$, and $g(K)=-K$. However, by Lemma 2 no such g could be of finite order.

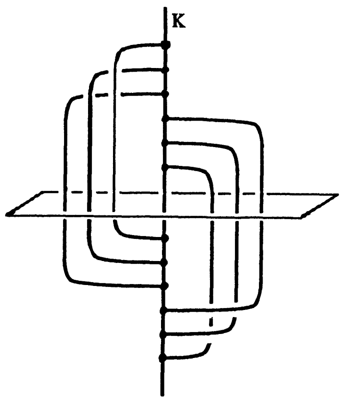

Fig. 4

These examples suggest that we might be able to say something more about finite order diffeomorphisms of Möbius ladders which are embedded in S^3. By Smith theory [Sm] we know that any finite order orientation reversing diffeomorphism of S^3 has either two points or a 2-sphere as its fixed point set; and any finite order orientation preserving diffeomorphism has either the empty set or a simple closed curve as its fixed point set. Let M_n be a Möbius ladder which is embedded in S^3.

Lemma 3. *Let $h: S^3 \to S^3$ be an orientation reversing diffeomorphism which is of finite order, with $h(M_n) = M_n$ and $h(K) = K$. Then the fixed point set of h consists of two points.*

Proof. By Smith Theory [Sm] if the fixed point set of h is not two points then it is a 2-sphere F. So we can pick the point x at infinity to be any fixed point of h which is not on M_n. Then h restricts to an orientation reversing periodic diffeomorphism $g: \mathbb{R}^3 \to \mathbb{R}^3$ with $g(M_n) = M_n$ and $g(K) = K$, and with fixed point set a plane P. Since P separates $\mathbb{R}^3$ into two components and g is orientation reversing, g must switch the two components of $\mathbb{R}^3 - P$. But since $g(K) = K$ the intersection of P and K must be non-empty. Hence $g(K) = -K$; which contradicts Lemma 2. Thus the fixed point set of h could not be a 2-sphere. □

Now we prove Theorem 3, which shows that Figs. 3 and 4 provide examples of the only possible orders for orientation reversing finite order diffeomorphisms of Möbius ladders embedded in S^3.

Theorem 3. *Let M_n be a Möbius ladder which is embedded in S^3 with loop K. Suppose that $h: S^3 \to S^3$ is an orientation reversing diffeomorphism with $h(M_n) = M_n$ and $h(K) = K$, and the order of h is some finite number p. If $h(K) = -K$ then $p = 2$, and if $h(K) = +K$ then $p = 4$.*

Proof. First suppose that $h(K) = -K$. Then h^2 fixes K pointwise. Hence h^2 also fixes each rung α_i pointwise. But h^2 is orientation preserving, so by Smith theory [Sm], if h^2 is not the identity map then the fixed point set of h^2 is either the empty set or a simple closed curve. Hence h^2 is the identity map.

Now suppose that $h(K) = +K$. Then h performs a cyclic permutation of the rungs α_i. By Lemma 3, the fixed point set of h is two points. Suppose $p = 2$, then $h(\alpha_i) = \alpha_i$ for all i. Hence h fixes a point of each α_i. Since $n \geqq 3$, this is a contradiction. Thus $p \neq 2$, so the map $g = h^2$ is not the identity. Since the fixed point set of g contains the fixed point set of h, by Smith Theory [Sm] the fixed point set of g must be a simple closed curve J. The order of h must be even, since h is orientation reversing. Thus $r = p/2$ is an integer, and the map $f = h^r$ has order two. Since $f(K) = +K$ we can use the same argument as above to show that f cannot be orientation reversing. So f is orientation preserving, and hence r must be even. Thus, in fact, $f = g^{p/4}$. This implies that the fixed point set of f contains the fixed point set of g. Since f is orientation preserving the fixed point set of f cannot contain more than a simple closed curve. Thus the fixed point set of f is J, so J intersects each rung α_i. Now this implies that g fixes a point of each α_i. Hence $g(\alpha_i) = \alpha_i$ for all i, and so the order of g is two. Therefore the order of h is four. □

Observe that, up to conjugacy, there is only one order 4 element of $\mathrm{Sym}(M_n)$ which preserves the orientation of K. Also, for n odd, there is only one conjugacy class of $\mathrm{Sym}(M_n)$ of order two which reverses the orientation of K. For n even, there are two conjugacy classes of $\mathrm{Sym}(M_n)$ of order two which reverse the orientation of K, one of which leaves no rung invariant and one of which leaves two rungs invariant. However, this latter symmetry actually fixes one of the invariant rungs pointwise. Hence if this symmetry were realizable by a finite order diffeomorphism $h: S^3 \to S^3$, it would contradict Lemma 3. Thus Theorem 3 completes our analysis of how elements of $\mathrm{Sym}(M_n)$ can be represented by orientation reversing elements of $\mathrm{Sym}(S^3, M_n)$ and $\mathrm{Sym}(\mathbb{R}^3, M_n)$.

3. Orientation Preserving Symmetries of Möbius Ladders

We consider how elements of $\mathrm{Sym}(M_n)$ can be represented by orientation preserving elements of $\mathrm{Sym}(S^3, M_n)$ and $\mathrm{Sym}(\mathbb{R}^3, M_n)$. All those elements of $\mathrm{Sym}(M_n)$ which are induced by rotations of K can be represented by periodic orientation preserving elements of $\mathrm{Sym}(S^3, M_n)$. In Fig. 5 we illustrate such an example with $n=3$. In order to facilitate our explanation of the diffeomorphism h, we have drawn a central axis A which is perpendicular to the plane containing the loop K. The action of h can be seen as the composition of a rotation by 120° about the axis A followed by a rotation by 120° about the loop K. An analogous example works for any n.

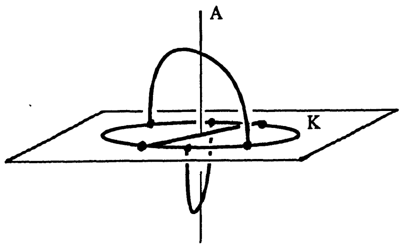

Fig. 5

In contrast, for $\mathbb{R}^3$ we have the following theorem.

Theorem 4. *Let M_n be a Möbius ladder which is embedded in $\mathbb{R}^3$ with loop K. Let $h: \mathbb{R}^3 \to \mathbb{R}^3$ be an orientation preserving finite order diffeomorphism with $h(M_n) = M_n$, and $h(K) = K$. If the order of h is even, then the order of h is two.*

Proof. Let r be the order of h and let J be the fixed point set of h. By Smith Theory [Sm], the fixed point set of an orientation preserving finite order diffeomorphism of $\mathbb{R}^3$ is a line. Thus for all $i<r$ the fixed point set of h^i is also J. We consider the cases where J intersects K and where J is disjoint from K separately. First, suppose J intersects K. Then the intersection of J and K is two points, and h reverses the orientation of K. In this case, h^2 fixes K pointwise in addition to J. Therefore h^2 is the identity.

Now suppose that J is disjoint from K. Then h preserves the orientation of K, and hence cyclically permutes the rungs α_i. By hypothesis r is assumed to be even, so $p=r/2$ is an integer. Now, h^p is a rotation of K of order two, and hence $h^p(\alpha_i)=\alpha_i$ for all i. Thus h^p fixes a point of each rung α_i. So J intersects every rung. But this implies that h fixes a point of each rung, and so h^2 fixes every rung pointwise. Thus again, h^2 fixes K pointwise in addition to J. So, in fact, $r=2$. □

For every n, up to conjugacy, $\mathrm{Sym}(M_n)$ has precisely one element of order two which respects the orientation of K. We see as follows that for every n there is an embedding of M_n in $\mathbb{R}^3$ such that this element can be realized by an orientation preserving element of $\mathrm{Sym}(\mathbb{R}^3, M_n)$ of order two. Let M_3 be the Möbius ladder illustrated in Fig. 5. Let $g: \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation by 180° about the central axis which is perpendicular to the plane containing the loop K. Then $g(M_3) = M_3$, $g(K) = +K$, the fixed point set of g is the central axis, and the order of g is two. Observe that for any n we can construct an analogous example.

Now we provide an example to show that we can have any odd order orientation preserving symmetry of a Möbius ladder in $\mathbb{R}^3$. Figure 6 illustrates a Möbius ladder M_3 which is invariant under a rotation of order three about a central axis. Observe that the same rotation will work for any number of rungs which is a multiple of three. Also, for any odd number r, let K be the boundary of a band with r half twists, then for any $n > 0$, we can construct an analogous Möbius ladder M_{nr} with loop K, and M_{nr} will be invariant under a rotation of order r.

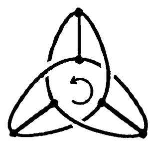

Fig. 6

Note that in an example which is constructed as in Fig. 6, the loop K will be knotted. This will not always be the case for every embedding of a Möbius ladder in $\mathbb{R}^3$ with an odd order symmetry. For example, the embedding of M_3 illustrated in Fig. 7 is invariant under a rotation of order three about a central axis. However, in Theorem 5 we will prove that, for n odd, if M_n is embedded in $\mathbb{R}^3$ so that it has an odd order symmetry, then at least one of three "special" cycles in M_n is knotted. We begin by introducing two special cycles, in addition to K, which are contained in M_n.

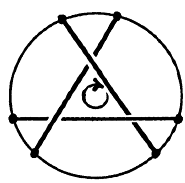

Fig. 7

Suppose that n is odd. Then K contains $2n$ edges, and its vertices occur at the endpoints of the rungs $\alpha_1, \ldots, \alpha_n$. Label the edges of K consecutively by $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$. Let $R = \alpha_1 \cup \ldots \cup \alpha_n \cup a_1 \cup \ldots \cup a_n$ and let $B = \alpha_1 \cup \ldots \cup \alpha_n \cup b_1 \cup \ldots \cup b_n$. Figure 8 illustrates the R and B for the Möbius ladder in Fig. 7.

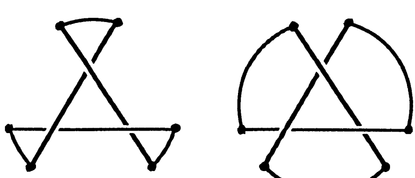

Fig. 8

By our construction it is clear that R is a collection of one or more simple closed curves. We see that R is precisely one simple closed curve as follows. Observe that each edge of K is connected by a rung to the $n+1^{\text{th}}$ subsequent edge. In particular, since n is odd, $n=2k+1$. Hence, $n+1=2(k+1)$. Now, since the edges of K are labelled alternately by a's and b's, the $n+1^{\text{th}}$ subsequent edge after a_i is a_{i+k+1} (counting the indices modulo n). Therefore for each i, a_i and a_{i+k+1} are in the same component of R. Now since n and $k+1$ are relatively prime, it follows that R has only one component. Thus, when n is odd R is one simple closed curve. Similarly for n odd, B is one simple closed curve.

Before we state our theorem concerning the curves R and B, we prove the following lemma.

Lemma 4. *Let L be an unknotted simple closed curve in S^3. Let $h: S^3 \to S^3$ be a finite order orientation preserving diffeomorphism such that $h(L)=L$, and the fixed point set of h is a simple closed curve A, which is disjoint from L. Then* $\operatorname{lk}(L, A)=1$.

Proof. Let Q denote the complement in S^3 of an open tubular neighborhood of L which is invariant ander h. Since L is unknotted Q is a solid torus. By the Equivariant Dehn's Lemma [MY] there is an embedded meridional disk D in Q with $h(D)=D$ or $h(D)\cap D=\emptyset$. Suppose $D\cap A=\emptyset$, then ∂D is trivial in $\pi_1(S^3-A)$. Hence L is trivial in $\pi_1(S^3-A)$. But this is impossible since $h(L)=L$. Thus A intersects D in $n>0$ points. Now since A is fixed pointwise by h, we must have $h(D)=D$. Thus $h|D$ is a periodic diffeomorphism of a disk with n fixed points. So $n=1$. Therefore $\operatorname{lk}(L, A)=1$. □

Theorem 5. *Let $n \geqq 3$ be odd, and let M_n be a Möbius ladder which is embedded in $\mathbb{R}^3$. Suppose there is a diffeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ with $h(M_n)=M_n$ and $h(K)=K$. Suppose further that h is orientation preserving with odd order. Then at least one of the simple closed curves R, B, or K is knotted.*

Proof. Let p be the order of h. Since K has $2n$ edges and p is odd, p must divide n. Thus $h(R)=R$ and $h(B)=B$. Also because p is odd, no individual rung is invariant under h. The fixed point set of an orientation preserving finite order diffeomorphism of $\mathbb{R}^3$ must be an embedded line by Smith Theory [Sm]. Extend h to a map $g: S^3 \to S^3$ by mapping the point at infinity to itself. Now the fixed point set of g is a simple closed curve A which is disjoint from each of K, R, and B. So we can consider the mod 2 algebraic linking number of A with each of these cycles. Let ω_1, ω_2, and ω_3 be the mod 2 algebraic linking numbers of A with R, B, and K respectively.

Suppose that all three loops R, B, and K are unknotted. Then, by Lemma 4, $\omega_1=\omega_2=\omega_3=1$. However, by our construction of R and B, in $H_1(S^3-A, \mathbb{Z}_2)$ we have $[R]+[B]=[K]$. So, as a sum in $\mathbb{Z}_2$, $\omega_1+\omega_2=\omega_3$. This contradiction implies that one of R, B, or K must be knotted. □

Now we shall consider the diffeomorphisms which are induced by reflections of K. In contrast with the elements induced by rotations of K, all of those elements of $\operatorname{Sym}(M_n)$ which are induced by reflections of K can be realized by periodic orientation preserving elements of $\operatorname{Sym}(\mathbb{R}^3, M_n)$. In Fig. 9 we illustrate a Möbius ladder M_4 embedded in $\mathbb{R}^3$ in such a way that there is an orientation preserving diffeomorphism $g: \mathbb{R}^3 \to \mathbb{R}^3$ with $g(M_4)=M_4$ and $g(K)=-K$. The diffeomorphism

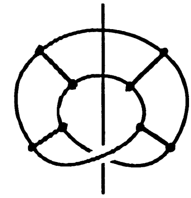

Fig. 9

$g: \mathbb{R}^3 \to \mathbb{R}^3$ is obtained by rotating by 180° about the axis A which is indicated in the Fig. 9. This diffeomorphism has order two, has fixed point set an embedded line, and $g(M_4) = M_4$ and $g(K) = -K$. For any even n we can construct an example which is analogous to this M_4. For any n which is odd we can construct a similar example but where the axis contains one of the rungs. This is illustrated for M_3 in Fig. 10. Therefore for any n, there is a Möbius ladder M_n which is embedded in $\mathbb{R}^3$ with an order two diffeomorphism $g: \mathbb{R}^3 \to \mathbb{R}^3$ with fixed point set an embedded line, where $g(M_n) = M_n$ and $g(K) = -K$.

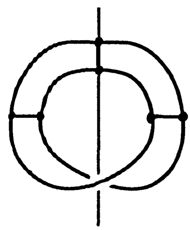

Fig. 10

4. Symmetries of Wheels

In order to try to generalize our results about chirality for Möbius ladders to a larger class of graphs, we introduce the following definitions.

Definition. A *wheel* is a graph consisting of a loop K and rungs $\alpha_1, \ldots, \alpha_n$, where K is a simple closed curve and the α_i are disjoint arcs with their endpoints at distinct points of K.

Definition. Let G be a wheel with loop K and rungs $\alpha_1, \ldots, \alpha_n$. Let B be the set of all rungs β_i with the property that the two components of $K - \beta_i$ contain equal numbers of endpoints of $\alpha_1, \ldots, \alpha_n$. If B contains $r > 0$ rungs, then we define the *Möbius hub* N_r of G to be the loop K together with all the rungs β_i in B.

Observe that the Möbius hub of a Möbius ladder is the Möbius ladder itself.

Lemma 5. *Let G be a wheel with Möbius hub N_r. If $r \geqq 3$, then N_r is a Möbius ladder.*

Proof. Let β be one of the rungs of N_r, and let C_1 and C_2 be the components of $K - \beta$, where K denotes the loop of G. Suppose that N_r is not a Möbius ladder, then without loss of generality there exists a rung γ of N_r such that γ has both endpoints in C_1. Now we can label the components D_1 and D_2 of $K - \gamma$ so that D_1 is contained in C_1 and D_2 contains C_2. By the definition of Möbius hub, C_1 and C_2 contain the same number of endpoints of the rungs $\alpha_1, \ldots, \alpha_n$ of the wheel G. But since D_1 is properly contained in C_1, there must be fewer endpoints of $\alpha_1, \ldots, \alpha_n$ in D_1 than there are in D_2. But this contradicts the fact that γ is also one of the rungs of the Möbius hub N_r. Therefore N_r must have actually been a Möbius ladder. □

Definition. Let G be a wheel with loop K, and let L_p be a subgraph of G containing the loop K and some rungs $\gamma_1, \ldots, \gamma_p$. Then L_p is said to be a *maximal Möbius subgraph* if both

1) L_p is a Möbius ladder, and

2) If α is a rung of G which is not a rung of L_p, then $L_p \cup \{\alpha\}$ is not a Möbius ladder.

Observe that since the Möbius hub of a wheel is a Möbius ladder it is contained in a (possibly non-unique) maximal Möbius subgraph. However, the Möbius hub of a wheel and a maximal Möbius subgraph of that wheel are, in general, different. Even assuming that $r \geqq 3$, the Möbius hub N_r of a wheel G is not necessarily a maximal Möbius subgraph. For example Fig. 11 illustrates a wheel where the Möbius hub N_3 is not a maximal Möbius subgraph. Here, for each β_i in N_3, the two components of $K - \beta_i$ each contain six endpoints of rungs of G. The rungs α_1 and α_2 are not in N_3 because, for $i=1$ and $i=2$, one component of $K - \alpha_i$ contains more endpoints than the other component. However, $N_3 \cup \alpha_1$ and $N_3 \cup \alpha_2$ are each maximal Möbius subgraphs of G. It is also easy to construct examples of wheels which have no Möbius hub yet have any number of maximal Möbius subgraphs.

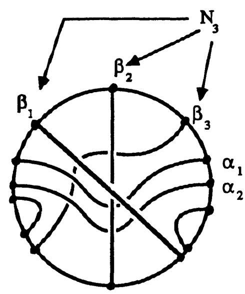

Fig. 11

Lemma 6. *Let G be a wheel with loop K. Suppose there is a homeomorphism h of the graph G such that $h(K) = K$. If G has Möbius hub N_r, then $h(N_r) = N_r$; also, if there is a p such that G contains a unique maximal Möbius subgraph L_p with p rungs, then $h(L_p) = L_p$.*

Proof. Let $A = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of all rungs of G and let B denote the subset of A consisting of rungs of N_r. Since $h(K) = K$, we must have $h(A) = A$. By the definition of the Möbius hub, B is the set of all rungs β_i such that the two components of $K - \beta_i$ contain equal numbers of endpoints of $\alpha_1, \ldots, \alpha_n$. Since h is a homeomorphism of G, in fact $h(B) = B$. Thus $h(N_r) = N_r$. Also, since L_p is the unique maximal Möbius subgraph with p rungs $h(L_p) = L_p$. □

Definition. A wheel G with loop K is *intrinsically chiral* if for any embedding of G in S^3 there is no orientation reversing diffeomorphism $h: S^3 \to S^3$ with $h(G) = G$ and $h(K) = K$.

We have shown in Theorem 2 that any Möbius ladder with an odd number of rungs is intrinsically chiral. This, together with Lemma 4, easily leads us to Theorem 6.

Theorem 6. *Let G be a wheel, and let $p \geqq 3$ be an odd number. Suppose that either the Möbius hub of G has p rungs, or G contains a unique maximal Möbius subgraph with p rungs. Then G is intrinsically chiral.*

Proof. Suppose that G is embedded in S^3 in such a way that there is an orientation reversing diffeomorphism $h: S^3 \to S^3$ with $h(G) = G$ and $h(K) = K$. By Lemma 4, if G has Möbius hub N_p, then $h(N_p) = N_p$, or if L_p is the unique maximal Möbius subgraph of G with p rungs, then $h(L_p) = L_p$. In either case, since $p \geqq 3$ and p is odd this contradicts Theorem 2. Therefore, in either case, G is intrinsically chiral. □

This theorem provides us with a way to construct many examples of intrinsically chiral wheels. However, not all intrinsically chiral wheels satisfy the hypotheses of this theorem. For example, Lemma 4 can also be used to construct various other types of intrinsically chiral wheels. An example of this other type is illustrated in Fig. 12. Here, the wheel G has no Möbius hub; however, it has a unique maximal Möbius subgraph L_4 with four rungs. By Lemma 4, any homeomorphism of h of G with $h(K) = K$, would also leave L_4 setwise invariant. But $G = L_4 \cup \beta$, thus $h(\beta) = \beta$. Hence also $h(\alpha) = \alpha$. Now let M_3 denote the wheel G after the rungs α and β have been removed. Then M_3 is a Möbius ladder with an odd number of rungs, and $h(M_3) = M_3$. Again this contradicts Theorem 2, so in fact G is intrinsically chiral.

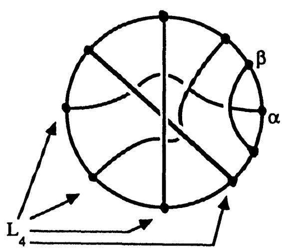

Fig. 12

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