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Classification of Supersingular Abelian Varieties

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0. Introduction

Let k be an algebraically closed field of characteristic $p>0$. An abelian variety X over k is called *supersingular* if $H^1_{\mathrm{cris}}(X/W)$ has Newton slopes all equal to $\frac{1}{2}$, or equivalently, if there is an isogeny $\varrho: E^g \to X$, where E is a supersingular elliptic curve, and $g=\dim(X)$. (This comes from a series of work by Eichler, Oort, Deligne, Shioda and Ogus. See [22, p. 113], [21, p. 35], [23, p. 586], [19, p. 59]). Oda and Oort [18] have studied the classification problem of supersingular abelian varieties in the case when $a(X)=1$ at the crystal level, where $a(X) =\dim_k \mathrm{Hom}(\alpha_p, X)$ [18, p. 595]. The other cases are relatively special but much more complicated.

In this paper we first study the classification problem of all cases at the crystal level (Sect. 1). This solves a problem left open by Oda and Oort [18]. Then, in Sect. 2, we study the fine moduli problem. Two kinds of additional structures are given to the families of supersingular abelian varieties so that they have fine moduli. The proofs are constructive.

In Sect. 3, we study what appears to be a difficult question: How can we recover a family of supersingular abelian varieties from its crystalline cohomology? This is solved by virtue of the so-called "α-sheaf".

In Sect. 4 we deal with the problem of how, for an arbitrary family of supersingular abelian varieties, we can modify it so that it has one of the additional structures of Sect. 2 (and hence is induced by a morphism to the fine moduli). For one of the structures, namely the "level structure", this is solved by Ogus.

Finally, we study the coarse moduli problem in Sect. 5. We prove that for any set of integer invariants, there is a coarse moduli space of supersingular abelian varieties having that set of integer invariants. As an example, we calculate the number of points in Oda-Oort's space which correspond to one isomorphism class of abelian varieties.

This paper contains the main part of the dissertation work directed by Professor Arthur Ogus, as well as some unpublished results of Ogus.

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1. Supersingular Dieudonné Crystals

We use the notation of [18].

Let $W=W(k)$ be the Witt ring of k, and $A=W[F,V]$ be the associative ring satisfying the following defining relations:

i) $FV=VF=p$;

ii) $Fa=a^{\sigma}F$, $Va=a^{\sigma^{-1}}V$ for all $a\in W$, where σ is the absolute Frobenius map.

Definition. A *Dieudonné crystal* is a left A-module which is free of finite rank as a W-module.

Let $A_{1,1}=A/A(F-V)$. A finite direct sum of (say g copies of) $A_{1,1}$ is called a *superspecial* Dieudonné crystal (of genus g). A Dieudonné crystal M is called *supersingular* (of genus g) if it is isomorphic to an A-submodule of a superspecial Dieudonné crystal (of genus g) of finite colength. A *trace map* [19, Sect. 6] of M is an isomorphism $\mathrm{tr}: \wedge^{2g}M\to W[-g]$ of Dieudonné crystals, where $W[-g]$ is free of rank 1 over W generated by some element x such that $Fx=Vx=p^g x$.

Clearly the isogeny ϱ in Sect. 0 induces a supersingular Dieudonné crystal structure on $H^1_{\mathrm{cris}}(X/W)$, and it has a canonical trace map coming from $\wedge^{2g}H^1_{\mathrm{cris}}(X/W)\simeq H^{2g}_{\mathrm{cris}}(X/W)$. If $g>1$, $H^1_{\mathrm{cris}}(X/W)$ is superspecial iff $X\simeq E^g$ for any supersingular elliptic curve E (cf. [23, p. 586]). In this case we also say that X is *superspecial*. Ogus proved the following Torelli theorem [19, Theorem 6.2].

Theorem. *If $g>1$, then the functor $(H^1_{\mathrm{cris}}, \mathrm{tr})$ defines a bijection*

$$\frac{\left\{\begin{array}{c}\textit{supersingular abelian varieties}\\ \textit{of dimension } g \textit{ over } k\end{array}\right\}}{\textit{isomorphisms}} \xrightarrow{\sim} \frac{\left\{\begin{array}{c}\textit{supersingular Dieudonné crystals}\\ \textit{of genus } g \textit{ together with a trace map}\end{array}\right\}}{\textit{isomorphisms}}$$

furthermore, for any two supersingular abelian varieties X, Y of dimension g, there is a canonical isomorphism

$$\mathrm{Hom}(Y,X)\otimes_{\mathbb{Z}}\mathbb{Z}_p\xrightarrow{\sim}\mathrm{Hom}_{A,\mathrm{tr}}(H^1_{\mathrm{cris}}(X/W),H^1_{\mathrm{cris}}(Y/W)).$$

The following lemmas are purely elementary, and the proofs are left to the reader.

Lemma 1.1. *Suppose that M is a supersingular Dieudonné crystal, and $N\subset M$ is an A-submodule such that $N+(F,V)M=M$. Then $N=M$.*

Let $\dot{M}=Ax_1\oplus Ax_2\oplus\ldots\oplus Ax_g$ be a superspecial Dieudonné crystal of genus g, where $Ax_i\simeq A_{1,1}$, i.e., the annihilator of x_i is $A(F-V)(1\leqq i\leqq g)$. To give an endomorphism of $\dot{M}$ (as an A-module) is equivalent to giving $x_1',\ldots,x_g'\in\dot{M}$ such that $(F-V)x_i'=0$ for all i, $1\leqq i\leqq g$. Let $x\in\dot{M}$. Then we can uniquely write

$$x=(a_1+b_1F)x_1+(A_2+b_2F)x_2+\ldots+(a_g+b_gF)x_g,$$

where $a_1,\ldots,a_g$, $b_1,\ldots,b_g\in W$. Hence $(F-V)x=0$ if and only if

$$0=(F-V)x=\sum_{i=0}^{g}(p(b_i^{\sigma}-b_i^{\sigma^{-1}})+(a_i^{\sigma}-a_i^{\sigma^{-1}})F)x_i,$$

i.e., $b_1^{\sigma}=b_1^{\sigma^{-1}}$, $a_1^{\sigma}=a_1^{\sigma^{-1}}$, or $a_i,b_i\in W(\mathbb{F}_{p^2})$ $(1\leqq i\leqq g)$.

Let $H = W(\mathbb{F}_{p^2})[F]/(F^2 - p, Fa - a^\sigma F, a \in W(\mathbb{F}_{p^2}))$. Then it is easy to check that H is a generalized quaternion algebra [7, Sect. 2.4] over $W(\mathbb{F}_p)$. Furthermore, there is an anti-automorphism $\phi: H \to H$ given by $\phi(a+bF) = a^\sigma - bF$. This gives a "*norm*" map $||: H \to W(\mathbb{F}_p)$ by

$$|a+bF| = (a+bF)\phi(a+bF) = aa^\sigma - pbb^\sigma.$$

Clearly $|\alpha\beta| = |\alpha| \cdot |\beta|$ for $\alpha, \beta \in H$.

Lemma 1.2. *The map* $||: H \to W(\mathbb{F}_p)$ *is surjective.*

Now $\tilde{M} \overset{\text{def}}{=} \ker(F - V: \dot{M} \to \dot{M})$ can be viewed as an H-module:

$$\tilde{M} = Hx_1 \oplus Hx_2 \oplus \ldots \oplus Hx_g.$$

Clearly $\dot{M} \simeq W \otimes_{W(\mathbb{F}_{p^2})} \tilde{M}$. Hence we say that $\tilde{M}$ is the "*skeleton*" of $\dot{M}$.

Let v be the p-adic valuation map on $K(\mathbb{F}_{p^2})$ [the quotient field of $W(\mathbb{F}_{p^2})$]. Then it is clear that $v(|a+bF|) = \min(2v(a), 2v(b)+1)$ for $a, b \in K(\mathbb{F}_{p^2})$. If

$$\alpha, \beta \in H \otimes_{W(\mathbb{F}_{p^2})} K(\mathbb{F}_{p^2}),$$

then clearly $\alpha\beta^{-1} \in H$ if and only if $v(|\alpha|) \geqq v(|\beta|)$. From this one can easily deduce that any H-submodule of a free H-module is again free, using an argument similar to that for modules over a DVR.

If $\dot{M}_1, \dot{M}_2$ are two superspecial A-submodules of $\dot{M}$, then clearly the corresponding skeletons $\tilde{M}_1, \tilde{M}_2$ are H-submodules of $\tilde{M}$. Hence

$$\dot{M}_1 + \dot{M}_2 \simeq (\tilde{M}_1 + \tilde{M}_2) \otimes_{W(\mathbb{F}_{p^2})} W \quad \text{and} \quad \dot{M}_1 \cap \dot{M}_2 \simeq (M_1 \cap M_2) \otimes_{W(\mathbb{F}_{p^2})} W$$

are superspecial because $\tilde{M}_1$ and $\tilde{M}_2$ are free H-modules. Furthermore, if $\dot{M}_1$ and $\dot{M}_2$ are of genus g, then $\tilde{M}_1$ and $\tilde{M}_2$ are of rank g over H, and so is $\tilde{M}_1 \cap \tilde{M}_2$. Hence $\dot{M}_1 \cap \dot{M}_2$ is of genus g also. Therefore we get (cf. [12, Theorem 3.1])

Lemma 1.3. *The sum and intersection of two superspecial subcrystals of $\dot{M}$ are again superspecial. If M is an arbitrary subcrystal of $\dot{M}$ of finite colength, then there is a smallest superspecial subcrystal $S'(M)$ containing M. Furthermore, there is a largest superspecial subcrystal (of genus g) $S_0(M)$ contained in M.*

Note that $S'(M)$ and $S_0(M)$ do not depend on $\dot{M}$. They have minimality and maximality respectively in $M \otimes_W K$, where K is the quotient field of $W(k)$.

For a $g \times g$ non-singular matrix over H, we can still define its "determinant" as in [7, Sect. 2.1]. There the "determinant" of a non-singular matrix is an element of $H-(0)/C$, where C is the commutator subgroup of H^*. In our case, $||$ clearly factors through $H-(0)/C$. Hence we can define "*determinant*" to be an element in $W(\mathbb{F}_p)$, i.e., it is 0 if the rows are linearly dependent, and for a non-singular matrix, if the "determinant" in the sense of [7] is αC, $\alpha \in H-(0)$, then we define our determinant to be $|\alpha|$.

For a matrix $T = (a_{ij} + b_{ij}F)$ over H corresponding to an endomorphism of $\dot{M}$, we can rewrite it as a W-linear transformation with respect to the basis

$x_1, Fx_1, \ldots, x_g, Fx_g$ of $\dot{M}$ over $W(\mathbb{F}_{p^2})$ as follows

$$\tilde{T}=\begin{pmatrix} a_{11} & b_{11} & \cdots & a_{1g} & b_{1g} \\ pb_{11}^{\sigma} & a_{11}^{\sigma} & \cdots & pb_{1g}^{\sigma} & a_{1g}^{\sigma} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{g1} & b_{g1} & \cdots & a_{gg} & b_{gg} \\ pb_{g1}^{\sigma} & a_{g1}^{\sigma} & \cdots & pb_{gg}^{\sigma} & a_{gg}^{\sigma} \end{pmatrix}$$

It is not hard to check that the determinant of T is just equal to $\det(\tilde{T})$. Note that H is isomorphic to the algebra of the matrices of the form $\begin{pmatrix} a & b \\ pb^{\sigma} & a^{\sigma} \end{pmatrix}$, $a, b \in W(\mathbb{F}_{p^2})$.

A matrix over H is invertible if and only if its determinant is not in $pW(\mathbb{F}_p)$. An automorphism of $\dot{M}$ preserves a given trace map if and only if the corresponding matrix over H has determinant 1. In fact,

$$\mathrm{tr} \circ \Lambda^{2g}(T) = |T| \cdot \mathrm{tr} : \Lambda^{2g}\dot{M} \to W[-g].$$

Lemma 1.4. *Let $a_1, \ldots, a_g \in k$. Then the following are equivalent.*

i) *$a_1, \ldots, a_g$ are linearly independent over $\mathbb{F}_{p^2}$;*

(ii)
$$D(a_1, \ldots, a_g) \overset{\text{def.}}{=} \begin{vmatrix} a_1 & \cdots & a_g \\ a_1^{p^2} & \cdots & a_g^{p^2} \\ a_1^{p^4} & \cdots & a_g^{p^4} \\ \vdots & \ddots & \vdots \\ a_1^{p^{2g-2}} & \cdots & a_g^{p^{2g-2}} \end{vmatrix} \neq 0.$$

In fact, $D(y_1, \ldots, y_g) = (-1)^{g(g-1)/2} \prod_{\substack{1 \leqq i \leqq g \\ \lambda_1, \ldots, \lambda_g \in \mathbb{F}_{p^2}}} (y_i + \lambda_{i+1} y_{i+1} + \ldots + \lambda_g y_g)$.

In the following, we will denote by $\bar{a}$ the image of $a \in W$ under $W \to W/pW \simeq k$.

Lemma 1.5. *Let*

$$v = (a_1 + b_1 F)x_1 + \ldots + (a_g + b_g F)x_g \in \dot{M}.$$

Then $\dot{M} = S'(Av)$ if and only if $D(\bar{a}_1, \ldots, \bar{a}_g) \neq 0$. And in this case, $F^{g-1}\dot{M} = (F, V)^{g-1}v$.

Lemma 1.6. *Let M be an arbitrary supersingular Dieudonné crystal. Then there exists $v \in M - FS'(M)$ such that $S'(Av) = S'(M)$.*

For an arbitrary M, let

$$S(M) = \{v \in M \otimes_W K \mid (F, V)^{g-1} v \subset M\}, \qquad S_0'(M) = (F, V)^{g-1} M.$$

By Lemma 1.5 and 1.6, there exists $v \in S(M)$ such that

$$(F, V)^{g-1} v = F^{g-1} S'(S(M)) \subset M.$$

Hence $S'(S(M)) \subset S(M)$, i.e., $S(M)$ is superspecial.

Corollary 1.7. *The relations $F^{g-1}S(M)=S_0(M)$, $F^{g-1}S'(M)=S'_0(M)$ hold.*

We will call $S(M)$ the *level structure* of M, and $S'(M)$ the *colevel structure* of M.

Remark 1.8. The dual W-module $M^\vee=\mathrm{Hom}_W(M,W)$ has a natural A-module structure. Namely, if $m\in M$, $f\in M^\vee$, then $(Ff)(m)=f(Vm)^\sigma$, $(Vf)(m)=f(Fm)^{\sigma^{-1}}$ [17, p. 83]. It is clear that $S(M)^\vee=S'_0(M^\vee)$, $S'(M)^\vee=S_0(M^\vee)$, as W-submodules of $\mathrm{Hom}_W(M,K)$.

Let $M^i=M\cap F^iS'(M)$. Then $F:M^i\to M^{i+1}$ induces a map

$$\bar{F}:\bar{M}^i=M^i/M^{i+1}\to\bar{M}^{i+1}=M^{i+1}/M^{i+2}.$$

Similarly V induces a map $\bar{V}:\bar{M}^i\to\bar{M}^{i+1}$. It is clear that $\bar{F}$ (resp. $\bar{V}$) is injective because it is induced by a bijection from $F^iS'(M)/F^{i+1}S'(M)$ to $F^{i+1}S'(M)/F^{i+2}S'(M)$.

Lemma 1.9. *For a given i,* $\mathrm{Im}\,\bar{F}\neq\mathrm{Im}\,\bar{V}:\bar{M}^i\to\bar{M}^{i+1}$ *and hence* $\dim_k\bar{M}^i<\dim_k\bar{M}^{i+1}$, *unless* $\dim_k\bar{M}^i=g$.

Corollary 1.10. *If* $M=Av$, *then* $S(M)=S'(M)$.

Conversely, if M is not generated by one vector, then $S(M)\neq S'(M)$. This is clear because $\dim_k\bar{M}^{g-2}=g$ in this case, hence $M^{g-2}\subseteqq S'_0(M)$ is already superspecial.

We will use the following notation:

$s'_i=\dim_k(M\cap F^{i-1}S'(M)/M\cap F^iS'(M))\quad(1\leqq i\leqq g)$,

$t'_i=\dim_k((F,V)M\cap F^iS'(M)/(F,V)M\cap F^{i+1}S'(M))\quad(1\leqq i\leqq g-1)$,

$s_i=\dim_k(M\cap F^{i-1}S(M)/M\cap F^iS(M))\quad(1\leqq i\leqq g)$,

$t_i=\dim_k(F^{-1}M\cap V^{-1}M\cap F^{i-1}S(M))/(F^{-1}M\cap V^{-1}M\cap F^iS(M))\quad(1\leqq i\leqq g-1)$.

Clearly s'_i, t'_i, s_i, t_i are integer invariants satisfying the following inequalities and equalities:

i) $1\leqq s'_i\leqq s'_{i+1}$, $s'_g=g$, and $s'_i<s'_{i+1}$ unless $s'_i=g$;
ii) $s'_i\leqq t'_i\leqq s'_{i+1}$;
iii) $s_i\leqq s_{i+1}$, $s_g=g$, and $s_i<s_{i+1}$ unless $s_{i+1}=0$;
iv) $s_i\leqq t_i\leqq s_{i+1}$;
v) $s_i(M)+s'_{g-1}(M^\vee)=g$;
vi) $a(M)=\sum_{i=1}^{g}s'_i-\sum_{i=1}^{g-1}t'_i$, where $a(M)=\dim_k(M/(F,V)M)$ [18, p. 598], and so on.

A sequence of integers $(s_1,\dots,s_g)=s$ will be called an *index* if $0\leqq s_1\leqq\dots\leqq s_g=g$, and $s_i<s_{i+1}$ unless $s_{i+1}=0$.

Proposition 1.11. *For any supersingular Diedudonné crystal M, there is a canonical filtration* $M^i=M\cap F^i\dot{M}$ $(0\leqq i\leqq g-1)$, *where* $\dot{M}=S'(M)=Ax_1\oplus\dots\oplus Ax_g$ *is superspecial. Then* $\bar{M}^i=M^i/M^{i+1}$ *can be viewed as a k-vector subspace of* $F^i\dot{M}/F^{i+1}\dot{M}$. *F (resp. V) induces an injective σ-linear (resp.* σ^{-1}*-linear) homomorphism* $\bar{M}^i\to\bar{M}^{i+1}$, *and there is* $v\in M-F\dot{M}$ *such that* $\dot{M}=S'(Av)$. *Conversely, given a superspecial crystal* $\dot{M}=Ax_1\oplus\dots\oplus Ax_g$, *and a k-linear subspace* $\bar{M}^i$ *of* $F^i\dot{M}/F^{i+1}\dot{M}$ *for each i* $(0\leqq i\leqq g-1)$ *such that* $(\bar{F},\bar{V})\bar{M}^i\subseteqq\bar{M}^{i+1}$, *and such that* $\bar{M}^0$ *contains a vector*

$(\bar{a}_1, \ldots, \bar{a}_g)$ *satisfying* $D(\bar{a}_1, \ldots, \bar{a}_g) \neq 0$, *there is a unique supersingular Dieudonné crystal* $M \subsetneq \dot{M}$ *such that* $S'(M) = \dot{M}$ *and* $M^i/M^{i+1} = \bar{M}^i$ *as* k-*subspaces of* $F^i\dot{M}/F^{i+1}\dot{M}$.

2. Flag Type Level Structures

We first fix some notation on group schemes. For the basic facts of group schemes, see [4–6] or [14].

Let S be a scheme of finite type over k. Let $\pi: G \to S$ be a group scheme with multiplication $m: G \times_S G \to G$, zero section $o: S \to G$ and inverse $\iota: G \to G$, all being morphisms over S. We will always assume that π is separated, hence o is a closed immersion. Let $\mathcal{M}$ be the ideal sheaf of o. Let $\omega_{G/S} = o^*\mathcal{M}$. Then canonically $\Omega^1_{G/S} \simeq \pi^*\omega_{G/S}$, and

$$\mathrm{Lie}(G/S) \simeq \mathcal{H}om_S(\omega_{G/S}, O_S) \simeq \omega^\vee_{G/S},$$

where $\mathrm{Lie}(G/S)$ is the sheaf of (left) invariant derivations.

We say $G \to S$ is an *abelian scheme* if π is flat and proper with geometrically integral fibers. It is called *supersingular* if each of its closed fibers over S is a supersingular abelian variety.

For convenience, we will use the following notation. For any scheme $\tau: X \to S$, and any base change $\sigma: T \to S$, denote $X \times_\sigma T = X \times_S T$ in order to emphasize σ. In particular, if $\sigma = F^n$, where $F: S \to S$ is the Frobenius map, we write $X^{(p^n)} = X \times_\sigma S$.

If π is flat, finite and commutative, then we can define its *Cartier dual* $G^\vee$ (see [14, III.14]). The *relative Frobenius map* $F_{G/S}: G \to G^{(p)}$ is the unique morphism such that $\mathrm{pr}_1 \circ F_{G/S}$ is the absolute Frobenius map of G, where pr_1 is the first projection of $G^{(p)} = G \times_F S$. The *Verschiebung map* $V_{G/S}: G^{(p)} \to G$ is defined by the dual of

$$F_{G^\vee/S}: G^\vee \to (G^\vee)^{(p)} \simeq (G^{(p)})^\vee .$$

We have [6, p. 29]

Lemma 2.1. *The relations* $F_{G/S} \circ V_{G/S} = p \cdot \mathrm{id}_{G^{(p)}}$, $V_{G/S} \circ F_{G/S} = p \cdot \mathrm{id}_G$ *hold.*

Lemma 2.2 [14, p. 138]. *There is a canonical isomorphism of sheaves of* O_S-*modules*

$$\mathcal{H}om_S(G, G_{a/S}) \xrightarrow{\sim} \mathrm{Lie}(G^\vee/S).$$

The left hand side is isomorphic to the subsheaf of O_S of sections t such that $m^*(t) = t \otimes 1 + 1 \otimes t$, $o^*(t) = 0$, which will be called the α-*sheaf* of G.

Let $X \to S$ be a scheme of finite type. By a G-*action* on X we mean an S-morphism $\varrho: G \times_S X \to X$ such that

i) $\varrho \circ (m \times_S \mathrm{id}_X) = \varrho \circ (\mathrm{id}_G \times_S \varrho): G \times_S G \times_S X \to X$;

ii) $\varrho \circ (o \times_S \mathrm{id}_X) = \mathrm{id}_X: X \to X$, via $X \simeq S \times_S X$.

The action ϱ is called *free* if $(\varrho, \mathrm{pr}_2): G \times_S X \to X \times_S X$ is a closed immersion. Furthermore, ϱ is called *affine* if there exists an open affine covering $\{U_i, i \in I\}$ of S, and an open affine covering $\{V_{ij}, j \in J_i\}$ of $f^{-1}(U_i)$ for each $i \in I$, such that $\varrho(G \times_S V_{ij}) = V_{ij}$ for every pair $i \in I, j \in J_i$.

Lemma 2.3 [14, p. 111]. *Suppose that $G\to S$ is flat and finite and that ϱ is affine. Then ϱ has a scheme-theoretic quotient $\tau: X\to Y$ (i.e., τ is the universal morphism such that $\tau\circ\varrho=\tau\circ \mathrm{pr}_2: G\times_S X\to Y$), where τ is finite and Y is of finite type over S. Furthermore, if ϱ is free, then τ is flat, and the quotient commutes with base change of S.*

Let $\mathscr{G}_S$ be the category of group schemes over S whose morphisms are homomorphisms. Then S is a beginning and ending object of $\mathscr{G}_S$. The kernel and cokernel are defined in the categorical sense. The kernel always exists: $\ker(f: G\to G')$ is the pullback of f and $o_G: S\to G'$. And Lemma 2.3 gives

Corollary 2.4. *Suppose H is a finite, flat, closed subgroup scheme of a commutative group scheme G of finite type over S. If the action of H on G via m is affine, then there exists a quotient group scheme G/H.*

Example. Suppose $f: G\to G'$ is an isogeny (i.e., surjective and quasi-finite) of abelian schemes. Then for any flat subgroup scheme H of $\ker(f)$, the action of H on G is affine. Indeed, by [8, p. 136], f is finite and flat, and H fixes $f^{-1}(U)$ for any open affine $U\subseteqq G'$.

Definition. A finite, flat group scheme $G\to S$ is called an *α-group* over S if $F_{G/S}=0$, $V_{G/S}=0$.

The following fact is well known, and the proof is left to the reader.

Lemma 2.5. *If G is an α-group over S, then locally over S, G is isomorphic to $S\times\left(\alpha_p\times\overset{n}{\dots}\times\alpha_p\right)$, where $\alpha_p\simeq \mathrm{Spec}\, k[t]/(t^p)$ with the group scheme structure given by $m^*(t)=t\otimes 1+1\otimes t$, $\iota^*(t)=-t$, $o^*(t)=0$.*

The above n, if it is a constant, is called the *α-rank* of G. It is equal to $\mathrm{rank}_{O_S}(\omega_{G/S})$ (Note that for any flat α-group G, $\omega_{G/S}$ is flat). Furthermore, $\mathrm{rank}_{O_S}(\pi_* O_G)=p^n$. Lemma 2.5 also shows that $\omega_{G/S}$ is canonically isomorphic to the α-sheaf of G. Therefore we obtain

Corollary 2.6. *There is an anti-equivalence of categories*

$$((\text{flat coherent sheaves of } O_S\text{-modules}))\leftrightarrow((\alpha\text{-groups over } S))$$
$$\text{the } \alpha\text{-sheaf of } G \leftarrow\!\shortmid G$$
$$\mathscr{F}\mapsto \mathrm{Spec}(\mathrm{Sym}(\mathscr{F})/\mathscr{F}^{[p]})$$

compatible with the functor $^\vee$, *where $\mathscr{F}^{[p]}$ is the ideal of* $\mathrm{Sym}(\mathscr{F})$ *generated by the* p^{th} *powers of the sections of $\mathscr{F}$. (If f is a section of $\mathscr{F}$, then $m^*(f)=f\otimes 1+1\otimes f$.)*

Remark 2.7. Suppose G is a finite commutative group scheme over S such that $F_{G/S}=0$. Then $\omega_{G/S}$ is flat iff $G\to S$ is flat. In this case, the canonical map from the α-sheaf of $G^\vee$ to $\omega_{G^\vee/S}$ is surjective.

For any supersingular abelian variety X of dimension g over k, Sect. 1 tells us that $H^1_{\mathrm{cris}}(X/W)$ has a natural filtration, defining an index $s=(s_1,\dots,s_g)$ [and also $t=(t_1,\dots,t_{g-1})$], which is a geometric invariant. This inspires us to make the following definition. We fix a supersingular elliptic curve E because the isomorphism class of E^g is independent of the choice of E when $g>1$, as we have seen in Sect. 1.

Definition. *A flag type level structure* of index s over S is a supersingular abelian scheme $A \to S$ together with isogenies (of supersingular abelian schemes) $\varrho_i: A_i^{(p)} \to A_{i-1} (1 \leqq i \leqq g-1)$ over S such that

i) $A_{g-1} = E^g \times S$, $A_0 = A$;

ii) $\ker \varrho_i$ is a flat α-group of α-rank s_i.

For a noetherian scheme S, let $\mathscr{C}_S = $((schemes over S)). We need the following lemma to construct the moduli spaces of supersingular abelian schemes with level structure.

Lemma 2.8. *Let $\mathscr{F}$ be a locally free sheaf of O_S-modules of* rank r. *Let $\mathscr{F}'$ be a coherent O_S-submodule of $\mathscr{F}$. Then the functor $P: \mathscr{C}_S \to$((sets)) defined by*

$$P(\tau: X \to S) = \{\textit{locally free quotient } O_S\textit{-modules } \mathscr{E} \textit{ of } \tau^*\mathscr{F} \textit{ of } \mathrm{rank}\, n\}$$

is represented by a relative Grassmannian $\mathbf{G}$. *Furthermore, the following hold.*

i) *Let $Q: \mathscr{C}_S \to$(sets)) be defined by*

$$Q(\tau: X \to S) = \{\textit{locally free quotients } h: \tau^*\mathscr{F} \twoheadrightarrow \mathscr{E} \textit{ of } \mathrm{rank}\, n \textit{ such that } \tau^*\mathscr{F}' \subseteqq \ker h\}$$

Then Q is represented by a closed subscheme $\mathbf{G}'$ of $\mathbf{G}$.

ii) *If $\mathscr{F} = \pi_* O_G$, where $\pi: G \to S$ is a finite flat group scheme, then the functor*

$$R: \mathscr{C}_S \to (sets)),$$

$$(X \to S) \mapsto \{\textit{flat closed subgroup schemes of } X \times_S G \textit{ over } X \textit{ of } \mathrm{rank}\, n\}$$

is represented by a closed subscheme $\mathbf{G}_G$ of $\mathbf{G}$.

iii) *Therefore, if $\mathscr{F}'$ defines a closed subgroup scheme G' of G (not necessarily flat), then the functor*

$$R': \mathscr{C}_S \to ((sets)),$$

$$(X \to S) \mapsto \{\textit{flat closed subgroup schemes of } X \times_S G' \textit{ over } X \textit{ of } \mathrm{rank}\, n\}$$

is represented by $\mathbf{G}' \cap \mathbf{G}_G (= \mathbf{G}' \times_{\mathbf{G}} \mathbf{G}_G)$.

Proof. [15, p. 32] gives the universality of the (absolute) Grassmannian, and the relative case comes from abstract nonsense. Then we need to check that Q, R are determined by algebraic conditions respectively. This is boring but without any difficulties. The last statement is just abstract nonsense. Q.E.D.

For an (arbitrary) abelian scheme A over S, we can define its *Verschiebung map* as follows. Since $F_{A/S}: A \to A^{(p)}$ is flat [8, p. 136], $\cdot p: A \to A$ is also flat, and since $\ker(F_{A/S}) \hookrightarrow \ker(\cdot p)$, there exists a unique $V: A^{(p)} \to A$ such that $V \circ F = \cdot p$. When $S = \mathrm{Spec}\, k$, *this coincides with the usual definition* of V, as the dual of $F_{A^\vee/S}$, where $A^\vee$ is the dual of A [14, Sect. 13].

Theorem 2.9. *Given an index s (see Sect. 1), the functor*

$$T_s: ((k\textit{-schemes})) \to ((sets)),$$

$$X \mapsto \{\textit{flag type level structures of index } s \textit{ over } X\}/\textit{isomorphisms}$$

is represented by a projective scheme S_s over k.

Proof. We use inverse induction to construct projective schemes $S_{g-1}, \ldots, S_0$ together with supersingular abelian schemes $\pi_i: B_i \to S_i$ $(0 \leqq i < g)$. Take $S_{g-1} = \mathrm{Spec}(k)$, $B_{g-1} = E^g$. Given $\pi_i: B_i \to S_i$, let

$$K_1 = \ker(F_{B_i^{(p)}/S_i}: B_i^{(p)} \to B_i^{(p^2)}), \qquad K_2 = \ker(V_{B_i/S_i}: B_i^{(p)} \to B_i),$$
$$K = K_1 \cap K_2 (= K_1 \times_{B_i^{(p)}} K_2).$$

Then K is a closed subgroup scheme of a flat group scheme K_1. Note that a flat finite closed subgroup scheme of $B_i^{(p)}$ is an α-group if and only if it is a subgroup scheme of K. Now by Lemma 2.8, the functor

$$R: \mathscr{C}_{S_i} \to ((\mathrm{sets})),$$
$$(X \to S_i) \mapsto \{\text{flat closed subgroup schemes of } X \times_{S_i} K \text{ of } \alpha\text{-rank } s_i\}$$

is represented by a relatively projective morphism $S_{i-1} \to S_i$, and a flat closed subgroup scheme G of $S_{i-1} \times_{S_i} K$ of α-rank s_i. Then let $B_{i-1} = B_i^{(p)} \times_{S_{i-1}} / G$.

Clearly $S_s = S_0$ is projective over k. We claim that S_s represents T_s.

Suppose for $S \to \mathrm{Spec}(k)$, a flag type level structure

$$\{A_i \to S,\ \varrho_i: A_i^{(p)} \to A_{i-1}\ (0 \leqq i < g)\}$$

is given. Let $\sigma_{g-1}: S \to S_{g-1} = \mathrm{Spec}(k)$ be the (unique) trivial morphism. Then $A_{g-1} \simeq B_{g-1} \times_{S_{g-1}} S$. Given $\sigma_i: S \to S_i$ such that $A_i \simeq B_i \times_{S_i} S$, by Lemma 2.8 there exists a unique $\sigma_{i-1}: S \to S_{i-1}$ such that $G \times_{S_{i-1}} S = \ker(\varrho_i: A_i^{(p)} \to A_{i-1})$ as subgroup schemes of $K \times_{S_i} S$. Hence

$$\begin{aligned} A_{i-1} &\simeq A_i^{(p)}/G \times_{S_{i-1}} S \simeq B_i \times_{\sigma_i \circ F_S} S/G \times_{S_{i-1}} S \simeq B_i \times_{F \circ \sigma_i} S/G \times_{S_{i-1}} S \\ &\simeq B_i^{(p)} \times_{\sigma_i} S/G \times_{\sigma_{i-1}} S \simeq (B_i^{(p)} \times_{S_i} S_{i-1}) \times_{\sigma_{i-1}} S/G \times_{\sigma_{i-1}} S \\ &\simeq B_{i-1} \times_{S_{i-1}} S. \end{aligned}$$

Thus $A_0 \simeq B_0 \times_{S_0} S$. To see the uniqueness of the morphism $S \to S_s$, note that for $0 \leqq i < g$, $S \to S_i$ is the composite of $S \to S_s$ and $S_s \to S_i$, and that the above argument shows the uniqueness of $S \to S_i$ inductively. Q.E.D.

Remark 2.10. For a supersingular abelian scheme $A \to S$, a "*level structure*" means an isogeny $f: E^g \times S \to A$ over S such that $\ker(f) \subset \ker(F^{g-1})$. Hence $\deg(f)$ is a power p^n, and $n \leqq g(g-1)$. It is clear that the functor

$$\hat{T}_k: \mathscr{C}_k \to ((\mathrm{sets})),$$
$$S \mapsto \{\text{level structures of degree } p^n \text{ over } S\}/\text{isomorphisms}$$

is represented by a projective scheme $\hat{S}_n$.

3. Approaches via Crystals

Let M be a crystal on S such that $pM = 0$. Then $F_{S/W}^* M$ is canonically determined by M_S, the value of M on the PD-thickening $(S, 0)$, where $F_{S/W}$ is the Frobenius map. Indeed, in this case, we need only consider PD-thickenings $U \hookrightarrow T$ such that

$pO_T=0$. Then there is a unique σ_T making the following diagram commutative:

$$\begin{tikzcd} U \arrow[r] \arrow[d, "F_U"'] & T \arrow[d, "F_T"] \arrow[dl, "\sigma_T"'] \\ U \arrow[r] & T \end{tikzcd}$$

Hence $F_T^* M_T \simeq \sigma_U^* M_U$. Furthermore, given any coherent sheaf $\mathscr{E}$ on S, one can obtain a crystal (via σ) on S which is killed by p [16, Sect. 4]. Such a crystal is called *degenerate*. Thus we get a right exact functor

$$((\text{coherent sheaves on } S)) \to ((\text{crystals on } S))$$

denoted by σ^*, by abuse of notation.

By an (F, V)-crystal on S we will mean a crystal M together with morphisms $F_M: F_{S/W}^* M \to M$ and $V_M: M \to F_{S/W}^* M$ such that $F_M \circ V_M = p \cdot \mathrm{id}_M$, $V_M \circ F_M = p \cdot \mathrm{id}_{F_{S/W}M}$.

Definition. A *crystal on S with flat type level structure* of index s consists of the following data.

i) (F, V)-crystals M_i $(0 \leqq i < g)$, such that

a) $M_{g-1} \simeq \varrho^* \dot{M}$, where $\varrho: S \to \mathrm{Spec}(k)$ is the projection and $\dot{M}$ is defined in Sect. 1; and

b) M_{i-1} is an (F, V)-subcrystal of $F_{S/W}^* M_i$ containing $\mathrm{Im}(V_{M_i})$ and $\mathrm{Im}(F_{F_{S/W}^* M_i})$;

ii) Flat coherent sheaves of O_S-modules N_i of rank s_i together with epimorphisms $h_i: (M_i/\mathrm{Im}(F_{M_i}))_S \twoheadrightarrow N_i$ and isomorphisms $n_i: F_{S/W}^* M_i/M_{i-1} \xrightarrow{\sim} \sigma^* N_i$.

For a group scheme G over S, we will denote its Dieudonné module by $\mathbf{D}(G)$. We need to quote [1, Sect. 4.3] as the following lemma.

Lemma 3.1. *Let G be a flat group scheme over S. If $F_{G/S}=0$, then canonically $\mathbf{D}(G) \simeq \sigma^*(\omega_{G/S})$. If $V_{G/S}=0$, then canonically $\mathbf{D}(G) \simeq \sigma^*(\mathrm{Lie}(G^\vee/S))$.*

Theorem 3.2. *By taking Dieudonné modules, there is a fully faithful functor $\mathbf{D}_S$ from the category of isomorphism classes of flag type level structures of index s over S to the category of isomorphism classes of crystals with flag type level structures of index s on S. If σ^* is faithful, then $\mathbf{D}_S$ is a natural equivalence.*

Proof. Look at the definition in Sect. 2. Let $M_i = R^1(\pi_i)_{\mathrm{cris}*}(O_{A_i}/W)$. Then i) in the above definition is clear. For ii), first note that $M_i/\mathrm{Im}(F_{M_i}) \simeq \mathbf{D}(\ker(F_{A_i}))$. Then using Lemma 3.1 we see that

$$(M_i/\mathrm{Im}(F_{M_i}))_S \simeq F_S^* \omega_{\ker(F_{A_i})/S} \simeq \omega_{(\ker(F_{A_i}))^{(p)}/S}$$

canonically. Also $\mathbf{D}(\ker(F_{A_i^{(p)}})) \to \mathbf{D}(\ker(\varrho_i))$ is canonically induced by

$$\omega_{\ker(F_{A_i^{(p)}})/S} \simeq \omega_{(\ker(F_{A_i}))^{(p)}/S} \to \omega_{\ker(\varrho_i)/S} \simeq N_i \overset{\text{def.}}{=} \text{the } \alpha\text{-sheaf of } \ker(\varrho_i).$$

Now assume that σ^* is faithful. We use induction to define a quasi-inverse of $\mathbf{D}_S$. Given the data in the definition of crystals with flag type level structure, suppose we have constructed $A_i \to S$. Let $K_1' = \ker(F_{A_i/S})$, $K_1 = \ker(F_{A_i^{(p)}/S}) \simeq K_1'^{(p)}$, $K_2 = \ker(V_{A_i/S})$, and $K = K_1 \cap K_2$. Then ii) gives an epimorphism $h_i: (M_i/\mathrm{Im}(F_{M_i}))_S \simeq \omega_{K_1/S} \twoheadrightarrow N_i$. Furthermore, $V_{K_1'/S}$ induces $V_{K_1'/S}^*: \omega_{K_1'/S} \to \omega_{K_1/S}$, and

$\operatorname{coker}(V^*_{K'_1/S}) \simeq \omega_{K/S}$. Since σ^* is faithful, b) implies that $h_i \circ V^*_{K'_1/S}: \omega_{K'_1/S} \to N_i$ is the zero map. Hence N_i is also a quotient of $\omega_{K/S}$. Let $\mathscr{F}_2$ be the α-sheaf of K_2, and $\mathscr{F}$ be the α-sheaf of K. By Remark 2.7, the induced morphism $\mathscr{F}_2 \to \omega_{K_2/S}$ is surjective, hence so is $\mathscr{F} \to \omega_{K/S}$. Therefore N_i is also a quotient of $\mathscr{F}_2$. This defines a closed subgroup scheme G of K. Clearly $N_i \simeq \omega_{G/S}$, so G is flat, again by Remark 2.7. Now let $A_{i-1} = A_i^{(p)}/G$.

Even when σ^* is not faithful, we can still recover a supersingular abelian scheme with flag type level structure from its crystal as above. Hence $\mathbf{D}_S$ is fully faithful. Q.E.D.

So far we have not taken account of the integer invariants t_i $(1 \leqq i < g)$ of Sect. 1. Let $t = (t_1, \ldots, t_{g-1})$. We can modify the definition in Sect. 2 by changing "index s" to "index (s, t)", and adding

iii) $\ker(F_{A_i^{(p)}/S}) \cap \ker(V_{A_i/S})$ is flat of α-rank t_i.

We can also define a functor

$$T_{s,t}: ((\text{reduced } k\text{-schemes})) \to ((\text{sets})),$$

$$S \mapsto \{\text{flag type level structures of index } (s,t) \text{ over } S\}/\text{isomorphisms}.$$

Then one can prove that $T_{s,t}$ is represented by a locally closed subscheme of S_s, just as in Theorem 2.9.

Again use induction. Suppose we have got a locally closed subscheme $S_{i,t}$ of S_i. Then it is easy to see that t_i is an upper-semicontinuous function on the set of closed points. Thus there is a greatest locally closed subset U of $S_{i,t}$ with reduced induced structure such that $K \times_{S_i} U$ is a flat α-group of α-rank t_i. Since giving a flat subgroup scheme of a flat α-group is equivalent to giving a flat quotient of its α-sheaf, we see that $S_{i-1,t}$ is a relative Grassmannian over U. In particular, if t is the smallest possible (i.e., such that $S_{s,t}$ is not empty) in the lexicographic order, then $S_{s,t}$ is a smooth open subset of S_s. By Sect. 1, one can easily see that S_s is not empty for any s. Hence we get

Proposition 3.3. *For any index s, S_s contains a non-empty smooth open subset.*

4. Supersingular Abelian Schemes

Suppose we have a supersingular abelian scheme $f: A \to S$ of dimension g. Then it is natural to ask: does it admit a flag type level structure? The answer is negative in general. (See the counter examples below.) However, it is still possible to get a flag type level structure after making some changes in S. We need the following lemma.

Lemma 4.1 (Ogus). *If S is normal, then there is an étale morphism $e: S' \to S$ such that there is an isogeny $E^g \times S' \to A \times_S S'$ (over S').*

*Proof.** Let $l \neq p$ be a prime, and consider the local systems $E_n \overset{\text{def.}}{=} R^1 F_* \mathbb{Z}/l^n\mathbb{Z}$ on S. According to a theorem of Grothendieck [9, Proposition 4.4.], A/S is isogeneous to a constant family (i.e., a product of an abelian variety over k with S) if and only if

* Unpublished proof of A. Ogus

E_n is constant for all $n>0$. Thus, it will suffice to prove that this condition can be achieved after some finite étale covering $S'\to S$.

The key case is when k is an algebraic closure of $\mathbb{F}_p$. In this case, we can find a field $\mathbb{F}_q\subset k$ and a descent $f_0: A_0\to S_0$ of f to varieties of finite type over $\mathbb{F}_q$. Let us choose m large enough so that $l^m\geqq 4g$ and a finite étale covering $S_0'\to S_0$ on which E_m is constant. Replace S_0 by S_0' to simplify the notation. Replacing q by a power, we can assume that $q=p^d$, with d even, and that $p^{d/2}\equiv 1 \pmod{l^m}$.

Let σ be a geometric point of S_0 and consider the exact sequence of groups

$$1\to\pi_1(S,\sigma)\to\pi_1(S_0,\sigma)\to\mathrm{Gal}(k/\mathbb{F}_q)\to 1\,.$$

Let $E=\varprojlim E_m$. Then $E(\sigma)$ is a $\mathbb{Z}_l[\pi_1(S_0,\sigma)]$-module, and is l-torsion free. It suffices to prove that the action ϱ of $\pi_1(S,\sigma)$ on $E(\sigma)\otimes\mathbb{Q}_l$ is trivial.

For each closed point s of S_0 and each Frobenius element F_s of $\pi_1(S_0,\sigma)$, the trace $A(s)$ of $\varrho(F_s)$ is the same as the trace of $F_{A_{\bar s}}^{\deg(s)}$ on $H^1(A_{\bar s},\mathbb{Q}_l)$. As $A_{\bar s}$ is a supersingular abelian variety, we know

i) $A(s)$ is an integer;
ii) $|A(s)|\leqq 2gp^{d(s)/2}$ (Riemann hypothesis);
iii) $p^{d(s)/2}$ divides $A(s)$ [by supersingularity, noting that $d|d(s)$, so $d(s)$ is even];
iv) $A(s)\equiv 2g \pmod{l^m}$ (since E_m is constant).

Let $B(s)=A(s)\cdot p^{-d(s)/2}$. Then $B(s)\in\mathbb{Z}$, $|B(s)|\leqq 2g$, and $B(s)\equiv 2g \pmod{l^m}$ since $p^{d(s)/2}$ is a power of $q^{1/2}$ and $q^{1/2}\equiv 1 \pmod{l^m}$. Then since $l^m\geqq 4g$, $B(s)=2g$ and $A(s)=2gp^{d(s)/2}$ for every s.

Now consider the representation ϱ' of $\pi_1(S_0,\sigma)$ attached to a constant family of supersingular abelian varieties. The argument above shows that ϱ and ϱ' have the same trace on Frobenius elements, and hence on all elements, by Chebataroff. It follows that ϱ' and ϱ have isomorphic semi-simplifications. In particular, the semi-simplification ϱ_{ss} of ϱ is trivial when restricted to $\pi_1(S,\sigma)$. Then $\varrho|_{\pi_1(S,\sigma)}$ admits a decomposition series whose successive quotients are constant. By a (deep!) theorem of Deligne [3, p. 383, Théorème 3.4.1], $\varrho|_{\pi_1(S,\sigma)}$ is in fact semi-simple, hence trivial. Q.E.D.

Theorem 4.2. *If S is integral, then there are a blowing up $b: S_b\to S$, a normalization $n: S_n\to S_b$, and an étale covering $e: S_e\to S_n$ such that $A^{(p^{2g-3})}\times_S S_e$ admits a flag type level structure.*

Proof. Let $K_2=\ker(V_{A/S})$, $K=\ker(V_{K_2^\vee/S})$. Letting m be the rank of

$$\mathrm{Im}(V^*_{K_2^\vee/S}:\omega_{K_2^\vee/S}\to\omega_{K_2^{\vee(p)}/S})\,,$$

there is an open dense subset $U\subset S$ on which $\mathrm{coker}(V^*_{K_2^\vee/S})$ is flat of rank $g-m$. By Lemma 2.8, the functor

$$R': \mathscr{C}_S\to((\mathrm{sets}))\,,$$

$$(X\to S)\mapsto\{\text{flat closed subgroup schemes of } K\times_S X \text{ over } X \text{ of } \alpha\text{-rank}\, g-m\}$$

is represented by a relatively projective morphism $b': S'\to S$ and a flat closed subgroup scheme $G\hookrightarrow K\times_S S'$ over S'. But $b'|_{b'^{-1}(U)}$ should be an isomorphism. Since $G^\vee$ is an α-group and is a quotient of $K_2^{\vee(p)}$, it gives an isogeny $A_1\to A^{(p)}\times_S S'$ with kernel $G^\vee$. Replace S' by the irreducible component (with the reduced induced structure) which maps surjectively to S. Then b' becomes a blowing up.

Take S' instead of S and A_1 instead of A, then repeat this procedure, and so on. We will finally obtain a blowing up $b: S_b \to S$ and an isogeny $i: A' \to A^{(p^{g-1})} \times_S S_b$ over S_b such that:

i) $\ker(i) \subsetneqq \ker(F^{g-1})$;

ii) $\ker(F_{A'^{(p)}/S_b}) \cap \ker(V_{A'/S_b})$ is flat of α-rank g on an open dense subset of S_b;

iii) There are isogenies $A' = A'_{g-1} \to A'_{g-2} \to \ldots \to A'_0 \simeq A^{(p^{g-1})} \times_S S_b$, such that every kernel is an α-group.

Condition ii) implies that $\ker(F_{A'^{(p)}/S_b}) = \ker(V_{A'/S_b})$, and hence A' is superspecial (i.e., all of the closed fibers are superspecial).

Now pull back A' via the normalization $n: S_n \to S_b$. Then using Lemma 4.1, we get an étale covering $e: S_e \to S_n$ and an isogeny over S_e:

$$h: E^g \times S_e \to A' \times_{S_b} S_e.$$

Let $H = \ker(h)$. First we have a decomposition $H \simeq H_1 \times_S H_2$, where H_1 is étale and H_2 is local. Since H_1 is a closed subgroup scheme of $\ker(\cdot\, t_{E^g \times S_e}) \simeq \ker(\cdot\, t_{E^g}) \times S_e$ for some integer $t (p \nmid t)$, we see that $H_1 \simeq H_0 \times S_e$, where H_0 is étale over k, and $H_1 \hookrightarrow E^g \times S_e$ is induced by a morphism $H_0 \to E^g$. Thus we may assume $H_1 = 0$ by taking E^g/H_0 (also superspecial) instead of E^g.

Suppose $H' = \ker(F_{H/S})$ has α-rank r on an open dense subset U. As above, we have a blowing up $b': S_{eb} \to S_e$ and a flat closed subgroup scheme $H_3 \subsetneqq H' \times_{S_e} S_{eb}$ of α-rank r. In fact, $H' \hookrightarrow \ker(F_{E^g}) \times S_e$ induces a closed immersion $\eta: S_{eb} \to G_{g,r} \times S_e$, where $G_{g,r}$ is the Grassmannian. If $x \in U$, then E^g/H'_x is still superspecial by Sect. 1. But there are only a finite number of subgroup schemes $\bar{H} \subset \ker(F_{E^g})$ of α-rank r such that $E^g/\bar{H}$ is superspecial. Thus $\mathrm{pr}_1 \circ \eta(b'^{-1}(U))$ is just one point. Therefore b' is an isomorphism and $H' \simeq H'_0 \times S_e$ for some $H'_0 \subset E^g$. Replacing E^g by E^g/H'_0 (also superspecial) and repeating this argument, h will finally become an isomorphism. Q.E.D.

Corollary 4.3. *If $A \to S$ has superspecial closed fibers and there is an isogeny $E^g \times S \to A$, then $A \simeq E^g \times S$.*

Remark 4.4. Letting $s_0 = (1, 2, \ldots, g)$, we note that S_{s_0} has special importance, since every supersingular abelian variety has a flag type level structure of index s_0 [18, Theorem 2.2]. By slightly modifying the proof of Theorem 4.2, one shows that under the conditions of Theorem 4.2, there are a blowing up $b: S_b \to S$, a normalization $n: S_n \to S_b$, and an étale covering $e: S_e \to S_n$ such that $A^{(p^{2g-3})} \times_S S_e$ has a flag type level structure of index s_0.

Example 1 (cf. [21, Remark 10]). This example shows the necessity of the étale covering in Theorem 4.2. Let $S = \mathrm{Spec}(k[x, y, (x-y)^{-1}]$, $G = \mathbb{Z}/2\mathbb{Z} = \{e, \sigma\}$. Define a G-action on S by letting e act as id_S, and σ act by switching x and y. Also define a G-action on $E^2 = E \times E$ by letting e act as id_{E^2}, and σ act by switching factors. Then we get a free action of G on $A = E^2 \times S$ compatible with the (free) action of G on S. By Lemma 2.3, we get quotients $A' = A/G$ and $S' = S/G$. It is easy to check that the abelian scheme structure of A over S induces an abelian scheme structure of A' over S'. Also S' is smooth. We claim that $A' \to S'$ does not admit a level structure, not even over an open dense subset of S'. Indeed, suppose we have an isogeny $f: E^2 \times S' \to A'$ over S'. Then by the above corollary, we may assume that f is an isomorphism. The projection $\varrho: S \to S'$ induces a morphism $g: A \to E^2 \times S$ over S such that $(\mathrm{id} \times \varrho) \circ g$ is the projection $A \to A'$. Here g must be an isomorphism, judging by its degree. Let x be a closed point of S. Then $g - g_x \times \mathrm{id}_S$ maps A_x to one point, hence by rigidity [14, p. 43], it factors through S, i.e., $g - g_x \times \mathrm{id}_S = 0$, or $g = g_x \times \mathrm{id}_S$. Now the projection $A \to A'$ is equal to both $g_x \times \varrho$ and $(g_x \times \varrho) \circ \sigma = (g_x \circ \sigma) \times \varrho$, so $g_x = g_x \circ \sigma$, a contradiction.

Example 2. This example shows the necessity of the blowing up in Theorem 4.2. Let $g=3$, and $f: E^2 \times \hat{S}_3 - \hat{A}$ over $\hat{S}_3$ represent $\hat{T}_3$ (see Sect. 2). Consider the morphism $\varrho: S_{s_0} \to \hat{S}_3$ induced by the level structure $E^2 \times S_{s_0} \to A_0$ over S_{s_0}. We claim that ϱ is surjective. Indeed, for the fiber over a closed point $x \in \hat{S}_3$, f induces a flag type level structure $\hat{A}_{xi} = E^2/((\ker F^{2-i}) \cap (\ker f_x))$ of index either $(1,2,3)$ or $(0,0,3)$. In the later case, $\ker(f_x) = \ker(F_{E^2})$, hence for any $G \subset \ker(F_{E^2})$ of α-rank 2, the flag type level structure $E^2 \to E^2/G \to E^2/\ker(F_{E^2})$ is compatible with the level structure f at x. Therefore $\varrho^{-1}(x)$ is of dimension $2(\simeq \mathbf{P}^2)$.

Now let us calculate S_{s_0}. We use the notation used in the proof of Theorem 2.9. First, $S_1 \simeq \mathbf{P}^2 \simeq \mathrm{Proj}\, k[x_0, x_1, x_2]$. Let $G = \ker(B_2^{(p)} \times S_1 \to B_1)$, $K = \ker(F_{B_1^{(p)}/S_1}) \cap \ker(V_{B_1/S_1})$. Then $G' = (\ker(F_{B_2^{(p)} \times S_1/S_1})/G)^{(p)}$ is a flat subgroup scheme of K of α-rank 1, and

$$\omega_{G'/S_1} \simeq (\ker(\omega_{B_2^{(p)} \times S_1/S_1} \to \omega_{G/S_1}))^{(p)} \simeq O_{S_1}(-1)^{(p)} \simeq O_{S_1}(-p).$$

Let $G'' = K/G'$. Clearly both $F_{B_1^{(p)}/S_1}$ and V_{B_1/S_1} factor through

$$j: B_1^{(p)} \to B_1^{(p)}/G' (\simeq B_2 \times S_1).$$

Let $F_{B_1^{(p)}/S_1} = F' \circ j$, $V_{B_1/S_1} = V' \circ j$. Then $G'' \simeq \ker(F') \cap \ker(V')$. Hence

$$\omega_{G''/S_1} \simeq \omega_{B_2 \times S_1/S_1}/(\mathrm{Im}(F'^*) \cap (\mathrm{Im}(V'^*)),$$

which is isomorphic to the cokernel of

$$\begin{aligned} &O_{S_1}(-1) \oplus O_{S_1}(-p^2) \to O_{S_1}^{\oplus 3} \\ &\qquad (u, v) \mapsto (x_0 u + x_0^{p^2} v, x_1 u + x_1^{p^2} v, x_2 u + x_2^{p^2} v). \end{aligned}$$

In particular, ω_{G''/S_1} is torsion free. Hence ω_{K/S_1} is also torsion free, by the exact sequence

$$0 \to \omega_{G''/S_1} \to \omega_{K/S_1} \to \omega_{G'/S_1} \to 0.$$

Since $S_0 \simeq \mathrm{Proj}_{O_{S_1}}(\omega_{K/S_1})$, we see that S_0 is irreducible.

Let S_n be the normalization of $\hat{S}_3$. We claim that for any étale covering $S_e \to S_n$, $\hat{A} \times_{\hat{S}_3} S_e$ does not admit a flag type level structure. Indeed, if it had, then we would get a surjective morphism $S_e \to S_0$ inducing $\hat{A} \times_{\hat{S}_3} S_e \simeq A_0 \times_{S_0} S_e$ over S_e. Since $S_e \to \hat{S}_3$ is finite, $\hat{A} \times_{\hat{S}_3} S_e \to S_e$ would have only a finite number of superspecial closed fibers. But we have seen that $A_0 \times_{S_0} S_e \to S_e$ has an infinite number of superspecial closed fibers, a contradiction.

Therefore the étale covering and the blowing up in Theorem 4.2 are both necessary.

5. Coarse Moduli Spaces

Let $A \to S$ be a supersingular abelian scheme. Then for any closed point $x \in S$, $G_x = \ker(F) \cap \ker(V)$ of A_x corresponds to $\bar{M}^0$ in Sect. 1, where $M = H^1_{\mathrm{cris}}(A_x/W)$. Hence the α-rank of G_x is equal to $s'_1(M)$. Therefore s'_1 is an upper-semicontinuous function of S_{cl}. Similarly s'_2 is an upper-semicontinuous function on the open subset of S_{cl} of all closed points whose fibers have minimal s'_1, and so on.

Let M be a supersingular Dieudonné crystal of genus g over W. Then it has a canonical flag type level structure $M_i = F^{g-1-i}S(M) + M$ $(0 \leqq i < g)$ whose index s is clearly the smallest possible in the lexicographic order. This flag type level structure is called *rigid*. Similarly we can define a rigid flag type level structure for a supersingular abelian variety.

Now consider the moduli space S_s. Any $\varrho \in \mathrm{Aut}(E^g)$ gives another flag type level structure of index s over S_s, hence gives an automorphism of S_s. Therefore $\mathrm{Aut}(E^g)$ acts on S_s. The action is finite since for any $\varrho, \varrho' \in \mathrm{Aut}(E^g)$ such that $\varrho - \varrho' \in p^g \,\mathrm{End}(E^g)$, $\varrho - \varrho' = 0$ on the closed subgroup schemes $\ker(A_i^{(p)} \to A_{i-1})$ $(1 \leqq i < g)$, hence ϱ and ϱ' act in the same way on S_s. Therefore the action has a quotient $\tilde{S}_s$ by Lemma 2.3. Similarly, using the method of Sect. 3, we see that $\mathrm{Aut}(\dot{M}, \mathrm{tr})$ acts on S_s and the quotient space is just $\tilde{S}_s$. But $\mathrm{Aut}(\dot{M})$ also acts on S_s and we get a quotient space $\hat{S}_s$ (which was described in [18, p. 606]). We use similar notation for an index (s, t). The actions may not be transitive on a set of all closed points with isomorphic A_0 [resp.., (M_0, tr) or M_0] fibers, but they are transitive if one of the fibers is rigid, because any isomorphism $M_0 \tilde{\to} M'_0$ is induced by an isomorphism $S(M) \tilde{\to} S(M'_0)$. Clearly the closed points with rigid fibers form an open subset, which we denote by an upper r. By Theorem 4.2, it is easy to show that $(\tilde{S}^r_{s,t})^{(p^{2g-3})}$ can be viewed as the "coarse moduli space" of supersingular abelian varieties of index (s, t). In other words, we have

Proposition 5.1. *If S is a normal scheme of finite type over k, and $\pi: A \to S$ is a supersingular abelian scheme whose closed fibers are all of the same index (s, t), then there is a unique morphism $f: S \to (\tilde{S}^r_{s,t})^{(p^{2g-3})}$ over k such that for any closed point x of S, $\pi^{-1}(x)$ is isomorphic to the supersingular abelian variety corresponding to $f(x)$.*

Proof. We may assume S is integral. By Theorem 4.2, there is an étale covering $e: S_e \to S$ such that $A^{(p^{2g-3})} \times_S S_e$ has a flag type level structure of index (s, t). (We don't need blowing up since the group scheme K in the proof of Theorem 4.2 is flat.) This gives a unique morphism $S_e \to S^r_{s,t}$, hence a morphism $f': S_e \to \tilde{S}^r_{s,t}$. Clearly f' maps every closed fiber of e to one point. Take another étale covering $S'_e \to S$ such that $S'_e \times_S S_e$ is a disjoint union of copies of S'_e. Then $f' \circ \mathrm{pr}_2: S'_e \times_S S_e \to \tilde{S}^r_{s,t}$ factors through S'_e set-theoretically, hence scheme-theoretically [10, Ex. II.4.2]. So f' factors through S by the following lemma (whose proof is easy and is left to the reader).

Lemma 5.2. *Suppose $X \to S$, $Y \to S$ are both flat surjective morphisms of finite type of Noetherian schemes. Let $Z = X \times_S Y$. Then S is the scheme-theoretic push-out of $\mathrm{pr}_1: Z \to X$ and $\mathrm{pr}_2: Z \to Y$.*

The only thing remaining to check now is the coincidence of $A_x = \pi^{-1}(x)$ with the supersingular abelian variety A'_x corresponding to $f(x)$. The above shows that

$A_x^{(p^{2g-3})} \simeq A'_x$. To get rid of (p^{2g-3}) one can consider $S^{(p^{3-2g})}$ instead of S. Then one gets $S^{(p^{3-2g})} \to \tilde{S}^r_{s,t}$, or $S \to (\tilde{S}^r_{s,t})^{(p^{2g-3})}$. Q.E.D.

It is natural to ask: What are the degrees (of the generic fibers) of the finite morphisms $S^r_{s,t} \to \tilde{S}^r_{s,t}$, $\tilde{S}^r_{s,t} \to \hat{S}^r_{s,t}$, etc.? As an example, we calculate the special case $s = s_0$ here. In this case t can only be $(1, 2, \ldots, g-1)$. We need the following proposition.

Proposition 5.3. *Let $M = Av$, where v satisfies Lemma 1.5 and is general enough (i.e., $\bar{v}$ is contained in some non-empty Zariski open subset of $k^{\oplus g}$). Then an automorphism $h \in \mathrm{Aut}_A(\dot{M})$ stabilizes M if and only if there is $\lambda \in W(\mathbb{F}_{p^2})^*$ $(= W(\mathbb{F}_{p^2}) - pW(\mathbb{F}_{p^2}))$, such that $(h - \lambda I)\dot{M} \subseteq F^{g-1}\dot{M}$, and $\lambda - \lambda^\sigma \in p^{\left[\frac{g-1}{2}\right]} W(\mathbb{F}_{p^2})$.*

The normal subgroup of all automorphisms satisfying the conditions in Proposition 5.3 will be denoted by H.

Lemma 5.4. *Let $T = (a_{ij})$, $T' = (b_{ij})$ be two $g \times g$ matrices over k, let n be an integer, $0 \leqq n \leqq 2g-3$, and $y_1, \ldots, y_g$, $y'_1, \ldots, y'_g$ be indeterminates. Let*

$$R(T, T', n, y_1, \ldots, y_g, y'_1, \ldots, y'_g) = \begin{vmatrix} \sum_{j=1}^{g} a_{j1} y_j^{p^n} + b_{j1} y_j'^{p^n} & \cdots & \sum_{j=1}^{g} a_{jg} y_j^{p^n} + b_{jg} y_j'^{p^n} \\ y_1 & \cdots & y_g \\ y_1^{p^2} & \cdots & y_g^{p^2} \\ \vdots & \ddots & \vdots \\ y_1^{p^{2g-4}} & \cdots & y_g^{p^{2g-4}} \end{vmatrix}$$

Then $R(T, T', n, y_1, \ldots, y_g, y'_1, \ldots, y'_g) = 0$ if and only if $T = a_{11} I$ and $T' = 0$ when n is even, or $T = T' = 0$ when n is odd.

Proof. Look at the Laplacian expansion of $R(T, T', n, y_1, \ldots, y_g, y'_1, \ldots, y'_g)$. If $b_{ij} \neq 0$, then there is only one term of the form $c y_i'^{p^n} \prod_{l<j} y_l^{p^{2i-2}} \prod_{l>j} y_l^{p^{2i-4}}$, $c \neq 0$, a contradiction. If $a_{ij} \neq 0$, $i \neq j$, then there is only one term of the form

$$c y_i^{p^{n+1}} \prod_{\substack{l \neq i, j \\ s_l < s_m \text{ if } l < m}} y_l^{p^{s_l}}, \quad c \neq 0,$$

also impossible. If n is odd and $a_{ii} \neq 0$, then there is only one term of the form

$$c y_i^{p^n} \prod_{i<j} y_l^{p^{2i-2}} \prod_{l>j} y_l^{p^{2i-4}}, \quad c \neq 0,$$

still impossible. Finally, if n is even and $i \neq j$, then there are exactly two terms of the form

$$c y_i^{p^n} y_j^{p^n} \prod_{\substack{l \neq i, j \\ s_l < s_m \text{ if } l < m}} y_l^{p^{s_l}},$$

namely $c = \pm a_{ii}$ and $c = \mp a_{jj}$. Hence $a_{ii} = a_{jj}$. Q.E.D.

Proof of Proposition 5.3. The proof of sufficiency is easy. We now prove necessity. Fix a basis $x_1, \ldots, x_g$ of $\dot{M}$. For every n, $0 \leqq n < g$, we want to obtain $\lambda_n \in W(\mathbb{F}_{p^2})^*$ such that $h \equiv \lambda_n I \pmod{F^n \dot{M}}$ and $\lambda_n - \lambda_n^\sigma \equiv 0 \pmod{p^{[n/2]}}$. We use induction on n.

Take $\lambda_0=1$. Suppose we have got λ_n. Then there are two possible cases.

i) n is even. In this case we may assume that $\lambda_n \in W(\mathbb{F}_p)^*$. Let $h'=h-\lambda_n I = p^{n/2}(a_{ij}+b_{ij}F)$. Then h stabilizing M implies that $h'v \in p^{n/2}\dot{M} \cap M = (F^n v, F^{n-1}Vv, \ldots, V^n v)$. Since

$$h'v = p^{n/2} \sum_{i,j=1}^{g} (a_i+b_iF)(a_{ij}+b_{ij}F)x_j$$

and

$$F^{n-l}V^l v = p^{n/2} \sum_{i=1}^{g} (a_i^{\sigma^{n-2l}} + b_i^{\sigma^{n-2l}} F)x_i,$$

modulo $F^{n+1}\dot{M}$ we get (still denoting by $\bar{a}$ the image of $a \in W$ in $W/pW \simeq k$)

$$0=\begin{vmatrix} \sum_{i=1}^{g} \bar{a}_i\bar{a}_{i1} & \cdots & \sum_{i=1}^{g} \bar{a}_i\bar{a}_{ig} \\ \bar{a}_1^{\sigma^{-n}} & \cdots & \bar{a}_g^{\sigma^{-n}} \\ \bar{a}_1^{\sigma^{-n+2}} & \cdots & \bar{a}_g^{\sigma^{-n+2}} \\ \vdots & \ddots & \vdots \\ \bar{a}_1^{\sigma^{-n+2g-4}} & \cdots & \bar{a}_g^{\sigma^{-n+2g-4}} \end{vmatrix} = R((\bar{a}_{ij}), (0), n, \bar{a}_1, \ldots, \bar{a}_g, \bar{b}_1, \ldots, \bar{b}_g)^{\sigma^{-n}}.$$

Hence by Lemma 5.4, $(\bar{a}_{ij})=\bar{a}_{11}I$ when v is general enough. Note that $\bar{a}_{ij} \in \mathbb{F}_{p^2}$ and there are only a finite number of $g \times g$ matrix over $\mathbb{F}_{p^2}$. Therefore we can take $\lambda_{n+1} \in W(\mathbb{F}_{p^2})$ such that $h \equiv \lambda_{n+1} I \,(\operatorname{mod} F^{n+1}\dot{M})$ and $\lambda_{n+1} \equiv \lambda_n \,(\operatorname{mod} p^{n/2})$.

ii) n is odd. Then we can write

$$h \equiv \lambda_n I + p^{\frac{n-1}{2}}(b_{ij})F \left(\operatorname{mod} p^{\frac{n+1}{2}}\right),$$

$$\lambda_n^{\sigma} - \lambda_n \equiv p^{\frac{n-1}{2}} \mu \left(\operatorname{mod} p^{\frac{n+1}{2}}\right), \quad \mu \in W(\mathbb{F}_{p^2}).$$

Let

$$v' = hv - \lambda_n v \equiv p^{\frac{n-1}{2}} \sum_{j=1}^{g} \left(\sum_{i=1}^{g} a_i b_{ij} + \mu b_j \right) F x_j \left(\operatorname{mod} p^{\frac{n+1}{2}} \dot{M}\right).$$

Since $v' \in F^n \dot{M} \cap M$ we have

$$0=\begin{vmatrix} \sum_{i=1}^{g} \bar{a}_i\bar{b}_{i1}+\bar{\mu}\bar{b}_1 & \cdots & \sum_{i=1}^{g} \bar{a}_i\bar{b}_{ig}+\bar{\mu}\bar{b}_g \\ \bar{a}_1^{\sigma^{-n}} & \cdots & \bar{a}_g^{\sigma^{-n}} \\ \vdots & \ddots & \vdots \\ \bar{a}_1^{\sigma^{-n+2g-4}} & \cdots & \bar{a}_g^{\sigma^{-n+2g-4}} \end{vmatrix} = R((\bar{b}_{ij}^{\sigma}), \bar{\mu}^{\sigma} I, n, \bar{a}_1, \ldots, \bar{a}_g, \bar{b}_1, \ldots, \bar{b}_g)^{\sigma^{-n}}.$$

Since n is odd and v is general enough, we have $\bar{\mu}=0$, $\bar{b}_{ij}^{\sigma}=0$. Therefore

$$h \equiv \lambda_n I \left(\operatorname{mod} p^{\frac{n+1}{2}}\right). \quad \text{Q.E.D.}$$

Now $S^r_{s_0}$ is smooth, and there is an open dense subset of S_{s_0} on which the action of $\mathrm{Aut}_A(\dot{M})/H$ is free and transitive for every set of closed points with isomorphic M_0 fibers. Therefore

$$\begin{aligned}&\text{the degree of } S^r_{s_0} \to \hat{S}^r_{s_0}\\ &= \#(\mathrm{Aut}_A(\dot{M})/H)\\ &= \begin{cases} p^{(g-2)(2g^2+g)+2g-3[g/2]}(p+1)(p^4-1)(p^6-1)\dots(p^{2g}-1) & \text{if } g>2\\ p^2(p^4-1) & \text{if } g=2 \end{cases}.\end{aligned}$$

Now consider the degree of $\tilde{S}^r_{s_0} \to \hat{S}^r_{s_0}$. We have seen that $\mathrm{Aut}_A(\dot{M}, \mathrm{tr})$ is the normal subgroup of $\mathrm{Aut}_A(\dot{M})$ consisting of all matrices of determinant 1. Let $H' = H \cap \mathrm{Aut}(\dot{M}, \mathrm{tr})$. Then the degree is equal to the order of $(\mathrm{Aut}(\dot{M})/H)/(\mathrm{Aut}(\dot{M}, \mathrm{tr})/H')$. When $g \neq 2$, letting f be the $2g^{\text{th}}$ power map of $(W(\mathbb{F}_p)/p^{[g/2]}W(\mathbb{F}_p))^* : f(a) = a^{2g}$, this is equal to

$$\begin{aligned}\frac{\#((W(\mathbb{F}_p)/p^{[g/2]}W(\mathbb{F}_v))^*)}{\#((W(\mathbb{F}_p)/p^{[g/2]}W(\mathbb{F}_p))^{2g*})} &= \#(\ker(f))\\ &= \#\{\text{solutions of } x^{2g}=1 \text{ in } \mathbb{Z}/p^{[g/2]}\mathbb{Z}\}.\end{aligned}$$

If $g=2$, we also get that the degree is equal to the number of solutions of $x^2=1$ in $\mathbb{F}_p$. Summarizing, we obtain

$$\text{degree of } \tilde{S}^r_{s_0} \to \hat{S}^r_{s_0} = \begin{cases} \mathrm{G.C.D.}(2g, (p-1)p^{[g/2]-1}) & \text{if } p \neq 2,\ g>2\\ 2\,\mathrm{G.C.D.}(2g, p^{[g/2]-2}) & \text{if } p=2,\ g>3\\ 2 & \text{if } p\neq 2,\ g=2\\ 1 & \text{if } p=2,\ g\leqq 3. \end{cases}$$

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