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Dubrovin Valuation Rings and Henselization

Adrian R. Wadsworth*, **

Department of Mathematics, University of California at San Diego, La Jolla, CA 92093, USA

1. Introduction: Dubrovin Valuation Rings

Valuation theory is increasingly being recognized as a useful tool in studying the arithmetic of finite dimensional division algebras. However, there are some serious obstacles in noncommutative valuation theory that have limited its application. Notably, a valuation on the center F of a division ring D (with $[D:F]<\infty$) need not extend to a valuation on D (though when it does extend, the extension is unique). In addition, even if one starts working with division rings, one is led inevitably to consider also matrices over division rings, which arise with tensor products or scalar extensions. But there is not yet a reasonable notion of valuation for matrix rings.

N.I. Dubrovin introduced a few years ago in [D 1] and [D 2] a generalized notion of valuation ring which overcomes both of these obstacles. By basing his approach on the notion of a place in the category of simple Artinian rings, he obtained a significantly larger class of rings than the classical valuation rings on division rings. For example, the Dubrovin valuation rings restricting to a discrete valuation ring V of the center are precisely the maximal orders over V (cf. (1.15) below). Although there is no actual valuation associated with Dubrovin's rings, they nonetheless possess many of the properties of valuation rings, and have excellent extension properties. In fact, if S is a central simple F-algebra (with $[S:F]<\infty$) and V is a valuation ring of the field F, then there is always a Dubrovin valuation ring B of S with $B\cap F=V$; moreover, any two such Dubrovin valuation rings are conjugate in S, hence isomorphic (cf. Theorem A in Sect. 2). Thus, Dubrovin valuation rings appear to be very natural objects for studying the internal structure of division algebras and central simple algebras.

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While the theory of Dubrovin valuation rings is still quite new, these rings have already proved useful in working with the classical noncommutative valuation rings. For example, some of the difficult theorems in [JW] and [M1] on division algebras over Henselian fields have been generalized in [M2] to results about Dubrovin valuation rings, and these more general theorems have turned to have easier and more natural proofs. I would expect many more applications of Dubrovin valuation rings as the theory is developed further.

We will here prove some theorems on the structure of Dubrovin valuation rings, and use Henselization techniques to show that they are closely related to the classical valuation rings on division rings. Among other things, we define a value group for Dubrovin valuation rings, prove an "Ostrowski theorem" for such rings, and give various characterizations of Dubrovin valuation rings integral over their centers. Our main results are all stated in Sect. 2, with the proofs given in Sects. 3–5.

I wish to thank K. Mathiak and J. Gräter for introducing me to Dubrovin valuation rings. I thank Gräter also for showing me his results with Brungs which appear in [BG2] – these were what convinced me to begin investigating Dubrovin's rings. In addition, I would like to thank J.-P. Tignol for some enlightening conversations. The main results of this paper were announced in [W2].

For the rest of this section, we recall the basic properties of Dubrovin valuation rings proved by Dubrovin in [D1] and [D2], and give some examples of such rings. This will provide some introduction to these rings, which may be unfamiliar to many readers. The properties quoted here will be used extensively later in the paper.

A few words on notation: If B is any ring, we write

B^* for the group of units of B;

$Z(B)$ for the center of B;

J(B) for the Jacobson radical of B;

$M_k(B)$ for the $k \times k$ matrix ring over B.

If S is an algebra over a field F, then $[S:F]$ denotes the dimension of S over F. The term "field" always means a commutative field. When there is a clearly defined monomorphism f from one object into another, we will routinely identify the domain of f with its image. Thus, canonical injections become inclusions, and canonical isomorphisms are written as equalities. This will occur particularly frequently in comparing residue rings and value groups of different valuation rings.

Dubrovin's definition in [D1] of a noncommutative valuation ring is based on the idea of a place in the category of simple Artinian rings. Let S be simple Artinian. We call a subring B of S a *Dubrovin valuation ring* (*of* S) if

(i) B has an ideal I such that B/I is simple Artinian;

(ii) for each $s \in S-B$ there are $b_1, b_2 \in B$, such that $b_1 s, sb_2 \in B-I$.

(Dubrovin called such a ring a noncommutative valuation ring, and the terminology in [BG2] is S-valuation ring.) Dubrovin showed [D1, Sect. 1, Proposition 3] that the ideal I is actually the Jacobson radical $J(B)$. Furthermore [D1, Sect. 1, Theorem 4], B is a left and right order in S, hence a prime left and right Goldie ring. In addition, it is easy to check that $Z(B) = B \cap Z(S)$ and $Z(B)$ is a valuation ring of the field $Z(S)$. Dubrovin proved the following further significant properties of a Dubrovin valuation ring B (cf. [D1, Sect. 1, Theorems 4, 7; Sect. 2, Theorem 4] and [D2, Sect. 1, Proposition 2; Sect. 2, Theorem 1]):

(1.1) (Bézout) Every finitely generated left (resp. right) ideal of B is principal.

(1.2) (semihereditary) Every finitely generated left (resp. right) ideal of B is projective as a B-module.

(1.3) B has the "k-chain property": There is an integer $k>0$ such that for any $n>k$ and any $a_1, \ldots, a_n \in B$, the left ideal $Ba_1 + \ldots + Ba_n$ is generated by k of the a_i. Likewise for right ideals. (The smallest such k turns out to be the matrix size of $B/J(B)$.)

(1.4) The two-sided ideals of B are linearly ordered by inclusion, while the left ideals (resp. right ideals) are in general not linearly ordered. Likewise, the $B-B$ bimodules of S are linearly ordered.

(1.5) (Morita invariance) If $S \cong M_m(D)$, then $B \cong M_m(C)$, where C is a Dubrovin valuation ring of D. Furthermore, for any natural number k, $M_k(B)$ is a Dubrovin valuation ring of $M_k(S)$. If $e \in B$ is idempotent ($e \neq 0$), then eBe is a Dubrovin valuation ring of eSe.

(1.6) (composition of places) If T is a subring of B with $J(B) \subseteq T$, then T is a Dubrovin valuation ring of S iff $T/J(B)$ is a Dubrovin valuation ring of the simple Artinian ring $B/J(B)$. (This property is particularly useful for building examples of Dubrovin valuation rings.)

(1.7) (overrings) Let A be any overring of B, i.e., a ring with $B \subseteq A \subseteq S$. Then, A is a Dubrovin valuation ring of S, $J(A)$ is a prime ideal of B, A is the left (and right) localization of B with respect to the elements of B regular mod $J(A)$, and $B/J(A)$ is a Dubrovin valuation ring of $A/J(A)$.

(1.8) (localization) Suppose $[S:F] < \infty$, where $F = Z(S)$. Let $V = B \cap F$. For any prime ideal Q of B, set $P = Q \cap V$. Then P is a prime ideal of V, and the central localization B_P (of B as a V-algebra with respect to $V-P$) is an overring of B with $J(B_P) = Q$. Thus, (combining this with (1.7)) there is a one-to-one correspondence between prime ideals of B, prime ideals of V, and overrings of B. (Distinct prime ideals P of V yield distinct overrings of B, as $B_P \cap F = V_P$.)

Consider, by way of comparison, the valuation rings R that arise from valuations on S when S is a division ring. Such a subring R of S is characterized by the following properties (cf. [S, p. 12]):

(1.9) (i) For every $s \in S^*$, $s \in R$ or $s^{-1} \in R$.
(ii) For every $s \in S^*$, $sRs^{-1} = R$.

We will call a ring R satisfying properties (1.9) (i) and (ii) an *invariant valuation ring* of the division ring S. (The name comes from (ii) – invariance under inner automorphisms.) As is easy and well-known, for such an R the left ideals are the same as the right ideals and are linearly ordered, $J(R)$ is the unique maximal ideal of R, and the residue ring $R/J(R)$ is a division ring. Also the associated valuation on S is completely determined by R: The value group of R is

$$\Gamma_R = S^*/R^*, \tag{1.10}$$

which is made into a (totally) ordered abelian group by setting $s_1R^* \leqq s_2R^*$ iff $s_1R \supseteq s_2R$. Then the associated valuation is $v: S^* \to \Gamma_R$ given by $s \mapsto sR^*$.) Clearly, every invariant valuation ring is a Dubrovin valuation ring.

A Dubrovin valuation ring B of a simple Artinian ring S need not be invariant, but it has the following features in common with invariant valuation rings. First, B has a *residue ring*

$$\bar{B} = B/J(B), \tag{1.11}$$

which is simple Artinian, though not necessarily a division ring. Second, although there is in general no valuation associated to B, we can still define a value group, as follows: Set

$$\mathrm{st}(B) = \{s \in S^* : sBs^{-1} = B\}, \tag{1.12}$$

the stabilizer of B under the action of S^*. Indeed, $\mathrm{st}(B)$ coincides with the normalizer of B^* in S^*. Then define the *value group* Γ_B of B by

$$\Gamma_B = \mathrm{st}(B)/B^*. \tag{1.13}$$

The elements of Γ_B are in one-to-one correspondence with the fractional two-sided ideals of the form $sB = Bs$ for $s \in \mathrm{st}(B)$. Note that Γ_B is an ordered group with respect to the ordering given by $sB^* \leqq tB^*$ iff $sB \supseteq tB$ $(s, t \in \mathrm{st}(B))$. This is a total ordering, by property (1.4). Of course, if B is an invariant valuation ring of a division ring, then Γ_B coincides with the usual value group of B.

Besides invariant valuation rings, we note the following significant examples of Dubrovin valuation rings.

(1.14) *Example.* If V is any commutative valuation ring and A is any Azumaya algebra over V, then A is a Dubrovin valuation ring. (See (3.4) below or [D2, Sect. 2, Proposition 1]).

(1.15) *Example.* Let V be a discrete (rank 1) valuation ring of a field F, and let S be a simple algebra with $Z(S) = F$ and $[S:F] < \infty$. Let B be a subring of S with $B \cap F = V$. Then B is a Dubrovin valuation ring of S iff B is a maximal order of V in S.

Proof. Dubrovin proved in [D1, Sect. 1, Theorem 4] that B is a Dubrovin valuation ring of S iff $B/J(B)$ is simple Artinian, every finitely-generated left or right ideal of B is principal, and B is a left and right order of S. If B is a maximal order of V in S, then B has all of these properties by [Re, Theorem 18.7, p. 179]; so B is a Dubrovin valuation ring of S. Conversely, suppose B is Dubrovin, and $B \cap F = V$. Since $[S:F] < \infty$, B is a p.i.-ring. Because V is Noetherian, Formanek's theorem [F, Theorem 1] shows B is a finitely-generated V-module. In addition, $BK = S$, as B is prime p.i. (cf. [Rw, Theorem 1.7.9, p. 53]). Thus, B is an order of V in S, and the overring property (1.7) shows B is a maximal order. □

One may view Dubrovin valuation rings in finite dimensional central simple algebras as a reasonable generalization to arbitrary commutative valuation base rings of the notion of maximal orders over discrete valuation rings. The results stated in the next section are known for maximal orders over discrete valuation rings; it would be interesting to see how much further the rich theory of maximal orders can be extended to Dubrovin valuation rings.

2. Statement of Theorems

In this section we state the main theorems of this paper (the lettered theorems) and some of their corollaries. The theorems and corollaries will be proved in Sects. 3–5. The similarities in approach to several of the proofs and the intertwinings of the arguments dictate that the theorems be proved together rather than sequentially.

A crucial property of Dubrovin valuation rings is that they give essentially unique extensions of valuation rings of the center:

(2.1) **Theorem.** *Let S be a simple Artinian ring and let $F=Z(S)$, with $[S:F]<\infty$. If V is any valuation ring of F, then there is a Dubrovin valuation ring B of S with $B\cap F=V$.*

Theorem A. *With S, F as in Theorem 2.1, if B and B_0 are two Dubrovin valuation rings of S with $B\cap F=B_0\cap F$, then there is a $u\in S^*$ with $uBu^{-1}=B_0$.*

Theorem 2.1 was proved by Dubrovin in [D2, Sect. 3, Theorem 2], with a more understandable proof given in [BG2, Theorem 3.8]. Theorem A was proved for $B\cap F$ of finite rank by Brungs and Gräter in [BG2, Theorem 5.4]; the proof we give below will have no such restriction.

Our main theorem, Theorem B below, describes what happens to a Dubrovin valuation ring with passage to the Henselization of the valuation on the center. To place this in context, we recall the corresponding situation for invariant valuation rings. It has long been known (cf. [S, Theorem 9, p. 53]) that if V is a Henselian valuation ring of a field F, then inside any F-division algebra D with $[D:F]<\infty$ there is a unique invariant valuation ring R extending V (i.e., $R\cap F=V$). However, if V is not Henselian, then V has at most one extension to an invariant valuation ring of D, but possibly no extension at all (cf. [W1, Corollary] or [Er2, Corollary 1]). Now, every field with valuation ring (F, V) has an essentially unique Henselization (F^h, V^h). Recently, Morandi has shown that the Henselization can be used to determine whether V extends to D:

(2.2) **Theorem.** *Let D be a division ring, let $F=Z(D)$ with $[D:F]<\infty$, and let V be a valuation ring of F. Let (F^h, V^h) be the Henselization of (F, V). Then,*

(i) *V extends to an invariant valuation ring R of D iff $D\otimes_F F^h$ is a division ring.*

(ii) *Suppose $D\otimes_F F^h$ is a division ring. Let R^h be the (unique) extension of V^h to an invariant valuation ring of $D\otimes_F F^h$. Then $R=R^h\cap D$, $\Gamma_R=\Gamma_{R^h}$, and $\bar{R}=\overline{R^h}$.*

Theorem 2.2 is proved in [M1, Theorem 2]*. When $D\otimes_F F^h$ is not a division ring there is still a Dubrovin valuation ring B of D extending V; our main theorem provides an analogue to Theorem 2.2 for B.

Before stating Theorem B we must set up some more notation. For the rest of this section fix a Dubrovin valuation ring B of a simple Artinian ring S, and let $F=Z(S)$ and $V=B\cap F$, a valuation ring of F. We always assume that $[S:F]<\infty$. Let (F^h, V^h) be the Henselization of the valued field (F, V). In addition to the

* Part (i) of this theorem is asserted in [Er2], but Ershov's proof has a gap I do not know how to fill

terminology $\bar{B}$, st(B), Γ_B introduced in (1.11)–(1.13) we write

π_B for the natural projection: $B \to \bar{B} = B/J(B)$;

$t_B =$ matrix size of $\bar{B}$ (i.e., $\bar{B} \cong M_{t_B}(E)$, where E is a division ring);

$n_B =$ matrix size of $S \otimes_F F^h$;

$s_B = n_B/t_B$ (which we will see is always an integer).

For any F-algebra C, $\mathrm{Aut}_F C$ denotes the group of F-automorphisms of C. But if C is a field, we also write $\mathscr{G}(C/F)$ for the Galois group $\mathrm{Aut}_F C$.

With B as above, note that for any $s \in \mathrm{st}(B)$, conjugation by s is a V-automorphism of B, so it induces a $\bar{V}$-automorphism of $\bar{B}$. That is, there is a well-defined homomorphism

$$\varphi_B : \mathrm{st}(B) \to \mathrm{Aut}_{\bar{V}} \bar{B} \quad \text{given by} \quad \varphi_B(s)(\bar{b}) = \overline{sbs^{-1}},$$

where $\bar{b} = \pi_B(b)$, for $b \in B$. Every automorphism of $\bar{B}$ maps $Z(\bar{B})$ onto $Z(\bar{B})$. Further, if $s \in B^* \cdot F^*$ then $\varphi_B(s)$ is the identity on $Z(\bar{B})$. Thus, as $\mathrm{st}(B)/(B^* \cdot F^*) = \Gamma_B/\Gamma_V$, φ_B induces a homomorphism

$$\theta_B : \Gamma_B/\Gamma_V \to \mathscr{G}(Z(\bar{B})/\bar{V}).$$

For any V-algebra A and any prime ideal P of V, we write A_P for the localization of A at P; so $A_P = A \otimes_V V_P$.

We can now give our main theorem. Let B be a Dubrovin valuation ring and let S, F, V, F^h, V^h be as described above, with $[S:F] < \infty$, and write

$$S \otimes_F F^h \cong M_{n_B}(D^h),$$

where D^h is a division ring. Since V^h is Henselian, there is a unique invariant valuation ring R of D^h with $R \cap F^h = V^h$.

Theorem B. (i) $\bar{B} \cong M_{t_B}(\bar{R})$ *where* $\bar{R}(=R/J(R))$ *is a division ring.*

(ii) $\Gamma_B = \Gamma_R$.

(iii) *Using the isomorphism* $Z(\bar{B}) \cong Z(\bar{R})$ *induced by* (i) *above, we have a commutative diagram:*

$$\begin{array}{ccc} \Gamma_B/\Gamma_V & \xrightarrow{\;=\;} & \Gamma_R/\Gamma_V \\ {\scriptstyle \theta_B}\downarrow & & {\scriptstyle \theta_R}\downarrow \\ \mathscr{G}(Z(\bar{B})/\bar{V}) & \xrightarrow{\;\cong\;} & \mathscr{G}(Z(\bar{R})/\overline{V^h}) \end{array}.$$

Corollary B. *With the notation as above, the maps* θ_B *and* φ_B *are surjective and* $Z(\bar{B})$ *is a normal (but not necessarily separable) extension field of* $\bar{V}$ *of finite degree. If* $Z(\bar{B})$ *is separable over* $\bar{V}$, *then it is actually abelian Galois over* $\bar{V}$. Γ_B *is an abelian group and* Γ_B/Γ_V *is finite.*

Theorem B allows us to extend the "Ostrowski theorem" to Dubrovin valuation rings: For B, S, F, V as above, define the *defect* of B by

$$\delta(B) = [S:F]/([\bar{B}:\bar{V}]\,|\Gamma_B:\Gamma_V|\,(n_B/t_B)^2).$$

Theorem C. *For* B *and* R *as in Theorem B,* $\delta(B) = \delta(R)$. *Consequently,* $\delta(B) = 1$ *if* $\mathrm{char}(\bar{V}) = 0$, *and* $\delta(B) = p^a$ *for some integer* $a \geq 0$ *if* $\mathrm{char}(\bar{V}) = p \neq 0$.

Note that if B is an invariant valuation ring $n_B = t_B = 1$ (cf. Theorem 2.2), so Theorem C reduces to the Ostrowski theorem for such valuation rings proved for V Henselian by Draxl [Dr, Theorem 2] and in general by Morandi [M1, Theorem 3].

In order to prove Theorems A and B and also to clarify the relation between the integer invariants n_B and t_B we must consider other Henselizations – at the localizations of V. Let $\{P_i\}_{i\in I}$ be the (linearly ordered) set of nonzero prime ideals of V. For each $i\in I$, let (F_i, V_i) be the Henselization of (F, V_{P_i}), and let $S\otimes_F F_i \cong M_{m_i}(D_i)$, where D_i is a division ring. So each $m_i \leqq [S:F]$. Note that for $i, s\in I$, if $P_i \subsetneqq P_s$ then V_{P_s} is a refinement of V_{P_i}, so we may view $F_i \subseteq F_s$ (cf. the discussion of Henselization in the next section); hence $m_i | m_s$. Thus, if $m\in\{m_i : i\in I\}$ there is a prime ideal P_j maximal such that $m_j = m$. (For, let $I_m = \{i\in I : m_i = m\}$. Then set $P_j = \bigcup_{i\in I_m} P_i$, which is a prime ideal as the P_i are linearly ordered. Since F_j is the direct limit of the F_i, $i\in I_m$, we have $m_j = m$.) We call such a P_j a *jump prime ideal* of V with respect to S.

So, P_j is a jump prime ideal iff for each $P_s \supsetneqq P_j$, $m_s > m_j$. We write $j(V, S)$ for the number of jump prime ideals, and call this the *jump rank* of V (re S). The jump rank is a convenient invariant for induction arguments, since it is always finite, even when the usual rank (= Krull dimension) of V is infinite. Since the prime ideals of our Dubrovin valuation ring B correspond to the primes of V, we say Q is a jump prime ideal of B if $Q\cap V$ is a jump prime ideal of V re S.

Now, let A be any ring with $B\subseteq A\subseteq S$. Then we know (cf. (1.7)) that A is a Dubrovin valuation ring of S and is a localization of B. Let $W = A\cap F$, a localization of V. We set $\tilde{V} = V/J(W)$, which is a valuation ring of the field $\bar{W} = W/J(W)$. Now, $Z(\bar{A})$ is a field extension of $\bar{W}$ of finite degree (see below). Set

(2.3) $\quad \ell_{B,A}$ = the number of extensions of $\tilde{V}$ to valuation rings of $Z(\bar{A})$.

Theorem D. *With B, S, F as above, let $Q_1 \subsetneqq Q_2 \subsetneqq \dots \subsetneqq Q_k$ be a finite set of prime ideals of V which includes all the jump primes of V re$S$. Set $\ell_i = \ell_{B_{Q_i}, B_{Q_{i-1}}}$. Then,*

$$n_B = t_B \ell_2 \ell_3 \dots \ell_k.$$

The next theorem and its corollary are crucial for the inductive proofs of Theorems A, B, and D.

Theorem E. *With B, S, F as above, let A be a ring with $B\subseteq A\subseteq S$, and let $W = A\cap F$. Set $\tilde{B} = B/J(A)$, which is a Dubrovin valuation ring of $\bar{A}$. Then,*

(i) *There is an exact sequence*

$$(\Gamma_{B,A}) \qquad 0\to \Gamma_{\tilde{B}} \to \Gamma_B \to \Gamma_A \to \mathcal{G}(Z(\bar{A})/\bar{W})/H \to 0,$$

where $H = \{\tau\in\mathcal{G}(Z(\bar{A})/\bar{W}) : \tau(\tilde{B}\cap Z(\bar{A})) = \tilde{B}\cap Z(\bar{A})\}$.

(ii) $t_B = t_{\tilde{B}} \geqq t_A$.

(iii) $n_B = n_{\tilde{B}}(n_A/t_A)\ell_{B,A}$, *i.e.*, $s_B = s_{\tilde{B}} s_A \ell_{B,A}$.

Corollary E. *With $B, S, F, A, \tilde{B}$ as in Theorem E, let Q be a prime ideal of B with $Q\supseteq J(A)$. If $Q/J(A)$ is a jump prime ideal of $\tilde{B}$, then Q is a jump prime ideal of B.*

The converse to Corollary E is not true in general: One can construct examples in which Q is a jump prime ideal of B even though $Q/J(A)$ is not a jump prime ideal of $\tilde{B}$.

The invariant t_B of a Dubrovin valuation ring B is bounded above by n_B and below by the matrix size of S. The rings achieving the extreme values of t_B are particularly interesting:

Theorem F. *The following are equivalent:*

(i) $t_B = n_B$.

(ii) *B is integral over V.*

(iii) *Every principal two-sided ideal of B is principal as a left ideal and as a right ideal of B.*

(iv) *Every two-sided ideal of B is generated by elements of* $\mathrm{st}(B)$.

(v) *For all rings A, E with $B \subseteqq E \subsetneqq A \subseteqq S$, $\ell_{E,A} = 1$.*

(vi) *$B \otimes_V V^h$ is a Dubrovin valuation ring of $S \otimes_F F^h$.*

(vii) *There is a Dubrovin valuation ring B^h of $S \otimes_F F^h$ with $B^h \cap S = B$ and $B^h \cap F^h = V^h$.*

(Yet another condition, (vi′), equivalent to those in Theorem F will be given at the beginning of Sect. 4.)

The equivalence of conditions (iii) and (iv) to the others in Theorem F is due to Morandi [M2], who has also found further interesting properties and characterizations of Dubrovin valuation rings integral over their centers. Note that condition (iv) says that the two-sided ideals of B are classified by the value group of B, just as for a commutative valuation ring. So this holds iff B is integral over $Z(B)$. Observe also that whenever V has rank 1 the conditions of Theorem F all hold, as Theorem D shows $t_B = n_B$.

Theorem G. *Suppose (in addition to the standing hypotheses) that S is a division ring. Then the following are equivalent:*

(i) $t_B = 1$.

(ii) *For each $s \in S^*$, $s \in B$ or $s^{-1} \in B$.*

(iii) *B has only finitely many different conjugates in S.*

(iv) *The set T of elements of S integral over V is a ring. $\left(\text{In fact, } T = \bigcap_{s \in S^*} sBs^{-1}.\right)$*

When these equivalent conditions hold, the number of conjugates of B is exactly n_B.

The rings described in Theorem G, which satisfy condition (i) of (1.9) but not (ii), have been studied extensively by Mathiak, Gräter, and others (cf. [Ma]). We call them *total valuation rings.* Mathiak has defined a value group for a total valuation ring B as $S^* \Big/ \bigcap_{s \in S^*} sB^*s^{-1}$. Gräter has proved [G, Theorem 3.4] that our value group Γ_B is isomorphic to the center of Mathiak's value group. Of the conditions in Theorem G, (ii)⇒(iv) and (ii)⇒(iii) were proved by Brungs and Gräter in [BG1, Theorems 1, 3], who also showed in [BG1, Theorem 1] that the number of conjugates of B satisfying (ii) is bounded by $\sqrt{[S:F]}$. The easy equivalence (i)⇔(ii) appears in [BG2, Lemma 2.2].

The following corollary is immediate from Theorems 2.2, F, and G:

Corollary G. *Suppose (in addition to the standing hypotheses) that S is a division ring. Then the following are equivalent:*

(i) $n_B = 1$.

(ii) *B is an invariant valuation ring.*

(iii) *B is a total valuation ring and B is integral over V.*

3. Preliminaries

In this section we give preliminary results to prepare for the proofs of the theorems stated in Sect. 2. We prove some special cases of these theorems, in preparation for the main argument, which begins in Sect. 4.

The following notation will be fixed throughout this section: B is a Dubrovin valuation ring of a simple Artinian ring S, $F = Z(S)$, and $V = B \cap F$. We adopt as a standing hypothesis that $[S:F] < \infty$. Note that (F, V) is a valued field, i.e., F is the quotient field of the valuation ring V. We will write $(F, V) \subseteq (F', V')$ if (F', V') is another valued field, with $F \subseteq F'$ and $V' \cap F = V$.

We begin with some general lemmas that prepare the way for a useful result (Proposition 3.3) on extending Dubrovin valuation rings.

(3.1) **Lemma.** *Suppose (K, Y) is a valued field, T is a K-algebra (containing K), and C is a subring of T with $C \cap K = Y$. Let $N \subseteq M$ be Y-modules. Then,*

(a) *M is a flat Y-module iff M is torsion-free.*

(b) *$N \otimes_Y C \subseteq M \otimes_Y C$, $M \subseteq M \otimes_Y C$, and $(N \otimes_Y C) \cap M = N$.*

Proof. (a) This is well-known, and is actually true if Y is an invariant valuation ring of a division ring K (with $[K:Z(K)] < \infty$) – cf. [JW, Sect. 2].

(b) Since C is a torsion-free, hence flat Y-module we may identify $N \otimes_Y C$ with its image in $M \otimes_Y C$. We have the exact sequence

$$\operatorname{Tor}_1^Y(M, C/Y) \to M \otimes_Y Y \to M \otimes_Y C.$$

Since $C \cap K = Y$, C/Y is a torsion-free, so flat, Y-module; hence $\operatorname{Tor}_1^Y(M, C/Y) = 0$. This shows the map $M = M \otimes_Y Y \to M \otimes_Y C$ is injective. Now set $N_1 = (N \otimes_Y C) \cap M \supseteq N$. Then N_1/N is the kernel of

$$M/N = (M/N) \otimes_Y Y \to (M/N) \otimes_Y C = (M \otimes_Y C)/(N \otimes_Y C).$$

Since $\operatorname{Tor}_1^Y(M/N, C/Y) = 0$, we have $N_1 = N$, as desired. □

(3.2) **Lemma.** *For any Dubrovin valuation ring B,* $\operatorname{st}(B) \cap (B - J(B)) = B^*$.

Proof. The inclusion $\supseteq$ is clear. For the reverse inclusion, take $s \in \operatorname{st}(B) \cap (B - J(B))$. Then $BsB = B$ as the ideals of B are linearly ordered and $B/J(B)$ is simple. But $BsB = Bs = sB$ as $s \in \operatorname{st}(B)$. Hence, $s \in B^*$. □

Let B be our Dubrovin valuation ring of S, and let B' be a Dubrovin valuation ring of a simple Artinian ring S', where $S \subseteq S'$. We say that B' is a *compatible extension* of B if $B' \cap S = B$, $J(B) \subseteq J(B')$, and $\operatorname{st}(B) \subseteq \operatorname{st}(B')$. When this occurs, $J(B') \cap B = J(B)$, as $J(B)$ is the maximal ideal of B, and we view $\bar{B} \subseteq \bar{B}'$ via the natural inclusion. Furthermore, clearly $\mathrm{st}(B')\cap S=\mathrm{st}(B)$ and $B'^{*}\cap\mathrm{st}(B)=B^{*}$; thus, the obvious homomorphism $\Gamma_B\to\Gamma_{B'}$ is injective and order-preserving, and we view $\Gamma_B\subseteq\Gamma_{B'}$. We say that B' is an *immediate compatible extension* of B if B' is a compatible extension of B with $\bar{B}'=\bar{B}$ and $\Gamma_{B'}=\Gamma_B$.

(3.3) **Proposition.** *With B, S, F, V as above, let K be a field with $K\subseteq F$ and $[F:K]<\infty$, and let $Y=V\cap K$. Let T be a simple K algebra with $[T:K]<\infty$. Suppose T contains a ring $C\supseteq Y$ which is a free Y-module of rank $[T:K]$. Set $\bar{C}=C/J(Y)C$. Suppose $S\otimes_K T$ and $\bar{B}\otimes_{\bar{Y}}\bar{C}$ are simple rings. Then $B\otimes_Y C$ is a Dubrovin valuation ring of $S\otimes_K T$ and is a compatible extension of B. Moreover, $\overline{B\otimes_Y C}=\bar{B}\otimes_{\bar{Y}}\bar{C}$ and $\Gamma_{B\otimes_Y C}=\Gamma_B$.*

Proof. Let $B'=B\otimes_Y C$ and $S'=S\otimes_K T=S\otimes_Y C$. We have $B'\subseteq S'$. Let $\{\alpha_1,\dots,\alpha_n\}$ be a base of C as a free Y-module. Then $\{\alpha_1,\dots,\alpha_n\}$ is also an F-base of T. Take any $\gamma\in S'$. Then γ has a unique representation

$$\gamma=\sum s_i\otimes\alpha_i$$

with $s_i\in S$; note that $\gamma\in B'$ iff each $s_i\in B$. Let $\varrho: B\otimes_Y C\to\bar{B}\otimes_{\bar{Y}}\bar{C}$ be the canonical epimorphism, mapping $\sum s_i\otimes\alpha_i$ to $\sum\bar{s}_i\otimes\bar{\alpha}_i$ for $s_i\in B$. Let $J=\ker(\varrho)$. Since $\bar{\alpha}_1,\dots,\bar{\alpha}_n$ is a $\bar{Y}$-base of $\bar{C}$, $\gamma\in J$ iff each $\bar{s}_i=0$, iff each $s_i\in J(B)$. By hypothesis B'/J is simple, and is Artinian since it is a finitely-generated $\bar{B}$-module.

To verify that B' is Dubrovin, take any $\gamma=\sum s_i\otimes\alpha_i\in S'-B'$. That is, some $s_i\in S-B$. Now, as B is Dubrovin, $\sum s_iB=rB$ for some $r\in S-R$ (cf. (1.1)). Write $s_i=rb_i$ with $b_i\in B$ and $r=\sum s_ic_i$ with $c_i\in B$. We have

$$\gamma=\sum s_i\otimes\alpha_i=(r\otimes 1)\left(\sum b_i\otimes\alpha_i\right).$$

Since B is Dubrovin there is a $d\in R$, such that $dr\in B-J(B)$. Then $d\otimes 1\in B'$ and $(d\otimes 1)\gamma=\sum drb_i\otimes\alpha_i$. Note that not all $drb_i\in J(B)$, as otherwise $dr=\sum drb_ic_i\in J(B)$, contradicting the choice of d. Hence, $(d\otimes 1)\gamma\in B'-J$. By a symmetric argument there is a $d'\otimes a\in B'$ with $\gamma(d'\otimes 1)\in B'-J$. Thus, B' is a Dubrovin valuation ring with $J(B')=J$ and $\bar{B}'\cong\mathrm{im}(\varrho)=\bar{B}\otimes_{\bar{Y}}\bar{C}$.

Note that $C\cap K=Y$ as C is integral over Y. Hence by (3.1b), $B'\cap S=B$. Since the inclusions $J(B)\subseteq J(B')$ and $\mathrm{st}(B)\subseteq\mathrm{st}(B')$ are clear, B' is a compatible extension of B.

To show $\Gamma_{B'}=\Gamma_B$ it suffices to verify $\mathrm{st}(B')=\mathrm{st}(B)\cdot B'^{*}$. For this, take any $\gamma\in\mathrm{st}(B')$ and write $\gamma=(r\otimes 1)\left(\sum b_i\otimes\alpha_i\right)$ just as before. Then each $b_i\in B$. Suppose every $b_i\in J(B)$. Then as $r=\sum rb_ic_i$ with $c_i\in B$, we have $0=r(1-\sum b_ic_i)$. Since $1-\sum b_ic_i\in 1+J(B)\subseteq B^{*}$, we have $r=0$, so $\gamma=0$, a contradiction. Hence, some $b_i\notin J(B)$; thus we have expressed $\gamma=(r\otimes 1)\varepsilon$, where $\varepsilon=\sum b_i\otimes\alpha_i\in B'-J(B')$, and $r\in S^{*}$ as $\gamma\in S'^{*}$. Take any $b\in B$. Then $(b\otimes 1)\varepsilon\in B'$, so B' contains

$$\gamma(b\otimes 1)\varepsilon\gamma^{-1}=(r\otimes 1)\varepsilon(b\otimes 1)(r^{-1}\otimes 1)=\sum rb_ibr^{-1}\otimes\alpha_i.$$

So, $rb_ibr^{-1}\in B$, for all $b\in B$ and all i. Hence,

$$B\supseteq\sum rb_iBr^{-1}=\left(\sum s_iB\right)r^{-1}=rBr^{-1}.$$

Because $rBr^{-1}\cap F=B\cap F$, this inclusion implies by (1.8) that $rBr^{-1}=B$. Hence, $r\in\mathrm{st}(B)$. Thus, $\varepsilon=(r\otimes 1)^{-1}\gamma\in\mathrm{st}(B')\cap(B'-J(B'))$; by Lemma 3.2, $\varepsilon\in B'^{*}$. Therefore, $\gamma=(r\otimes 1)\varepsilon\in\mathrm{st}(B)\cdot B'^{*}$, as desired. □

The first two corollaries are known ([D2, Sect. 2, Proposition 1] and (1.5) above) except for the value group formulas.

(3.4) **Corollary.** *Let C be any Azumaya algebra over any commutative valuation ring V. Then C is a Dubrovin valuation ring of its ring of quotients and $\Gamma_C = \Gamma_V$.*

Proof. Proposition 3.3 applies with $B=V$ and $K=F$. □

(3.5) **Corollary.** *For a Dubrovin valuation ring B and any n, $M_n(B)$ is a Dubrovin valuation ring with $\overline{M_n(B)} = M_n(\bar{B})$ and $\Gamma_{M_n(B)} = \Gamma_B$.*

Proof. Apply. Proposition 3.3 with $C=M_n(V)$. □

(3.6) **Corollary.** *With the notation defined before* (3.1) *let K be a subfield of F with $[F:K]<\infty$, and let $Y=V\cap K$. Let $(K,Y)\subseteq(N,U)$ with N algebraic over K, and N and F linearly disjoint over K. Suppose,*

(i) *for each field L with $K\subseteq L\subseteq N$ and $[L:K]<\infty$, $[\overline{U\cap L}:\bar{Y}]=[L:K]$;*

(ii) *$Z(\bar{B})$ and $\bar{U}$ are linearly disjoint over $\bar{Y}$.*

Then $B\otimes_Y U$ is a Dubrovin valuation ring of $S\otimes_K N$ which is a compatible extension of B, with $\overline{B\otimes_Y U}=\bar{B}\otimes_{\bar{Y}}\bar{U}$ and $\Gamma_{B\otimes_Y U}=\Gamma_B$.

Proof. Take any field L with $K\subseteq L\subseteq N$ and $[L:K]<\infty$, and let $C=U\cap L$. Since C has maximal residue degree over Y, C is a free Y-module and the ramification index must be 1; hence $J(C)=J(Y)\cdot C$. Consequently, $C/J(Y)\cdot C$ is the field $\bar{C}=C/J(C)$. Hypothesis (ii) assures that $Z(\bar{B})$ and $\bar{C}$ are linearly disjoint over $\bar{Y}$, so $Z(\bar{B})\otimes_{\bar{Y}}\bar{C}$ is a field and $\bar{B}\otimes_{\bar{Y}}\bar{U}\cong\bar{B}\otimes_{Z(\bar{B})}(Z(\bar{B})\otimes_{\bar{Y}}\bar{C})$ is simple. Likewise, $S\otimes_K L$ is simple, as L and $F=Z(S)$ are linearly disjoint over K. Thus, Proposition 3.3 applies to $B\otimes_Y C$. The desired properties of $B\otimes_Y U$ follow by easy direct limit arguments, as U is the direct limit of the C's. □

For arbitrary valued fields (F,V) the Henselization of V plays much the same rôle as the completion plays for rank 1 valuation rings. We recall now the facts we need about Henselization. (For background on Henselian valuation rings, see e.g. [R1], [R2], or [E].)

A *Henselization* of a valued field (F,V) is a valued field (F^h,V^h) such that $(F,V)\subseteq(F^h,V^h)$, V^h is Henselian, and for any Henselian valued field (F',V') with $(F,V)\subseteq(F',V')$ there is a unique F-homomorphism $f:F^h\to F'$ such that $f^{-1}(V')=V^h$. Every valued field (F,V) has a Henselization which is unique up to F-isomorphism (cf. [E, p. 131]). We use the notation $(F^h,V^h)=(F,V)^H$ to indicate that (F^h,V^h) is a Henselization of (F,V). It is known [E, (17.11), (17.19)] that F^h is separable over F and V^h is an immediate extension of V, i.e., $\overline{V^h}=\bar{V}$ and $\Gamma_{V^h}=\Gamma_V$.

Now, supposee $(F^h,V^h)=(F,V)^H$, and let K be an algebraic extension field of F and Y an extension of V to K. Then (cf. [E, (17.16), (17.13)]),

(3.7) *there is a compositum $K\cdot F^h$ of K and F^h over F such that $(K\cdot F^h,Y')=(K,Y)^H$, where Y' is the unique extension of V^h to $K\cdot F^h$. Moreover, there is a 1–1 correspondence between the extensions of V to K and the equivalence classes of composites of K and F^h over F.*

(The references mentioned in [E] cover only K/F separable in (3.7). But the generalization to K/F algebraic is easy.) Two extreme cases of (3.7) will arise frequently: As F^h is separable over F, K is linearly disjoint to F^h over F iff V has a unique extension to K. On the other hand, if $[K:F]<\infty$, and V has $[K:F]$ different extensions to K, then there is an embedding $(K, Y)\subseteq(F^h, V^h)$. For, the composites of K over F are the indecomposable summands of $K\otimes_F F^h$. In this case there are $[K:F]$ summands, so each is F^h.

We will exploit heavily the relationship between Henselizations of comparable valuation rings of the same field. Let (F, V) be a valued field, let $P\neq(0)$ be a prime ideal of V, and let $W=V_P$. Let $(F_1, W_1)=(F, W)^H$ and $(F_2, V_2)=(F, V)^H$. Let $W_2=V_{2P}$, i.e., the localization of V_2 at P. This W_2 is the valuation ring of F_2 with $V_2\subseteq W_2$ and $W_2\cap F=W$. Since V_2 is Henselian, so is W_2 (cf. [R1, p. 210, Proposition 9]). Therefore, by the universal mapping property for the Henselization, we may view $(F_1, W_1)\subseteq(F_2, W_2)$. Let $V_1=V_2\cap F_1\subseteq W_1$; let $\tilde{V}=V/J(W)$, a valuation ring of $\bar{W}=W/J(W)$, and likewise let $\tilde{V}_i=V_i/J(W_i)\subseteq\bar{W}_i$, $i=1,2$. Observe that when we identify $\bar{W}_1=\bar{W}$, then $\tilde{V}_1=\tilde{V}$. Now view (F_2, W_2) as an extension of (F_1, W_1). The key fact we will use is:

(3.8) *(F_2, W_2) is the (unique) unramified extension of (F_1, W_1) such that $(\bar{W}_2, \tilde{V}_2)=(\bar{W}_1, \tilde{V}_1)^H$.*

This can be verified using the universal mapping property for Henselization (as in the proof of [M1, Theorem 2]) or by an argument using decomposition groups. That (F_2, W_2) is unramified over (F_1, W_1) means $\bar{W}_2$ is separable over $\bar{W}_1$ and that for each field K, $F_1\subseteq K\subseteq F_2$ with $[K:F_1]<\infty$, $[\overline{W_2\cap K}:\bar{W}_1]=[K:F_1]$.

Certain special cases of the theorems and corollaries in Sect. 2 are needed in order to prove the theorems in general. We now give those partial results:

(3.9) **Proposition.** (a) *If Theorem B holds for a Dubrovin valuation ring B, then Corollary B and Theorem C hold for B.*

(b) *Theorem E (i) holds for the ring B if Theorem B holds for the ring A and Theorem A holds for $\tilde{B}$ in $\bar{A}$.*

(c) *Theorem E (ii) holds.*

(d) *Theorem E (iii) holds for the ring B if Theorem B holds for the ring A.*

(e) *Corollary E holds for the ring B if Theorem B holds for the ring A.*

(f) *Implications* (vi)$\Rightarrow$(vii) *and* (vii)$\Rightarrow$(ii) *of Theorem F hold for any Dubrovin valuation ring B.*

Before proving Proposition 3.9, we recall a result we need of Brungs and Gräter [BG2, Lemma 4.1]. (This is a special case of Theorem A).

(3.10) **Lemma.** *Let D be a division ring with $[D:Z(D)]<\infty$, and let B be a Dubrovin valuation ring of D, and R an invariant valuation ring of D. If $B\cap Z(D)=R\cap Z(D)$, then $B=R$.*

Proof of Proposition 3.9. (a) The properties in Corollary B are all known for any invariant valuation ring of a finite-dimensional division algebra (cf. [JW, Proposition 1.7], or for the Henselian case, see [DK, p. 96] or [Er1, Proposition 1]). Since these properties hold for the R in Theorem B, the theorem assures that they also hold for B. (Note that the surjectivity of θ_B implies the surjectivity of φ_B, since the Skolem-Noether Theorem shows $\mathrm{Aut}_{Z(\bar{B})}\bar{B}\subseteq\mathrm{im}(\varphi_B)$.)

The fact that Γ_B is abelian also follows from Dubrovin's analysis of the ideals in a Dubrovin valuation ring [D2, Sect. 2, Proposition 4]. As for Theorem C, assuming Theorem B we have

$$\begin{aligned}\delta(B) &= [S:F]/([\bar{B}:\bar{V}]\,|\Gamma_B:\Gamma_V|\,(n_B/t_B)^2)\\ &= n_B^2[D^h:F^h]/(t_B^2[\bar{R}:\overline{V^h}]\,|\Gamma_R:\Gamma_{V^h}|\,(n_B/t_B)^2)\\ &= \delta(R),\end{aligned}$$

as $n_R=t_R=1$. The Ostrowski theorem holds for $\delta(B)$ since it holds for $\delta(R)$ by [Dr, Theorem 2].

(b) We have Dubrovin valuation rings B, A with $B\subseteq A\subseteq S$ and $\tilde{B}\subseteq\bar{A}$. The assumptions are that Theorem B holds for A and that any Dubrovin valuation ring of $\bar{A}$ contracting to $\tilde{B}\cap Z(\bar{A})$ in $Z(\bar{A})$ is conjugate to $\tilde{B}$. Note that Corollary B holds for A by (a) above. The maps φ_A, π_A, θ_A are the ones defined in Sect. 2 (but now for A). Let $W=A\cap F$, and let $Y=\tilde{B}\cap Z(\bar{A})$, a valuation ring of $Z(\bar{A})$.

Observe that $\mathrm{st}(B)\subseteq\mathrm{st}(A)$ as A is a central localization of B by (1.8). Note that if $s\in\mathrm{st}(A)$, then $s\in\mathrm{st}(B)$ iff $\varphi_A(s)\,(\tilde{B})=\tilde{B}$. This holds since $B=\pi_A^{-1}(\tilde{B})$. The inclusion $\mathrm{st}(B)\hookrightarrow\mathrm{st}(A)$ induces a homomorphism $\alpha:\Gamma_B\to\Gamma_A$ with image $A^*\cdot\mathrm{st}(B)/A^*$. We claim that

$$A^*\cdot\mathrm{st}(B)=\{s\in\mathrm{st}(A):\varphi_A(s)\,(Y)=Y\}\,.$$

For, the inclusion $\subseteq$ is clear. To see $\supseteq$ take any $s\in\mathrm{st}(A)$ such that $\varphi_A(s)\,(Y)=Y$. By Theorem A, $\tilde{B}$ and $\varphi_A(s)\,(\tilde{B})$ are conjugate in $\bar{A}$, so there is an $a\in A^*$ with $\varphi_A(a)\,(\tilde{B})=\varphi_A(s)\,(\tilde{B})$. Then $a^{-1}s\in\mathrm{st}(B)$ as $\varphi_A(a^{-1}s)\,(\tilde{B})=\tilde{B}$. Hence, $s=a(a^{-1}s)\in A^*\cdot\mathrm{st}(B)$, establishing the claim.

Now, the map $\theta_A:\Gamma_A/\Gamma_W\to\mathcal{G}(Z(\bar{A})/\bar{W})$ is surjective by Corollary B. The claim just proved shows the image of $A^*\cdot\mathrm{st}(B)$ in Γ_A/Γ_W is $\theta_A^{-1}(H)$, where $H=\{\tau\in\mathcal{G}(Z(\bar{A})/\bar{W}):\tau(Y)=Y\}$. This yields the exactness of the diagram of Theorem E (i) at Γ_A and at $\mathcal{G}(Z(\bar{A})/\bar{W})/H$.

The kernel of α is $(\mathrm{st}(B)\cap A^*)/B^*$. Now, the restriction of π_A to A^* is a group epimorphism: $A^*\to\bar{A}^*$ with kernel $1+J(A)\subseteq 1+J(B)\subseteq B^*$. Clearly $\pi_A(\mathrm{st}(B)\cap A^*)=\mathrm{st}(\tilde{B})$ and $\pi_A(B^*)=\tilde{B}^*$. Hence, via π_A, $\ker(\alpha)\cong\mathrm{st}(\tilde{B})/\tilde{B}^*=\Gamma_{\tilde{B}}$, showing the diagram is exact at Γ_B and $\Gamma_{\tilde{B}}$. This proves Theorem E (i).

(c) With B, A, $\tilde{B}$ as in Theorem E, note that $t_B=t_{\tilde{B}}$ as $\bar{\tilde{B}}=\bar{B}$. Also, $t_{\tilde{B}}\geqq t_A$ because $\tilde{B}$ is $t_A\times t_A$ matrices over some Dubrovin valuation ring, by (1.5) applied to $\tilde{B}$ in $\bar{A}$.

(d) With B, A, $\tilde{B}$, W as in Theorem E, let $\tilde{V}=V/J(W)=\tilde{B}\cap\bar{W}$, a valuation ring of $\bar{W}$. Let $Z(\bar{A})_{\mathrm{sep}}$ denote the separable closure of $\bar{W}$ in $Z(\bar{A})$, which by Corollary B is abelian Galois over $\bar{W}$. (Corollary B holds for A by (a) above since we are assuming Theorem B holds for A.) Let $L'\subseteq Z(\bar{A})_{\mathrm{sep}}$ be the decomposition field of $\tilde{B}\cap Z(\bar{A})_{\mathrm{sep}}$ over $\tilde{V}$, i.e., the fixed field of

$$\{\tau\in\mathcal{G}(Z(\bar{A})_{\mathrm{sep}}/\bar{W}):\tau(\tilde{B}\cap Z(\bar{A})_{\mathrm{sep}})=\tilde{B}\cap Z(\bar{A})_{\mathrm{sep}}\}\,.$$

Because $Z(\bar{A})$ is normal over $\bar{W}$,

$$\begin{aligned}[L':\bar{W}] &= \text{the number of extensions of } \tilde{V} \text{ to } Z(\bar{A})_{\mathrm{sep}}\\ &= \text{the number of extensions of } \tilde{V} \text{ to } Z(\bar{A})\\ &= \ell_{B,A}\,.\end{aligned}$$

Because $Z(\bar{A})$ and L' are normal over $\bar{W}$ and $\tilde{B}\cap L'$ extends uniquely to $Z(\bar{A})$, each extension of $\tilde{V}$ to L' extends uniquely to $Z(\bar{A})$. So, there are $\ell_{B,A}$ extensions of $\tilde{V}$ to L'. Set $\ell=\ell_{B,A}$.

Let $(F_1, W_1)=(F, W)^H$, and write $S\otimes_F F_1\cong M_{n_A}(D_1)$, where D_1 is an F_1-central division ring. After identifying $\bar{W}=\bar{W_1}$, let $V_1=\pi_{W_1}^{-1}(\tilde{V})$, a valuation ring of F_1 with $V_1\cap F=V$. Let A_1 be the invariant valuation ring of D_1 such that $A_1\cap F_1=W_1$, which exists as W_1 is Henselian. By Theorem B for A, $\bar{A}\cong M_{t_A}(\bar{A_1})$. We identify $Z(\bar{A_1})$ with $Z(\bar{A})$.

Because W_1 is Henselian there is an "inertial lift" L of L' in D_1 (cf. [JW, proof of Theorem 2.9]); that is, L is a field, $F_1\subseteq L\subseteq D_1$, with L separable over F_1, such that $[L:F_1]=[L':\bar{W}]=\ell$, and, setting $W_L=A_1\cap L$, $\bar{W_L}=L'$ in $Z(\bar{A_1})$. Let $D_L=C_{D_1}(L)$, the centralizer of L in D_1, and let $A_L=A_1\cap D_L$, an invariant valuation ring of D_L. Since $\bar{W_L}$ lies in $Z(\bar{A_1})$ and is separable over $\bar{W_1}$, [JW, Lemma 1.8(a)] shows that $\bar{A_L}=\bar{A_1}$. Let $\tilde{V_L}=\tilde{B}\cap\bar{W_L}$, and let $V_L=\pi_{W_L}^{-1}(\tilde{V_L})$, a valuation ring of L with $V_L\cap F_1=V_1$.

Let $(F_2, V_2)=(F, V)^H$ and let $W_2=V_{2J(W)}$ (the localization of V_2 such that $W_2\cap F=W$). As noted before (3.8) we may view $(F_1, W_1)\subseteq(F_2, W_2)$ with $V_2\cap F_1=V_1$. Let $\tilde{V_i}=V_i/J(W_i)$, $i=1,2$. Then (3.8) says (F_2, W_2) is the unramified extension of (F_1, V_1) such that $(\bar{W_2}, \tilde{V_2})=(\bar{W_1}, \tilde{V_1})^H$. Because $\tilde{V_1}$ has $[L:F_1]$ extensions to $\bar{W_L}=L'$, V_1 has $[L:F_1]$ extensions to L. So, by (3.7), as $(F_2, V_2)=(F_1, V_1)^H$, we may view $(L, V_L)\subseteq(F_2, V_2)$.

Set $A_3=A_L\otimes_{W_L}W_2$, a subring of $S_3=D_L\otimes_L F_2$. To show A_3 is a Dubrovin valuation ring of S_3, we invoke Corollary 3.6 with $B=A_L$ (which is invariant, hence Dubrovin), $K=F=L$, $Y=W_L$, and $(N, U)=(F_2, W_2)$. Condition (i) of (3.6) holds as (F_2, W_2) is unramified over (L, W_L). Also $(\bar{W_2}, \tilde{V_2})=(\bar{W_L}, \tilde{V_L})^H$, since

$$(\bar{W_1}, \tilde{V_1})\subseteq(\bar{W_L}, \tilde{V_L})\subseteq(\bar{W_2}, \tilde{V_2})=(\bar{W_1}, \tilde{V_1})^H.$$

Because V_L extends uniquely from $\bar{W_L}$ to $Z(\bar{A_L})$ (recalling $\bar{W_L}=L'$ and $Z(\bar{A_L})=Z(\bar{A_1})=Z(\bar{A})$), by (3.7) $Z(\bar{A_L})$ and $\bar{W_2}$ are linearly disjoint over $\bar{W_L}$. Thus, by Corollary 3.6 A_3 is a Dubrovin valuation ring of S_3, with $\bar{A_3}=\bar{A_L}\otimes_{\bar{W_L}}\bar{W_2}$.

Let $S_3\cong M_k(D_3)$, where D_3 is a division ring with $Z(D_3)=Z(S_3)=F_2$. Then by the Morita property (1.5) $A_3\cong M_k(A'_3)$, where A'_3 is a Dubrovin valuation ring of D_3 with $A'_3\cap F_2=A_3\cap F_2=W_2$. Because W_2 is Henselian, there is an invariant valuation ring of D_3 extending W_2; by (3.10) that invariant ring is A'_3. Hence $\bar{A'_3}$ is a division ring, showing that k is the matrix size of $\bar{A_3}$.

To compute $n_{\tilde{B}}$ we need a Henselization of $Z(\bar{A})$ with respect to $\tilde{B}\cap Z(\bar{A})$. Since $(\tilde{B}\cap Z(\bar{A}))\cap L'=\tilde{V_L}$ and $(L', \tilde{V_L})^H=(\bar{W_2}, \tilde{V_2})$, by (3.7) the field $Z(\bar{A})\otimes_{\bar{W_L}}\bar{W_2}$ is the desired Henselization. Because $\bar{A}\cong M_{t_A}(\bar{A_1})$,

$$\begin{aligned} n_{\tilde{B}} &= t_A\cdot \text{matrix size of } \bar{A_1}\otimes_{Z(\bar{A})}(Z(\bar{A})\otimes_{\bar{W_L}}\bar{W_2})\\ &= t_A\cdot \text{matrix size of } \bar{A_3} \qquad (\text{as } \bar{A_1}=\bar{A_L})\\ &= t_A\cdot k. \end{aligned}$$

But we also have

$$\begin{aligned} S\otimes_F F_2 &\cong [(S\otimes_F F_1)\otimes_{F_1} L]\otimes_L F_2\\ &\cong M_{n_A}(D_1\otimes_{F_1} L)\otimes_L F_2\\ &\cong M_{n_A\ell}(D_L\otimes_L F_2)\\ &\cong M_{n_A\ell k}(D_3). \end{aligned}$$

Hence, $n_B = n_A \ell k = n_A \ell_{B,A} n_{\bar{B}}/t_A$, proving Theorem E (iii).

(e) With the notation of Corollary E, let $E = B_{(Q \cap V)}$. Then E is a Dubrovin valuation ring of S with $B \subseteq E \subseteq A$ (cf. (1.8)), and $\tilde{B} = B/J(A) \subseteq \tilde{E} = E/J(A)$ are Dubrovin valuation rings of $\bar{A}$ with $J(\tilde{E}) = Q/J(A)$. Since we are assuming Theorem B holds for A, we have just shown Theorem E (iii) holds. By applying this theorem for B in A and again for E in A, we obtain

$$(3.11) \qquad n_B/n_E = (n_{\tilde{B}}/n_{\tilde{E}}) \cdot (\ell_{B,A}/\ell_{E,A}).$$

Observe that $\ell_{B,A} \geqq \ell_{E,A}$ since $\tilde{V} = V/J(W)$, as a refinement of $\tilde{T} = (E \cap F)/J(W)$, has at least as many extensions to $Z(\bar{A})$ as does $\tilde{T}$. In addition, $n_B \geqq n_E$ since the Henselization of F with respect to $E \cap F$ lies in the Henselization with respect to V. Likewise, $n_{\tilde{B}} \geqq n_{\tilde{E}}$. Thus, if $n_B = n_E$, then $n_{\tilde{B}} = n_{\tilde{E}}$ and $\ell_{B,A} = \ell_{E,A}$.

We prove the contrapositive of Corollary E. Suppose Q is not a jump prime ideal of B. Then there is a prime ideal Q' of B, $Q' \supsetneqq Q$, such that if $E' = B_{(Q' \cap V)}$, $n_{E'} = n_E$. We have $B \subseteq E' \subseteq E \subseteq A$. Formula (3.11) and the subsequent remarks are still valid with E' replacing B. Thus, $n_{E'} = n_E$ implies $n_{\tilde{E}'} = n_{\tilde{E}}$. Since $\tilde{E}' \supsetneqq \tilde{E}$ this implies that $J(\tilde{E})$ is not a jump prime ideal of $\tilde{B}$, as desired.

(f) For (vi)$\Rightarrow$(vii) of Theorem F, suppose $B \otimes_V V^h$ is a Dubrovin valuation ring of $S \otimes_F F^h$, where $(F^h, V^h) = (F, V)^H$. Set $B^h = B \otimes_V V^h$. Note that $S \otimes_F F^h = S \otimes_V V^h$. Then, $B^h \cap S = B$ by (3.1) (b), so $B^h \cap F = B \cap F = V$. Hence, $B^h \cap F^h$ is a ring containing V^h which contracts to V in F. Thus, $B^h \cap F^h = V^h$, yielding (vii) of Theorem F.

(vii)$\Rightarrow$(ii) Suppose there is a Dubrovin valuation ring B^h of $S \otimes_F F^h$ with $B^h \cap S = B$ and $B^h \cap F^h = V^h$. Write $S \otimes_F F^h = M_n(D^h)$, where D^h is a division ring and $n = n_B$. Let R be the invariant valuation ring of D^h with $R \cap F^h = V^h$. Then $B^h \cong M_n(R)$ by (1.5) and (3.10). Because R is integral over V^h (e.g., by [W1, Corollary]), R is locally finite, by Shirshov's theorem on integral p.i.-rings [Rw, pp. 206–207]. Hence, $B^h \cong M_n(R)$ is locally finite, so integral over V^h. For any $\beta \in B^h$, let f be the minimal (monic) polynomial of β over F^h, and let g be any monic polynomial in $V^h[X]$ with $g(\beta) = 0$. For any root γ of f in any extension field of F^h, $g(\gamma) = 0$, so γ is integral over V^h. Thus, the coefficients of f, as polynomials in the γ's are also integral over V^h. Hence, $f \in V^h[X]$ as V^h is integrally closed. Now, take any $b \in B$. The minimal polynomial of b over F is the same as its minimal polynomial (viewing $b \in S \otimes_F F^h$) over F^h. Because $B \subseteq B^h$, this polynomial has coefficients in $V^h \cap F = V$. Therefore, b is integral over V, proving (ii) of Theorem F. The proof of Proposition 3.9 is now complete. □

The next two lemmas will be used in proving that $B \otimes_V V^h$ is a Dubrovin valuation ring when B is integral over V.

(3.12) Lemma. *With B, S, F, V as defined before* (3.1), *let K be a subfield of F, $Y = V \cap K$, and let $(K, Y) \subseteq (K', Y')$. If $S \otimes_K K'$ is simple Artinian and $B \otimes_Y Y'$ is a Dubrovin valuation ring of $S \otimes_K K'$, then $B \otimes_Y Y'$ is a compatible extension of B. Furthermore, if $\bar{B} \otimes_{\bar{Y}} \bar{Y}'$ is simple, then $\overline{B \otimes_Y Y'} = \bar{B} \otimes_{\bar{Y}} \bar{Y}'$, and if also $J(Y) \cdot Y' = J(Y')$, then $J(B \otimes_Y Y') = J(B) \otimes_Y Y'$.*

Proof. Let $B' = B \otimes_Y Y'$, viewed as a subring of $S \otimes_K K' = S \otimes_Y Y'$. By (3.1), we have $B' \cap S = B$. Also, $J(B) \subseteq J(B) \otimes_Y Y' \subseteq J(B')$, as $J(B) \otimes_Y Y'$ is a proper ideal of B' by (3.1), while the linear ordering of ideals of B' (1.4) assures that $J(B')$ is the unique maximal ideal of B'. Clearly, $\mathrm{st}(B)\subseteq\mathrm{st}(B')$. So B' is a compatible extension of B. Consider the epimorphism $\gamma: B'\to\bar{B}\otimes_Y\overline{Y'}=\bar{B}\otimes_{\bar{Y}}\overline{Y'}$. If $\mathrm{im}(\gamma)$ is simple, then $\ker(\gamma)$ is the unique maximal ideal of B', which is $J(B')$, and $\overline{B'}=B'/J(B')=\mathrm{im}(\gamma)$. If further $J(Y)\cdot Y'=J(Y')$, then $\bar{B}\otimes_Y\overline{Y'}=\bar{B}\otimes_Y Y'$; so $J(B')=\ker(\gamma)=J(B)\otimes_Y Y'$, as desired. □

Here is the setup for the next lemma: Let $B\subseteq A$ be Dubrovin valuation rings of a simple Artinian ring S. Let $F=Z(S)$ (with $[S:F]<\infty$), $V=B\cap F$, $W=A\cap F$, and $\tilde{B}=B/J(A)$, a Dubrovin valuation ring of $\bar{A}$. Let K be a subfield of F with $[F:K]<\infty$ and let $U=V\cap K$, $X=W\cap K$, and $\tilde{U}=U/J(X)$. Let $(K',U')=(K,U)^H$ and $(K_1,X_1)=(K,X)^H$, and set $X'=X\cdot U'$, a valuation ring of K' with $X'\cap K=X$. As noted in (3.8) we may view $(K_1,X_1)\subseteq(K',X')$. Set $\tilde{U}'=U'/J(X')$. Note that by (3.8) $(\overline{X'},\tilde{U}')=(\bar{X},\tilde{U})^H$.

(3.13) **Lemma.** *In the setup just described, assume B is integral over U. If $A\otimes_X X_1$ and $\tilde{B}\otimes_{\tilde{U}}\tilde{U}'$ are both Dubrovin valuation rings, then $B\otimes_U U'$ is a Dubrovin valuation ring.*

Proof. Because B is integral over U, we must have A (a central localization of B) integral over X and $\tilde{B}$ integral over $\tilde{U}$. Recall [E, (13.3)] that the integral closure of X in F is the intersection of all the extensions of X to valuation rings of F. So, because $W\subseteq A$ is integral over X, W must be the unique extension of X to F. Hence $F\otimes_K K_1$ is a field by (3.7). This assures $S\otimes_K K_1$ is a simple Artinian ring. Let $S_1=S\otimes_K K_1$, $A_1=A\otimes_X X_1$, and $B_1=B\otimes_U U_1$, where $U_1=U'\cap K_1$. Let $P=J(X)\subseteq U$, so $A=B\cdot W=B_P$. Because $A=A_P$, $X=U_P$, and $X_1=U_{1P}$, we have $A_1=A\otimes_U U_1$ and likewise $J(A)\otimes_X X_1=J(A)\otimes_U U_1$. Note also that $\bar{A}\otimes_{\bar{X}}\overline{X_1}=\bar{A}$, since $\overline{X_1}=\bar{X}$. Because we are assuming A_1 is a Dubrovin valuation ring, Lemma 3.12 (applied to A, S, F, W, ...) says $\overline{A_1}=\bar{A}$ and $J(A_1)=J(A)\otimes_X X_1=J(A)\otimes_U U_1\subseteq B_1$. Set $\tilde{U}_1=U_1/J(X_1)$; so $\tilde{U}_1=\tilde{U}$ in $\bar{A}_1=\bar{A}$. Then, as $J(X)\cdot U_1=J(X)\cdot X_1=J(X_1)$, we have $B_1/J(A_1)=\tilde{B}\otimes_U U_1=\tilde{B}\otimes_{\tilde{U}}\tilde{U}_1=\tilde{B}$, which is a Dubrovin valuation ring of $\overline{A_1}$. Hence, by (1.6) B_1 is a Dubrovin valuation ring of S_1. Set $\tilde{B}_1=B_1/J(A_1)=\tilde{B}$.

Let $A'=A_1\otimes_{X_1}X'$ and $B'=B_1\otimes_{U_1}U'=B\otimes_U U'$. Since $(\overline{X'},\tilde{U}')=(\overline{X_1},\tilde{U}_1)^H$ and $\tilde{B}_1=\tilde{B}$ is integral over $\tilde{U}_1=\tilde{U}$, the argument used above to see S_1 is simple Artinian applies here to see that $\overline{A_1}\otimes_{\overline{X_1}}\overline{X'}$ is simple Artinian, i.e. $Z(\overline{A_1})$ and $\overline{X'}$ are linearly disjoint over $\overline{X_1}$. Then, as (K',X') is an unramified extension of (K_1,X_1), Corollary 3.6 shows A' is a Dubrovin valuation ring of S' with $\overline{A'}=\overline{A_1}\otimes_{\overline{X_1}}\overline{X'}$. Since $J(X_1)\cdot X'=J(X')$ we may invoke Lemma 3.12 again, obtaining $J(A')=J(A_1)\otimes_{\overline{X_1}}X'=J(A_1)\otimes_{U_1}U'\subseteq B'$. Furthermore, since $J(X_1)\cdot U'=J(X_1)\cdot X'=J(X')$, we have $B'/J(A')=\tilde{B}_1\otimes_{U_1}U'=\tilde{B}\otimes_{\tilde{U}}\tilde{U}'$, which by hypothesis is a Dubrovin valuation ring. Hence, by (1.6) B' is a Dubrovin valuation ring of S', as desired. □

4. The Main Argument

The basic argument for proving the theorems stated in Sect. 2 is an induction on jump rank. We give that argument in this section, while deferring the rest of the proofs to Sect. 5. Throughout this section B will be some fixed Dubrovin valuation ring of an F-central simple algebra S (with $[S:F]<\infty$), and $V=B\cap F$, a valuation ring of F. A will be some Dubrovin valuation ring with $B\subseteqq A\subseteqq S$, and $W=A\cap F$. While B will be fixed, the choice of A will vary depending on the context. We further set $\tilde{B}=B/J(A)$, which is a Dubrovin valuation ring of $\bar{A}=A/J(A)$; set $\tilde{V}=V/J(W)=\tilde{B}\cap\bar{W}$, a valuation ring of $\bar{W}$; and set $Y=\tilde{B}\cap Z(\bar{A})$, a valuation ring of the field $Z(\bar{A})$ with $Y\cap\bar{W}=\tilde{V}$. The following diagram indicates the inclusion relations among these rings:

$$
\begin{array}{ccc}
F & \subseteqq & S \\
| & & | \\
V & \subseteqq & B
\end{array}
\qquad
\begin{array}{ccccc}
\bar{W} & \subseteqq & Z(\bar{A}) & \subseteqq & \bar{A} \\
| & & | & & | \\
\tilde{V} & \subseteqq & Y & \subseteqq & \tilde{B}
\end{array}
$$

In order to prove Theorem F (ii)$\Rightarrow$(vi), it is necessary for the inductive process to prove the following stronger result:

(4.1) **Proposition.** *Let B be a Dubrovin valuation ring of a simple Artinian ring S. Let K be a subfield of $Z(S)$, with $[S:K]<\infty$. Let $U=B\cap K$, a valuation ring of K, and let (K', U') be the Henselization of (K, U). Suppose B is integral over U. Then $B\otimes_U U'$ is a Dubrovin valuation ring of $S\otimes_K K'$, and $B\otimes_U U'$ is an immediate compatible extension of B.*

We now prove Theorem A, Theorem B, Theorem D, Theorem F (i)$\Rightarrow$(vi), Proposition 4.1, and Theorem G (iii)$\Rightarrow$(i). In the proof of Theorem D, we assume $Q_1, Q_2, \ldots, Q_k$ are precisely the jump prime ideals of V with respect to S.

Proof. The proof is by a primary induction on the dimension $[S:F]$ and a secondary induction on the jump rank $j(V,S)$ of V with respect to S. Note that if $[S:F]=1$, then $B=V$ and everything holds trivially. Thus, we may assume $[S:F]>1$. We break the rest of the proof four parts: I. $\operatorname{rank}(V)=1$; II. $\operatorname{rank}(V)>1$, $j(V,S)=1$, and V has a minimal nonzero prime ideal; III. $j(V,S)=1$ and V has no minimal nonzero prime ideal; IV. $j(V,S)>1$. The primary induction hypothesis will be invoked only in part IV. For Theorem F we will actually prove (i)$\Rightarrow$(vi′), where (vi′) reads:

(vi′) *$B\otimes_V V^h$ is a Dubrovin valuation ring of $S\otimes_F F^h$, and is an immediate compatible extension of B.*

I

Assume $\operatorname{rank}(V)=1$. The conjugacy theorem (Theorem A) was proved for B in this case by Brungs and Gräter [BG2, Theorem 5.2]. Let $(\hat{F}, \hat{V})$ be the completion of (F, V) (with respect to the topology of the ideals of V), and let $\hat{S}=S\otimes_F\hat{F}$. We identify S and $\hat{F}$ with their images in $\hat{S}$. Let $\hat{B}$ be a Dubrovin valuation ring of $\hat{S}$ with $\hat{B}\cap\hat{F}=\hat{V}$. It is shown in [D2, Sect. 3, Lemma 1] and more convincingly in [BG2, Lemma 3.4] that $\hat{B}\cap S$ is a Dubrovin valuation ring of S contracting to V in F. By Theorem A, B and $\hat{B}\cap S$ are conjugate, so we may assume $B=\hat{B}\cap S$. The proof that $\hat{B}\cap S$ is Dubrovin in [D2] or [BG2] shows that $J(\hat{B})\cap S=J(B)$ and $\bar{B}=\bar{\hat{B}}$.

Because $\operatorname{rank}(V)=1$, we can take the Henselization (F^h, V^h) of (F, V) to be: F^h is the separable closure of F in $\hat{F}$ and $V^h=\hat{V}\cap F^h$ (cf. [E, (17.18)]). Let $S^h=S\otimes_F F^h \subseteq \hat{S}$, and let $B^h=\hat{B}\cap S^h$. Since $(\hat{F}, \hat{V})$ is the completion of (F^h, V^h), the arguments quoted in the preceding paragraph show that B^h is a Dubrovin valuation ring of S^h with $B^h \cap F^h = V^h$, $J(\hat{B})\cap S^h = J(B^h)$, and $\overline{B^h}=\overline{\hat{B}}$. Hence, $B^h\cap S=B$, $J(B^h)\cap S=J(B)$, and $\bar{B}=\overline{B^h}$.

We next show $B^h=B\cdot V^h$. Clearly $B\cdot V^h \subseteq B^h$. Let $\{b_1, \ldots, b_m\}\subseteq B$ be any F-base of S. Take any $\beta\in B^h$ and write $\beta=\sum b_i\gamma_i$ with $\gamma_i\in F^h$. Since F is dense in $\hat{F}$ as $\operatorname{rank}(V)=1$, F is dense in F^h. Hence, there exist $c_i\in F$ with $\gamma_i-c_i\in V^h$, $1\leqq i\leqq m$. Then $\sum b_ic_i=\beta-\sum b_i(\gamma_i-c_i)\in B^h\cap S=B$. Hence, $\beta=\sum b_ic_i+\sum b_i(\gamma_i-c_i)\in B\cdot V^h$. So, $B^h=B\cdot V^h$. Since $B\otimes_V V^h$ embeds in $S\otimes_V V^h$ by (3.1), this shows $B^h=B\otimes_V V^h$. Thus, B is integral over V by (3.9) (f), and B^h is a compatible extension of B.

To see $\Gamma_{B^h}=\Gamma_B$, i.e. $\operatorname{st}(B^h)=\operatorname{st}(B)\cdot B^{h*}$, take any $\delta\in\operatorname{st}(B^h)$ and write $\delta=\sum b_i\varepsilon_i$ and $\delta^{-1}=\sum b_i\gamma_i$, with the b_i as above and ε_i, $\lambda_i\in F^h$. There is an ideal $I\neq(0)$ of V^h, such that $\lambda_i I\subseteq J(V^h)$, for each i. Since F is dense in F^h we may choose $e_i\in F$ with $\varepsilon_i-e_i\in I$, each i. Let $d=\sum b_ie_i\in S$. Then,

$$\delta^{-1}(d-\delta)=\sum_i\sum_j b_ib_j\lambda_i(e_j-\varepsilon_j)\in B\cdot J(V^h)\subseteq J(B^h).$$

Hence $1+\delta^{-1}(d-\delta)\in 1+J(B^h)\subseteq B^{h*}$, so that

$$d=\delta[1+\delta^{-1}(d-\delta)]\in\operatorname{st}(B^h)\cdot B^{h*}\cap S=\operatorname{st}(B^h)\cap S=\operatorname{st}(B).$$

Thus, $\delta=d[1+\delta^{-1}(d-\delta)]^{-1}\in\operatorname{st}(B)\cdot B^{h*}$, showing $\Gamma_{B^h}=\Gamma_B$. Therefore, B^h is an immediate compatible extension of B; so (vi′) of Theorem F holds.

We can now prove Theorem B. We have $S^h\cong M_n(D^h)$ for some F^h-central division ring D^h, with $n=n_B$. Since V^h is Henselian, V^h extends to an invariant valuation ring R of D^h. Then $M_n(R)$ is a Dubrovin valuation ring of $M_n(D^h)$. By Theorem A, $B^h\cong M_n(R)$ by a V^h-isomorphism. Thus, $\bar{B}\cong\overline{B^h}\cong\overline{M_n(R)}\cong M_n(\bar{R})$, and hence $t_B=n=n_B$. Furthermore, $\Gamma_B=\Gamma_{B^h}\cong\Gamma_{M_n(R)}=\Gamma_R$, using Corollary 3.5. The middle isomorphism is the identity on Γ_{V^h}, and hence can be viewed as an equality in the divisible hull of $\Gamma_{V^h}=\Gamma_V$. We obtain the commutative diagram of Theorem B (iii) by combining the corresponding diagrams from B to B^h (as B^h is immediate over B) and from R to $M_n(R)$ to B^h. Theorem D holds as $n_B=t_B$ and there are no ℓ_i since $k=1$.

Now, let U, K and U', K' be as in Proposition 4.1, and suppose B is integral over U. Then V is integral over U, so V is the unique extension of U to F. By (3.7) $F\otimes_K K'$ is a field, and $F\otimes_K K'$ can be identified with F^h so that $V^h\cap K'=U'$. Then $S^h=S\otimes_F F^h=S\otimes_K K'=S\otimes_U U'$, and $B\otimes_U U'$ embeds in S^h by (3.1). Because $\operatorname{rank}(U)=\operatorname{rank}(V)=1$, F is dense in F'. Hence, the argument above proving $B^h=B\cdot V^h$ applies here to show $B^h=B\cdot U'=B\otimes_U U'$. This yields Proposition 4.1.

For Theorem G (iii)$\Rightarrow$(i) suppose S is a division ring and $t_B>1$. (So $S\neq F$.) Recall that in $M_k(D)$ for any division ring D (except the field with two elements) and any $k\geqq 2$, the only proper subrings invariant under all inner automorphisms are central. (For this, see [H, Theorem 3], [K, Satz 3], or for $k>2$ [Ro, Corollary 1].) Therefore, as $B^h\subseteq S^h$, of matrix size t_B,

$$V\subseteq\bigcap_{s\in S^{h*}}(sB^hs^{-1}\cap S)\subseteq B^h\cap Z(S^h)\cap S=V^h\cap S=V,$$

so equality holds throughout. Each $sB^h s^{-1} \cap S$ is a Dubrovin valuation ring of S contracting to V, so a conjugate of B. There must be infinitely many such conjugates, since for any $s_1, \dots, s_k \in S^*$ elementary localization theory shows $\left(\bigcap_{i=1}^{k} s_i B s_i^{-1}\right) \cdot F = S \neq F$. This completes part I.

Before going to part II, we prove further cases of Proposition 4.1:

(4.2) *Proposition 4.1 holds if* rank(U) *is finite or if B is an invariant valuation ring, or if $B \cong M_n(B')$, where B' is an invariant valuation ring.*

Proof of (4.2). The case rank$(U)=1$ was settled in part I above. Proposition 4.1 then holds whenever rank$(U) < \infty$ by induction on rank(U), with the induction step provided by Lemma 3.13.

Next, assume that S is a division ring and that B is an invariant valuation ring of S. Let $\{s_1, s_2, \dots, s_\ell\}$ be a base of S as a K-vector space, and let $s_i s_j = \sum_{k=1}^{\ell} a_{ijk} s_k$ with $a_{ijk} \in K$. Let K_0 be any subfield of K such that all $a_{ijk} \in K_0$ and K_0 is finitely-generated over the prime subfield. Let S_0 be the K_0-vector space (and algebra) spanned by $\{s_1, s_2, \dots, s_\ell\}$. Then, $K_0 \subseteq Z(S_0)$ as $K \subseteq Z(S)$, and $S_0 \otimes_{K_0} K = S$; hence, S_0 has no zero divisors, and as $[S_0 : K_0] = [S : K] < \infty$, S is a division ring. Let $B_0 = B \cap S_0$, which is an invariant valuation ring of S_0; let $U_0 = U \cap K_0 = B_0 \cap K_0$, a valuation ring of K_0. For any given $b \in B_0$, let $f \in K_0[X]$ be the minimal polynomial of b over K_0. Then, as $S = S_0 \otimes_{K_0} K$, f is also the minimal polynomial of b over K. But because $b \in B$ is integral over U, which is integrally closed, the coefficients of f lie in U. Hence $f \in (U \cap K_0)[X] = U_0[X]$; thus, B_0 is integral over U_0. Let $(K_0', U_0') = (K_0, U_0)^H$. Now, the rank of U_0 is finite (by, e.g. [B, Sect. 10, No. 3, Corollary 2]) since the transcendence degree of K_0 over the prime field is finite. Hence, by the finite rank case already proved, $B_0 \otimes_{U_0} U_0'$ is a Dubrovin valuation ring of $S_0 \otimes_{K_0} K_0'$. Because K is the direct limit of such fields K_0, and B (resp. U, U') is the direct limit of the corresponding B_0 (resp. U_0, U_0'), $B \otimes_U U'$ is the direct limit of the (compatible) Dubrovin valuation rings $B_0 \otimes_{U_0} U_0'$. It follows easily that $B \otimes_U U'$ is a Dubrovin valuation ring (in fact an invariant valuation ring, in view of Theorem 2.2).

Since we now have Proposition 4.1 for any invariant valuation ring, observe that it also follows for $B = M_n(R)$ where R is an invariant valuation ring. For, $B \otimes_U U' = M_n(R \otimes_U U')$ which is Dubrovin by (1.5) as $R \otimes_U U'$ is Dubrovin. This yields (4.2). □

The following general setup occurs repeatedly in the rest of the proof:

(4.3) *Setup.* Given B, S, F, V as usual, P will be a specified prime ideal of V, $A = B_P$, and $W = V_P = A \cap F$. Let $(F_1, W_1) = (F, W)^H$, and let $S_1 = S \otimes_F F_1$. Set $\tilde{V}_1 = \tilde{V}$ viewed in $W_1 = \bar{W}$, and let $V_1 = \pi_{W_1}^{-1}(\tilde{V}_1)$, the valuation ring of F_1 with $V_1 \subseteq W_1$ and $V_1 \cap F = V$. Set $A_1 = A \otimes_W W_1$. It will in every case be known that $n_A = t_A$ and that A_1 is a Dubrovin valuation ring of S_1 which is an immediate compatible extension of A, with $A_1 \cap F_1 = W_1$. Set $\tilde{B}_1 = \tilde{B}$ viewed in $\overline{A_1} = \bar{A}$, and let $B_1 = \pi_{A_1}^{-1}(\tilde{B}_1)$, a Dubrovin valuation ring of S_1 by (1.6), with $B_1 \cap F_1 = V_1$. Evidently, B_1 is a compatible extension of B, with $\overline{B_1} = \bar{B}$.

II

Now suppose $\operatorname{rank}(V)>1$, but $j(V,S)=1$. Suppose further that V has a minimal prime ideal $P \neq (0)$. Use this P to form A, W, $F_1, \ldots$ as in Setup 4.3 above. The properties of A_1 specified in Setup 4.3 are known from part I. Write $S_1 \cong M_n(D_1)$ where D_1 is an F_1-central division ring and $n=n_A=t_A$. Then by (1.5) $B_1 \cong M_n(B_1')$, where B_1' is a Dubrovin valuation ring of D_1. Assume for convenience that a set of matrix units has been chosen in B_1 so that $B_1=M_n(B_1')$ (and $S_1=M_n(D_1)$). Set $A_1'=B'_{1P}$, which assures $A_1=B_{1P}=M_n(A_1')$.

Set $(F_2, V_2)=(F, V)^H$. (This is the (F^h, V^h) of Theorem B.) As noted in Sect. 3, we may view $(F_1, V_1) \subseteq (F_2, V_2)$. Set $D_2=D_1 \otimes_{F_1} F_2$ and $S_2=S_1 \otimes_{F_1} F_2=M_n(D_2)$. Since $S_2 \cong S \otimes_F F_2$ the matrix size of S_2 is n_B. But $n_B=n_A=n$ because $j(V,S)=1$. Hence, D_2 is a division ring (namely, the D^h of Theorem B). We write R for the invariant valuation ring of D_2 with $R \cap F_2=V_2$. Then $R \cap D_1$ is an invariant valuation ring, and because $(F_2, V_2)=(F_1, V_1)^H$, Morandi's Theorem 2.2 says R is an immediate extension of $R \cap D_1$. Note also that by (3.10) $R \cap D_1=B_1'$, since $(R \cap D_1) \cap F_1=V_1$ $=B_1' \cap F_1$. Set $B_2=M_n(R)$, a Dubrovin valuation ring of S_2 with $B_2 \cap F_2=R \cap F_2$ $=V_2$; B_2 is a compatible extension of $B_1=M_n(B_1')$, hence a compatible extension of B. In view of Corollary 3.5, B_2 is actually an immediate compatible extension of B_1. Thus,

$$\bar{B}=\bar{\tilde{B}}=\overline{\tilde{B}_1}=\overline{B_1}=\overline{B_2}=M_n(\bar{R}).$$

Since R is invariant, $\bar{R}$ is a division ring; hence $t_B=n=n_B$. This proves (i) of Theorem B, and also Theorem D as there are no ℓ_i, since we are assuming the Q_i are only the jump prime ideals of V.

Since B_1' is an invariant valuation ring it is integral over $B_1' \cap F_1=V_1$ (cf. [W1, Corollary]). Set $\tilde{B}_1'=B_1'/J(A_1') \subseteq \overline{A_1'}$. Then $\tilde{B}_1' \cap Z(\overline{A_1'})$ is a valuation ring integral over $\tilde{V}_1$. So $\tilde{B}_1' \cap Z(\overline{A_1'})$ is the unique extension of $\tilde{V}_1$ from $\overline{W_1}$ to $Z(\overline{A_1'})$, i.e., $\ell_{B_1', A_1'}=1$.

We can now prove Theorem A for B: Let B_0 be another valuation ring of S with $B_0 \cap F=V=B \cap F$. Let $A_0=B_{0_P}$, so that $A_0 \cap F=W=A \cap F$. Since $\operatorname{rank}(W)=1$, we saw in part I that A_0 and A are conjugate. Thus, we may assume that $A_0=A$. Let $\tilde{B}_0=B_0/J(A)$, a Dubrovin valuation ring of $\bar{A}$. Since $\bar{A}=\overline{A_1}=M_n(\overline{A_1'})$, [BG2, Theorem 2.4] or [D1, Sect. 1, Theorem 7] says $\tilde{B}_0$ is conjugate to $M_n(C)$ for some Dubrovin valuation ring C of $\overline{A_1'}$. Then, as $C \cap \overline{W_1}=\tilde{B}_0 \cap \bar{W}=\tilde{V}=\tilde{V}_1$ and $\ell_{B_1', A_1'}=1$, we have $C \cap Z(\overline{A_1'})=\tilde{B}_1' \cap Z(\overline{A_1'})$. So, (3.10) yields $C=\tilde{B}_1'$, since $\tilde{B}_1'$ is invariant. Thus, $\tilde{B}_0$ is conjugate to $M_n(\tilde{B}_1')=\tilde{B}$ in $\bar{A}$. Therefore, as $B=\pi_A^{-1}(\tilde{B})$ and $B_0=\pi_A^{-1}(\tilde{B}_0)$, B and B_0 are conjugate in A, proving Theorem A for B.

In the previous paragraph we saw that the conjugacy Theorem A holds for $\tilde{B}=\tilde{B}_1$ in $\bar{A}=\overline{A_1}$. Since Theorem B holds for A and for A_1 by part I, Proposition 3.11(b) shows that we have the exact sequences $(\Gamma_{B,A})$ and (Γ_{B_1,A_1}) of Theorem E(i). Because B_1 is a compatible extension of B, there is a map of complexes $(\Gamma_{B,A}) \to (\Gamma_{B_1,A_1})$, to which we apply the 5-lemma to see $\Gamma_{B_1}=\Gamma_B$. Thus, $\Gamma_B=\Gamma_{B_1}=\Gamma_{B_2}=\Gamma_R$ as $B_2=M_n(R)$ (recall Corollary 3.5); this yields (ii) of Theorem B. Since the equality $\Gamma_B=\Gamma_R$ follows from identifications corresponding to inclusions $B \subseteq B_2$ and $R \subseteq B_2$, Theorem B (iii) follows at once. Note also we have now shown that B_1 is an immediate compatible extension of B. Since B_2 is immediate compatible over B_1, B_2 is also immediate compatible over B.

Because $B_2 \cap S = B$ and $B_2 \cap F_2 = V_2$, Proposition (3.9) (f) shows B is integral over V. Since $\tilde{B} = \tilde{B_1} \cong M_n(\tilde{B_1'})$ where $\tilde{B_1'}$ is an invariant valuation ring, (4.2) shows $\tilde{B} \otimes_{\bar{V}} \tilde{V_2}$ is a Dubrovin valuation ring. Since we also know $A \otimes_W W_1 = A_1$ is a Dubrovin valuation ring, Lemma 3.13 shows $B \otimes_V V_2$ is a Dubrovin valuation ring. Since $B \otimes_V V_2 \subseteq B_2$ and $(B \otimes_V V_2) \cap F_2 = V_2 = B_2 \cap F_2$ it follows from (1.8) that $B \otimes_V V_2 = B_2$, which is an immediate compatible extension of B, yielding (vi′) of Theorem F. Likewise, suppose $K, U, X, K_1, X_1, K', U', \tilde{U}'$ are as defined just before Lemma 3.13, and suppose B is integral over U. Then, $A \otimes_X X_1$ is Dubrovin by part I and $\tilde{B} \otimes_{\tilde{U}} \tilde{U}'$ is Dubrovin by (4.2), so $B \otimes_U U'$ is Dubrovin by (3.13). Thus, Proposition 4.1 holds.

It remains to prove (iii)$\Rightarrow$(i) of Theorem G for this B. Suppose S is a division ring and $t_B > 1$. Now, $\tilde{B}$ is a Dubrovin valuation ring of $\bar{A}$, and the matrix size of $\bar{A}$ is $t_A = n_A = n_B = t_B > 1$. Consequently, the argument of part I (invoking [H], etc.) shows that $\tilde{B}$ has infinitely many conjugates. Each lifts via π_A^{-1} to a different conjugate of B. This completes part II.

Before going on to part III, let us verify that Theorem B and Theorem D hold if B is an Azumaya algebra over V. For this, let $B^h = B \otimes_V V^h$ where $(F^h, V^h) = (F, V)^H$, and let $S^h = S \otimes_F F^h \cong M_n(D^h)$ where $n = n_B$ and D^h is a division ring. Then, B^h is an Azumaya algebra over V^h (cf. [DI, p. 61]), so by Corollary 3.4, $\Gamma_B = \Gamma_V = \Gamma_{V^h} = \Gamma_{B^h}$. Also, $\overline{B^h} = \bar{B} \otimes_{\bar{V}} \overline{V^h} = \bar{B}$, so B^h is an immediate compatible extension of B. Now, as B^h is Dubrovin, $B^h \cong M_n(B^{h'})$, where $B^{h'}$ is a Dubrovin valuation ring of D^h with $B^{h'} \cap F^h = V^h$. Because V^h is Henselian there is an invariant valuation ring R of D^h with $R \cap F^h = V^h$. By (3.10) $B^{h'} = R$. Thus, $\bar{B} = \overline{B^h} = M_n(\bar{R})$ (so $n = t_B$, as $\bar{R}$ is a division ring). We have established (i) and (ii) of Theorem B, and (iii) holds trivially since $Z(\bar{B}) = \bar{V}$ as B is Azumaya. Further, if E, A are rings with $B \subseteq E \subseteq A \subseteq S$, then A is Azumaya since it is a central localization of B (cf. (1.8)). Hence, $Z(\bar{A}) = \overline{A \cap F}$, which assures that $\ell_{E,A} = 1$. This yields Theorem D, since $n_B = n = t_B$ and all the $\ell_i = 1$.

III

Suppose now that $\mathrm{rank}(V) > 1$ and $j(V, S) = 1$, but V has no minimal nonzero prime ideal. The correspondence between prime ideals of B and V (cf. (1.8)) assures that $(0) = \bigcap Q$ as Q ranges over the nonzero prime ideals of B. We now invoke a variation of the Azumaya algebra argument of [D2, Sect. 3] and [BG2, Sect. 3]. If $[S:F] = k^2$, then B has p.i.-degree k. Then there is a homogeneous polynomial f with integer coefficients such that f is an identity for all algebras of p.i.-degree $< k$, but $f(b_1, \dots, b_m) \neq 0$ for some $b_1, \dots, b_m \in B$. So, there is a prime ideal Q of B with $f(b_1, \dots, b_m) \notin Q$. Let $P = Q \cap V$, and use this P in Setup 4.3 to form A, W, F_1, S_1, A_1, etc. Then, $J(A) = Q$. For $\bar{b}_i = \pi_A(b_i)$ we have $f(\bar{b}_1, \dots, \bar{b}_m) = \overline{f(b_1, \dots, b_m)} \neq 0$ in $\bar{A}$. Therefore, $\bar{A}$ has p.i.-degree $k =$ p.i.-degree of A. By the Artin-Procesi theorem [C, Theorem 9, p. 465], A is an Azumaya algebra over W. The properties of A_1 required for Setup 4.3 follow because A is an Azumaya algebra. With this choice of A nearly all the arguments of part II carry through without change. The necessary information about A and A_1 (which in part II was obtained from part I results) is contained in the observations above about Azumaya algebras.

The only argument from part II which does not carry over is the proof of the conjugacy Theorem A for B. For this, let B_0 be another valuation ring of S with $B_0 \cap F = V = B_1 \cap F$. Then, just as with B, there exists a prime ideal P_0 of V, with

$B_{0_{P_0}}$ Azumaya. Pick a prime ideal $P_1 \neq (0)$ of V with $P_1 \subseteq P \cap P_0$. Redefine A to be B_{P_1} and set $A_0 = B_{0_{P_1}}$. Then A and A_0 are Azumaya algebras over V_{P_1} since they are central localizations of the Azumaya algebras B_P and $B_{0_{P_0}}$. Because V_{P_1} is a valuation ring the functorial map of Brauer groups $\mathrm{Br}(V_{P_1}) \to \mathrm{Br}(F)$ is injective by [Sa, Lemma 1.2]. Hence, $[A_0] = [A]$ in $\mathrm{Br}(V_{P_1})$. Since projective V_{P_1}-modules are free this implies $M_\ell(A_0) \cong M_\ell(A)$ for some ℓ; hence, by the cancellation theorem for Azumaya algebras over semilocal rings (cf. [OS, Corollary 1]), $A_0 \cong A$ by a V_{P_1}-algebra isomorphism. This isomorphism extends by central localization to an F-automorphism of S, which by Skolem-Noether is inner. Consequently, A and A_0 are conjugate. The rest of the argument for conjugacy of B and B_0 is the same as in part II. This completes part III.

IV

Now assume $j(V, S) > 1$. We argue by induction on the jump rank (within the primary induction on $[S:F]$). Let $Q_1 \subsetneqq Q_2 \subsetneqq \ldots \subsetneqq Q_k = J(V)$ be the jump prime ideals of V with respect to S. For some i, $1 \leqq i < k$, let $A = B_{Q_i}$ and let $W = A \cap F = V_{Q_i}$. Specific choices of i will be made later. Using this A and W define $\tilde{B}$, $\tilde{V}$, Y, as described at the beginning of Sect. 4. The jump prime ideals of W re S are clearly $Q_1, Q_2, \ldots, Q_i$, so $j(W, S) = i < j(V, S)$. Hence, by induction Theorem B holds for the ring A. Consequently, by Proposition 3.9(e), we may invoke Corollary E to see that $j(Y, \bar{A}) \leqq j(V, S) - i < j(V, S)$. In addition, Theorem C applies to A by (3.9)(a), yielding $[\bar{A} : Z(\bar{A})] \leqq [\bar{A} : \bar{W}] \leqq [S:F]$. Consequently, the results we want to prove about B hold by induction for $\tilde{B}$ as well as for A.

To prove the conjugacy Theorem A for B, let B_0 be another Dubrovin valuation ring of S with $B_0 \cap F = V = B \cap F$. Let $A_0 = B_{0_{Q_i}}$, which is a Dubrovin valuation ring of S with $A_0 \cap F = W = A \cap F$. Then A and A_0 are conjugate in S, since Theorem A holds for A by induction. Hence, we may assume $A_0 = A$. Let $\tilde{B}_0 = B_0/J(A)$ and $Y_0 = \tilde{B}_0 \cap Z(\bar{A})$. Then Y and Y_0 are each valuation rings of $Z(\bar{A})$ extending $\tilde{V}$ in $\bar{W}$. Now, by (3.9)(a), Corollary B applies for A. Hence, $Z(\bar{A})$ is normal over $\bar{W}$, so there is a $\tau \in \mathscr{G}(Z(\bar{A})/\bar{W})$ with $\tau(Y_0) = Y$. Since θ_A is surjective (by Corollary B again) τ can be induced by conjugation by some element of $\mathrm{st}(A)$. After conjugating B_0 by such an element, we may assume $Y_0 = Y$. By Theorem A for $\tilde{B}$, which holds by induction, $\tilde{B}_0$ and $\tilde{B}$ are conjugate in $Z(\bar{A})$. Hence, B_0 and B are conjugate, completing the proof of Theorem A.

We can also settle Proposition 4.1. For, since B is integral over U, A is integral over X and $\tilde{B}$ integral over $\tilde{U}$. By induction, Proposition 4.1 holds for A and for $\tilde{B}$, hence it holds for B by Lemma 3.13.

We next dispose of Theorem G (iii)$\Rightarrow$(i). For this, suppose S is a division ring and $t_B > 1$. If $t_A = 1$, then $\bar{A}$ is a division ring with Dubrovin valuation ring $\tilde{B}$ and $t_{\tilde{B}} = t_B > 1$. By induction, $\tilde{B}$ has infinitely many different conjugates. Their inverse images in A yield infinitely many different conjugates of B. On the other hand, if $t_A > 1$, then by induction A has infinitely many different conjugates. For any $s \in S^*$, $(sBs^{-1})_{Q_i} = sB_{Q_i}s^{-1} = sAs^{-1}$. Thus, whenever $sAs^{-1} \neq tAt^{-1}$ we have $sBs^{-1} \neq tBt^{-1}$. So, the infinitely many conjugates of A yield infinitely many conjugates of B, as desired.

Now drop the assumption that S is a division ring. We can quickly settle Theorem D (assuming the Q_i are just the jump primes of V re S). For this choose Q_i (for the construction of A, W, etc.) to be the next to last jump prime Q_{k-1}. Then,

$j(Y, \bar{A}) \leqq k-(k-1)=1$, so that $n_{\tilde{B}}=t_{\tilde{B}}$ by parts I–III above. Of course also $t_{\tilde{B}}=t_B$. Note that the jump prime ideals of W re S are $Q_1, \ldots, Q_{k-1}$, and that the $\ell_2, \ldots, \ell_{k-1}$ of Theorem D are the same for A as for B, while the ℓ_k for B is $\ell_{B,A}$. Theorem B holds for A by induction, hence we have Theorem E(iii) for B by (3.9)(d). Thus,

$$\begin{aligned} n_B &= n_{\tilde{B}}(n_A/t_A)\ell_{B,A} \\ &= t_B(\ell_2 \ldots \ell_{k-1})\ell_k, \end{aligned}$$

as desired.

We now prove Theorem B under the assumption that $n_B=t_B$. For this, any choice of Q_i may be made $(1 \leqq i<k)$ for defining A, W, etc. Since Theorem B for A and Theorem A for $\tilde{B}$ hold by induction, Proposition 3.9 yields Theorem E for B in A. Hence, (with $s_B=n_B/t_B$) we have

$$s_B = s_{\tilde{B}} s_A \ell_{B,A}.$$

By assumption $s_B=1$, while $s_{\tilde{B}}$ and s_A are positive integers by Theorem D (which holds by induction), as is $\ell_{B,A}$. This forces

$$n_{\tilde{B}}=t_{\tilde{B}}, \quad n_A=t_A, \quad \text{and} \quad \ell_{B,A}=1.$$

The last equation means Y is the unique extension of $\tilde{V}$ to $Z(\bar{A})$. Hence, the exact sequence $(\Gamma_{B,A})$ of Theorem E(i) is actually the short exact sequence

$$(\Gamma_{B,A}) \qquad 0 \to \Gamma_{\tilde{B}} \to \Gamma_B \to \Gamma_A \to 0.$$

Now, using Q_i for the P, define F_1, W_1, S_1, V_1, A_1, B_1, etc. as in Setup 4.3. Since $n_A=t_A$, the required properties of A_1 follow from Theorem F (i)$\Rightarrow$(vi′), which holds for A by induction. Also, since $j(W_1, S_1)=1$ as W_1 is Henselian, Theorem B holds for A_1 by induction, so by (3.9) Theorem E holds for B_1 in A_1. The exact sequence (Γ_{B_1,A_1}) is actually short exact for the same reason as for $(\Gamma_{B,A})$ (as $\tilde{V}_1=\tilde{V}$ in $Z(\bar{A}_1)$ $=Z(\bar{A})$). Furthermore, the compatibility of B in B_1 assures there is a map of complexes $(\Gamma_{B,A}) \to (\Gamma_{B_1,A_1})$, which by the 5-lemma is an isomorphism. Hence, $\Gamma_{B_1}=\Gamma_B$ and B_1 is an immediate compatible extension of B.

Let $(F_2, V_2)=(F, V)^H=(F^h, V^h)$. As noted in Sect. 3, we may view $(F_1, V_1) \subseteqq (F_2, V_2)$. Let $S_2=S \otimes_F F_2 = S_1 \otimes_{F_1} F_2$; let $W_2=V_{2Q_i}$, so $W_2 \cap F_1 = W_1$; and let $\tilde{V}_2=V_2/J(W_2) \subseteqq \bar{W}_2$. By (3.8), W_2 is an unramified extension of W_1 with $(\bar{W}_2, \tilde{V}_2)$ $=(\bar{W}_1, \tilde{V}_1)^H$. Also, since $\tilde{V}_1=\tilde{V}$ has a unique extension to $Z(\bar{A}_1)$, $Z(\bar{A}_1)$ is linearly disjoint to $\bar{W}_2$ over $\bar{W}_1$ (cf. (3.7)). Set $A_2=A_1 \otimes_{W_1} W_2 \subseteqq S_2$. By Corollary 3.6, A_2 is a Dubrovin valuation ring of S_2 which is compatible with A_1, and $\Gamma_{A_2}=\Gamma_{A_1}$ and $\bar{A}_2$ $=\bar{A}_1 \otimes_{\bar{W}_1} \bar{W}_2$. View $\bar{A}_2$ as $\bar{A}_1 \otimes_{Z(\bar{A}_1)} (Z(\bar{A}_1) \otimes_{\bar{W}_1} \bar{W}_2)$. Note that by (3.7) the unique extension, call it Y_2, of $\tilde{V}_2$ to $Z(\bar{A}_1) \otimes_{\bar{W}_1} \bar{W}_2 (=Z(\bar{A}_2))$ is the Henselization of $\tilde{B}_1 \cap Z(A_1)$. Since $\tilde{B}_1=\tilde{B}$, $n_{\tilde{B}}=t_{\tilde{B}}$, and $j(Y, \bar{A})<j(V, S)$, Theorem F (i)$\Rightarrow$(vi′) holds by induction for $\tilde{B}_1$. Hence, if we set $\tilde{B}_2=\tilde{B}_1 \otimes_{\tilde{V}_1} \tilde{V}_2$, $\tilde{B}_2$ is a Dubrovin valuation ring of $\bar{A}_2$ which is an immediate compatible extension of $\tilde{B}_1$, with $\tilde{B}_2 \cap Z(\bar{A}_2)=Y_2$. Then $\pi_{A_2}^{-1}(\tilde{B}_2)$ is a Dubrovin valuation ring of S_2 contracting to $B_1 \cap S=B$ in S and to V_2 in F_2. Hence, B is integral over V by (3.9)(f).

Let $B_2=B \otimes_V V_2$, a Dubrovin valuation ring of S_2 by Proposition 4.1 (which was proved above for B); B_2 is a compatible extension of B. Since $B_2 \subseteqq \pi_{A_2}^{-1}(\tilde{B}_2)$ and both these Dubrovin valuation rings contract to V_2 in F_2, we must have B_2 $=\pi_{A_2}^{-1}(\tilde{B}_2)$ (cf. (1.8)). So, $\tilde{B}_2=B_2/J(A_2)$. An analogous argument shows $B_1 \otimes_V V_2$ $=\pi_{A_2}^{-1}(\tilde{B}_2)=B_2$ which yields that B_2 is a compatible extension of B_1. Now, as W_2

and Y_2 are Henselian, $j(W_2, S_2) = j(Y_2, \overline{A_2}) = 1$, so by induction Theorem B holds for A_2 and Theorem A holds for $\tilde{B}_2$. Thus, Proposition (3.9)(b) shows that Theorem E(i) holds for B_2 in A_2. The diagram (Γ_{B_2,A_2}) is a short exact sequence, as the Henselian valuation $\tilde{V}_2$ has a unique extension to $Z(\overline{A_2})$. Moreover, from the compatibility of B_1 in B_2, there is a map of complexes $(\Gamma_{B_1,A_1}) \to (\Gamma_{B_2,A_2})$, which is an isomorphism by the 5-lemma. Hence, $\Gamma_{B_2} = \Gamma_{B_1}$. Since $\overline{B_1} = \overline{\overline{B_1}} = \overline{\overline{B_2}} = \overline{B_2}$, B_2 is an immediate compatible extension of B_1, hence of B. (This yields Theorem F (i)$\Rightarrow$(vi$'$) for B.)

Now, $S_2 \cong M_n(D_2)$, where $n = n_B$ and D_2 is an F_2-central division ring. Since V_2 is Henselian, there is an invariant valuation ring of D_2, called R in Theorem B, such that $R \cap F_2 = V_2$. Because R is the only Dubrovin valuation ring of D_2 contracting to V_2, by (1.5) and (3.10) $B_2 \cong M_n(R)$. Thus, $\bar{B} = \overline{B_2} \cong M_n(\bar{R})$ where $n = n_B = t_B$, and $\Gamma_B = \Gamma_{B_2} = \Gamma_R$ by Corollary 3.5. This yields (i) and (ii) of Theorem B. Finally, in the diagram below,

$$\begin{array}{ccccc} \Gamma_B/\Gamma_V & \xrightarrow{=} & \Gamma_{B_2}/\Gamma_{V_2} & \xrightarrow{=} & \Gamma_R/\Gamma_{V_2} \\ \downarrow{\scriptstyle \theta_B} & & \downarrow{\scriptstyle \theta_{B_2}} & & \downarrow{\scriptstyle \theta_R} \\ \mathscr{G}(Z(\bar{B})/\bar{V}) & \xrightarrow{\cong} & \mathscr{G}(Z(\overline{B_2})/\overline{V_2}) & \xrightarrow{\cong} & \mathscr{G}(Z(\bar{R})/\overline{V_2}) \end{array}$$

the left square is commutative as B_2 is compatible with B, and the right square is commutative as $M_n(R)$ is compatible with R and $B_2 \cong M_n(R)$. The commutative outer rectangle establishes (iii) of Theorem B.

It remains to prove Theorem B in case $t_B < n_B$. We have shown Theorem D, which says

$$n_B = t_B \ell_2 \ldots \ell_k,$$

where $\ell_j = \ell_{B_{Q_j}, B_{Q_{j-1}}}$. Since $t_B < n_B$ some $\ell_j > 1$. Choose i, $1 \leqq i < k$, so that $\ell_2 = \ldots = \ell_i = 1$, but $\ell_{i+1} > 1$. (If $\ell_2 > 1$, set $i = 1$.) For this part of the argument, set $A = B_{Q_i}$ and $W = A \cap F$, with the i just selected. We have $\ell_{B,A} \geqq \ell_{A_{Q_{i+1}},A} = \ell_{i+1} > 1$, since $\tilde{V}$ has at least as many extensions to $Z(\bar{A})$ as (the coarser valuation ring) $V_{Q_{i+1}}/J(W)$. Note also that $j(W, S) = i < j(V, S)$ and by Theorem D for A, $n_A = t_A \ell_2 \ldots \ell_i = t_A$.

Now define F_1, W_1, S_1, V_1, A_1, B_1, etc. as in Setup 4.3, using Q_i for the P. Theorem F (i)$\Rightarrow$(vi$'$), which holds for A by induction, assures that A_1 has the required properties. For the same reasons as in the $n_B = t_B$ case above, Theorem E(i) applies to B in A and to B_1 in A_1, yielding exact (but no longer short exact) sequences $(\Gamma_{B,A})$ and (Γ_{B_1,A_1}). Because B_1 is a compatible extension of B there is a map of complexes $(\Gamma_{B,A}) \to (\Gamma_{B_1,A_1})$, which is an isomorphism by the 5-lemma (Recall that $\tilde{V}_1 = \tilde{V}$ and $\tilde{B}_1 = \tilde{B}$ in $\overline{A_1} = \bar{A}$.) Hence, B_1 is an immediate compatible extension of B.

We have $S_1 \cong M_m(D_1)$, where $m = n_A$ and D_1 is a division ring. Then by (1.5), with appropriate choice of idempotents, $B_1 = M_m(B_1')$ and $S_1 = M_m(D_1)$ where B_1' is a Dubrovin valuation ring of D_1; further, $A_1 = M_m(A_1')$ where $A_1' = B'_{1Q_i}$. We have $\bar{B} = \overline{B_1} = M_m(\overline{B_1'})$, $\Gamma_B = \Gamma_{B_1} = \Gamma_{B_1'}$ by Corollary 3.5, and a commutative diagram (from the compatibility of B_1 with B and with B_1'):

$$\begin{array}{ccccc} \Gamma_B/\Gamma_V & \xrightarrow{=} & \Gamma_{B_1}/\Gamma_{V_1} & \xrightarrow{=} & \Gamma_{B_1'}/\Gamma_{V_1} \\ \downarrow{\scriptstyle \theta_B} & & \downarrow{\scriptstyle \theta_{B_1}} & & \downarrow{\scriptstyle \theta_{B_1'}} \\ \mathscr{G}(Z(\bar{B})/\bar{V}) & \xrightarrow{\cong} & \mathscr{G}(Z(\overline{B_1})/\overline{V_1}) & \xrightarrow{\cong} & \mathscr{G}(Z(\overline{B_1'})/\overline{V_1}) \end{array} .$$

Since $[D_1 : F_1] = [S : F]/m^2$, Theorem B holds for B_1' by the primary induction if $m > 1$. In this case Theorem B follows for B by what we have just seen.

Thus, we may assume $m = 1$, i.e., $S_1 = D_1$. Because W_1 is Henselian, there is an invariant valuation ring of S_1 contracting to W_1; by (3.10) that ring is A_1.

Now, let $Z(\bar{A})_{\text{sep}}$ be the separable closure of $\bar{W}$ in $Z(\bar{A})$, and let $L' \subseteq Z(\bar{A})_{\text{sep}}$ be the decomposition field of $\tilde{B} \cap Z(\bar{A})_{\text{sep}}$ over $\tilde{V}$, so that $[L' : \bar{W}] = \ell_{B,A} > 1$. Just as in the proof of (3.9)(d), since L' is normal over $\bar{W}$ (as $Z(\bar{A})_{\text{sep}}$ is abelian Galois over $\bar{W}$), $\tilde{V}$ has $\ell_{B,A}$ different extensions from $\bar{W}$ to L', each of which extends uniquely to $Z(\bar{A})$. Identify L' with its image in $\bar{A}_1 = \bar{A}$. Since W_1 is Henselian there is a field L, $F_1 \subseteq L \subseteq S_1$, which is an inertial lift of L' over $\bar{W}_1 = \bar{W}$. That is, L is separable over F, with $[L : F_1] = [L' : \bar{W}] = \ell_{B,A}$, and $\bar{W}_L = L'$ in $\bar{A}_1$, where W_L is the valuation ring $A_1 \cap L$. Let S_L be the centralizer of L in S_1, and let $A_L = A_1 \cap S_L$. Then A_L is an invariant valuation ring of S_L with $\bar{A}_L = \bar{A}_1$ by [JW, Lemma 1.8(a)]. Let $B_L = B_1 \cap S_L$, $\tilde{B}_L = B_L/J(A_L)$, $V_L = B_L \cap L$, and $\tilde{V}_L = V_L/J(W_L)$. Then B_L is a Dubrovin valuation ring of S_L, since $B_L = \pi_{A_L}^{-1}(\tilde{B}_1)$. Furthermore, B_1 is a compatible extension of B_L. We also have $[S_L : L] < [S_L : L] \cdot [L : F_1]^2 = [S : F]$. Consequently, by the primary induction, Theorem B holds for B_L. We work back from B_L to B.

As in the proof of (3.9)(d), we may view $(L, V_L) \subseteq (F, V)^H$. So, since $(L, V_L)^H = (F, V)^H = (F^h, V^h)$ we have the same R for B as for B_L in Theorem B. Thus,

$$\bar{B} = \bar{\tilde{B}} = \bar{\tilde{B}}_L = \bar{B}_L \cong M_t(\bar{R}),$$

where $t = t_{B_L} = t_{\tilde{B}_L} = t_{\tilde{B}} = t_B$. As to the value groups, consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_{\tilde{B}_L} & \longrightarrow & \Gamma_{B_L} & \longrightarrow & \Gamma_{A_L} & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & \Gamma_{\tilde{B}_1} & \longrightarrow & \Gamma_{B_1} & \longrightarrow & \Gamma_{A_1} & \longrightarrow & \mathscr{G}(Z(\bar{A}_1)/\bar{W}_1)/H \longrightarrow 0.
\end{array}$$

The rows of the diagram are the exact sequences (Γ_{B_L, A_L}) and (Γ_{B_1, A_1}) given by Theorem E (i). (This theorem applies to B_L in A_L and B_1 in A_1 by Proposition 3.9(b), since Theorem B holds for A_L and A_1 (as W_L and W_1 are Henselian) and Theorem A holds for $\tilde{B}_L = \tilde{B}_1 = \tilde{B}$ by induction.) The top row is short exact because $\tilde{V}_L = \tilde{V}$ has a unique extension to $Z(\bar{A}_1) = L'$. In the bottom row

$$H = \{\tau \in \mathscr{G}(Z(\bar{A}_1)/\bar{W}_1) : \tau(\tilde{B}_1 \cap Z(\bar{A}_1)) = \tilde{B}_1 \cap Z(\bar{A}_1)\} = \mathscr{G}(Z(\bar{A}_1)/L').$$

The diagram is commutative because B_1 is a compatible extension of B_L. Furthermore, [JW, Lemma 1.8 (a)] shows that Γ_{A_L} maps onto $\theta_{A_1}^{-1}(\mathscr{G}(Z(\bar{A}_1)/L')$ in $\Gamma_{A_1}/\Gamma_{V_1}$. Thus, in the diagram above the image of $(\Gamma_{A_L} \to \Gamma_{A_1})$ contains the image of $(\Gamma_{B_1} \to \Gamma_{A_1})$. The diagram therefore shows $\Gamma_{B_1} = \Gamma_{B_L}$. Thus, $\Gamma_B = \Gamma_{B_1} = \Gamma_{B_L} = \Gamma_R$. Finally, (iii) of Theorem B follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
\Gamma_B/\Gamma_V & \longrightarrow & \Gamma_{B_1}/\Gamma_{V_1} & \longrightarrow & \Gamma_{B_L}/\Gamma_{V_L} & \longrightarrow & \Gamma_R/\Gamma_{V^h} \\
\Big\downarrow{\theta_B} & & \Big\downarrow{\theta_{B_1}} & & \Big\downarrow{\theta_{B_L}} & & \Big\downarrow{\theta_R} \\
\mathscr{G}(Z(\bar{B})/\bar{V}) & \longrightarrow & \mathscr{G}(Z(\bar{B}_1)/\bar{V}_1) & \longrightarrow & \mathscr{G}(Z(\bar{B}_L)/\bar{V}_L) & \longrightarrow & \mathscr{G}(Z(\bar{R})/\overline{V^h})
\end{array}$$

The first two squares of the diagram are commutative by the compatibility of B_1 over B and B_1 over B_L. The right square is commutative by Theorem B (iii) for B_L. This completes the proof of Theorem B, completing part IV of the main proof. □

We have now fully proved Theorem A and Theorem B. Thus, Corollary B, Theorem C, Theorem E, and Corollary E follow by Proposition. 3.9. We have proved Theorem D assuming that the Q_i are the jump prime ideals of V re S. This assumption was needed in part IV only to assure that $j(V_{Q_{k-1}}, S) < j(V, S)$, allowing us to apply Theorem B to $A = B_{Q_{k-1}}$ by induction on jump rank. Since we have now proved Theorem B in general, the argument in part IV (now by induction on k instead of jump rank) proves Theorem D as originally stated. We have also proved Proposition 4.1.

5. Proofs of Theorems *F* and *G* Completed

Proof of Theorem G. As we noted after the statement of Theorem G, (i)$\Leftrightarrow$(ii)$\Rightarrow$(iv) and (ii)$\Rightarrow$(iii) were proved by Brungs and Gräter. They also showed in [BG1, Theorem 4] that if the set of elements of S integral over V is a ring, then V extends to a total valuation ring B_1 of S. Since B is conjugate to B_1 by Theorem A, B is also a total valuation ring of S. This yields (iv)$\Rightarrow$(ii) of Theorem G, while (iii)$\Rightarrow$(i) was proved in Sect. 4 above.

It remains only to prove the formula for the number of conjugates when (i)–(iv) hold. We do this by induction on the jump rank. Let c_B be the number of conjugates of B. Note first that if $n_B = 1$, then B is an invariant valuation ring by Theorem 2.2 and Proposition 3.10; so, $c_B = 1 = n_B$. Now assume $n_B > 1 = t_B$. Then $j(V, S) > 1$ by Theorem D. Let P be the smallest jump prime ideal of V re S, and let $A = B_P$. With this P and A define W, $\tilde{V}$, $\tilde{B}$, and Y as at the beginning of Sect. 4. Since $j(W, S) = 1$, $n_A = t_A \leqq t_B = 1$ by Theorem D and Theorem E(ii). Hence $n_A = 1$, so that A is an invariant valuation ring. Therefore, all the conjugates of B lie in A, and are thus determined by their images in $\bar{A}$. That is, c_B equals the number of Dubrovin valuation rings $\tilde{B}_i$ of $\bar{A}$ with $\tilde{B}_i \cap \bar{W} = \tilde{V}$. For any such $\tilde{B}_i$, $\tilde{B}_i \cap Z(\bar{A})$ is one of the $\ell_{B,A}$ extensions of $\tilde{V}$ to $Z(\bar{A})$. If $\tilde{B}_i \cap Z(\bar{A}) = Y = \tilde{B} \cap Z(\bar{A})$, then $\tilde{B}_i$ is a conjugate of $\tilde{B}$ in $\bar{A}$. The number $c_{\tilde{B}}$ of such conjugates is $n_{\tilde{B}}$ by induction. (Note that $j(Y, \bar{A}) < j(V, S)$ by Corollary E. The hypotheses of Theorem G hold for $\tilde{B}$ as $\bar{A}$ is a division ring since A is invariant, and $t_{\tilde{B}} = t_B = 1$.) Since $Z(\bar{A})$ is normal over $\bar{W}$ by Corollary B (or [JW, Proposition 1.7]), $\mathscr{G}(Z(\bar{A})/\bar{W})$ acts transitively on the extensions of $\tilde{V}$ to $Z(\bar{A})$. Each automorphism in $\mathscr{G}(Z(\bar{A})/\bar{W})$ extends to a $\bar{W}$-automorphism of $\bar{A}$ by the surjectivity of θ_A in Corollary B. Hence, each extension of $\tilde{V}$ to $Z(\bar{A})$ is the center of $c_{\tilde{B}}$ Dubrovin valuation rings of $\bar{A}$. Therefore,

$$c_B = c_{\tilde{B}} \cdot \ell_{B,A} = n_{\tilde{B}} \cdot \ell_{B,A} = n_B,$$

as desired, where the last equality is given by Theorem E (iii). $\square$

Proof of Theorem F. We have already proved (vi) $\Rightarrow$ (vii) $\Rightarrow$ (ii) (in Proposition 3.9(f)), (ii) $\Rightarrow$ (vi) (in Proposition 4.1), and (i) $\Rightarrow$ (vi′) (in Sect. 4). Clearly (vi′) $\Rightarrow$ (vi). To complete the proof of Theorem F we now show (ii) $\Rightarrow$ (v) $\Rightarrow$ (i) and (vi′) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (v).

(ii)$\Rightarrow$(v) Suppose B is integral over V, and take any (Dubrovin valuation) rings E, A with $B \subsetneqq E \subsetneqq A \subsetneqq S$. Let $T = E \cap F$, $W = A \cap F$, $\tilde{T} = T/J(W) \subsetneqq \bar{W} \subseteqq \bar{A}$, $\tilde{E} = E/J(A) \subsetneqq \bar{A}$, and $P = J(T) \subsetneqq V$. Because B is integral over V, $E = B_P$ is integral over $T = V_P$. Hence $\tilde{E}$ is integral over $\tilde{T}$, which is a valuation ring of the field $\bar{W}$. Since the valuation ring $\tilde{E}\cap Z(\bar{A})$ is integral over $\tilde{T}$, it is the only extension of $\tilde{T}$ to $Z(\bar{A})$. Thus, $\ell_{E,A}=1$, proving (v).

(v)$\Rightarrow$(i) This is immediate from Theorem D since by hypothesis all the $\ell_i=1$.

(vi')$\Rightarrow$(iv) Let $B^h=B\otimes_V V^h$, which we are assuming is a Dubrovin valuation ring of $S\otimes_F F^h$. We have $S\otimes_F F^h\cong M_n(D^h)$ where $n=n_B$ and D^h is a division ring. Let R be the invariant valuation ring of D^h extending the Henselian valuation ring V^h of F^h. After making a convenient choice of matrix units (and copy of D^h), we may assume $B^h=M_n(R)$. We first prove (iv) for B^h. Every ideal I of B^h has the form $M_n(I')$ where I' is an ideal of R. Then $I'-\{0\}$ is a generating set of I as an ideal of B^h, and $I'-\{0\}\subseteq D^{h*}=\mathrm{st}(R)\subseteq \mathrm{st}(B^h)$.

To prove (iv) for B first note that $\Gamma_{B^h}=\Gamma_B$ since we are assuming B^h is an immediate compatible extension of B. Now, take any ideal I of B. If $\{\beta_i\}\subseteq \mathrm{st}(B^h)$ is a generating set of the ideal $I\otimes_V V^h$ of B^h, write each $\beta_i=b_iu_i$ with $b_i\in \mathrm{st}(B)$ and $u_i\in B^{h*}$. Let I_1 be the ideal of B generated by $\{b_i\}$. Then the ideal of B^h generated by I_1 is $I_1\otimes_V V^h$ but is also $I\otimes_V V^h$. Hence, by Lemma 3.1(b),

$$I=(I\otimes_V V^h)\cap V=(I_1\otimes_V V^h)\cap V=I_1,$$

which proves (iv) for B.

(iv)$\Rightarrow$(iii) Let $I=BbB$. By (iv), I is generated as an ideal by some $\{s_i\}\subseteq \mathrm{st}(B)$. Because the ideals of B are linearly ordered by inclusion (cf. (1.4)), we have $I=\bigcup_i Bs_iB$. So, there is an s_j with $b\in Bs_jB$. Then $I=Bs_jB=Bs_j=s_jB$, as $s_j\in \mathrm{st}(B)$, proving (iii).

(iii)$\Rightarrow$(v) Take any $b\in B$, $b\neq 0$. We have from (iii) $BbB=Bc=dB$ for some c, $d\in B$. Since every nonzero two-sided ideal of B contains a regular element, $c\in S^*$. We have $cB\subseteq dB=Bc$, so $cBc^{-1}\subseteq B$. Since $cBc^{-1}\cap F=B\cap F=V$, the correspondence (1.8) between overrings of cBc^{-1} and localizations implies $cBc^{-1}=B$. Hence, $c\in \mathrm{st}(B)$ and $BbB=Bc=cB$. Likewise, as any nonzero $t\in S$ can be thrown into B by multiplying by a nonzero element of F, BtB can be generated by an element of $\mathrm{st}(B)$ as a cyclic left and right B-module.

Now consider rings E, A with $B\subseteq E\subseteq A\subseteq S$. For any $t\in \mathrm{st}(E)$ we have just seen there is an $s\in \mathrm{st}(B)$, such that $BtB=Bs=sB$. Then, as $tE=EtE$, we have $tE=BtE=BtBE=sBE=sE$, yielding $t=su$ with $u\in E^*$. Thus, $\mathrm{st}(E)=\mathrm{st}(B)\cdot E^*$. Likewise, $\mathrm{st}(A)=\mathrm{st}(B)\cdot A^*=\mathrm{st}(B)\cdot E^*\cdot A^*=\mathrm{st}(E)\cdot A^*$. Let $\tilde{E}=E/J(A)\subseteq \bar{A}$, $T=E\cap F$, $W=A\cap F$, and $\tilde{T}=T/J(W)=\tilde{E}\cap \bar{W}$. By Corollary B, $Z(\bar{A})$ is normal over $\bar{W}$ and θ_A maps $\mathrm{st}(A)$ onto $\mathscr{G}(Z(\bar{A})/\bar{W})$. Hence, $\mathrm{st}(A)$ acts transitively on the set of extensions of $\tilde{T}$ to $Z(\bar{A})$. But in this action every element of $\mathrm{st}(E)\cdot A^*$ sends $\tilde{E}\cap Z(\bar{A})$ to itself. Thus, $\tilde{E}\cap Z(\bar{A})$ is the only extension of $\tilde{T}$ to $Z(\bar{A})$. So, $\ell_{E,A}=1$, proving (v). This completes the proof of Theorem F. □

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