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Scalar Curvatures on S^n

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0. Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geqq 3$, and $R(x)$ a smooth function on M^n. One would like to ask if $R(x)$ can be the scalar curvature of some metric $\tilde{g}$ that is pointwise conformal to the original metric g. This is an interesting problem in geometry. If we let $\tilde{g} = u^{4/(n-2)} g$, then it is equivalent to finding a positive solution of the differential equation

$$\begin{cases} -\Delta_g u + \dfrac{n-2}{4(n-1)} R_g u = \dfrac{n-2}{4(n-1)} R u^\tau, \\ u > 0 \end{cases} \tag{0.1}$$

where $\tau = (n+2)/(n-2)$ and R_g is the scalar curvature of the metric g.

In case R is a constant, this is the well-known Yamabe Problem and the affirmative answer was completed by Schoen [6]. Then the attention is turned to non-constant $R(x)$. Recently, Escobar and Schoen [4] obtained some good results in this respect.

It is known that the problem becomes difficult when $M^n = S^n$ with the standard metric g_0. In this situation, Kazdan and Warner [5] found an obstruction to the solvability of the corresponding equation:

$$\begin{cases} -\Delta_{g_0} u + v_n u = K(x) u^\tau \\ u > 0, \end{cases} \tag{$*$}$$

where $v_n = n(n-2)/4$ and $K(x) = [(n-2)/4(n-1)] R(x)$. More obstructions were found recently by Bourguignon and Ezin [2]. These show that for quite a few functions $K(x)$ Problem $(*)$ has no solution. Then for which K can one solve $(*)$? This has been an interesting problem for years.

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In paper [4], under the assumptions:

K_0) K is positive somewhere [known to be a necessary condition for the sovability of (*)];

K_1) (Symmetry condition) $K(hx)=K(x)$ for any $h\in\Gamma$, $x\in S^n$; where Γ is some finite group of isometries on S^n;

K_2) (Flatness condition) there is $p\in S^n$, such that

$$K(p)=\max_{S^n} K \quad \text{and} \quad \nabla^j K(p)=0, \quad j=1,2,\dots,n-2;$$

Escobar and Schoen proved that if Γ consists of the isometries acting *without* fixed point, then Problem (*) has a solution. Their main idea is to estimate the quotient by Green's function on S^n/Γ, a manifold *without* boundary.

Now, if the isometries in Γ *have* fixed point, S^n/Γ is a manifold *with* boundary. In the following, we will deal with this situation. To overcome the boundary difficulty, we construct a symmetric (in the sense of condition $\mathcal{K}_1$)) "Green's" function on the whole S^n for estimating. Combined with the other nonlinear analysis techniques, imposing some conditions on K at certain fixed points of Γ, we prove the existence of solutions for Problem (*). This generalizes some of results in paper [4].

We have recently learnt an announcement of interesting results of Bahri and Coron [1] on the S^3 problem. They replaced the symmetry conditions by conditions on the critical points of K.

After completing our paper, we were told about the results of Vaugon [7] on this problem. However, as we will show in our Remark 2.2, our results are much stronger than his.

Throughout this paper, we write the Laplacian and the gradient on the standard S^n as Δ and ∇ respectively.

Our Main Existence Results

Let $\Gamma=\{h_1,\dots,h_m\}$ where $h_i(i=1,\dots,m)$ are isometries on S^n.

Let $F_j=\{x\in S^n|\{h_1x,\dots,h_mx\}$ has exactly j distinct components$\}$ $j=1,\dots,m$. Clearly, $S^n=\bigcup_{j=1}^{m} F_j$, and $\bigcup_{j=1}^{m-1} F_j$ is the fixed points set under the action of the isometries of Γ. We will denote F_Γ the set F_1, the common fixed points set of all the isometries in Γ.

We revise the flatness condition K_2) in [4] as $\mathcal{K}_2$). Let $D_j=\bigcup_{i=1}^{j} F_i$. For any $j=2,\dots,m$, there is $p_j\in D_j$, such that

$$K(p_j)=\max_{D_j} K$$

and

$$\nabla^i K(p_j)=0, \quad i=1,\dots,n-2.$$

It is easily seen that, if the fixed point set $\bigcup_{j=1}^{m-1} F_j$ is empty, then $\mathcal{K}_2$) is just the flatness condition K_2) in [4]. As we will show in Lemma 1.3, condition $\mathcal{K}_2$) is satisfied automatically for $n=3$.

Theorem. *Assume* $K_0)$, $K_1)$, *and* $\mathscr{K}_2)$. *Then Problem* $(*)$ *has a solution if one of the following conditions is satisfied:*

1) $F_\Gamma = \emptyset$.
2) $\max_{F_\Gamma} K \leqq 0$.
3) *There is* j_0, $2 \leqq j_0 \leqq m$, *such that*

$$j_0^{2/(n-2)} \max_{F_\Gamma} K \leqq \max_{D_{j0}} K .$$

4) $0 < \max_{F_\Gamma} K < \frac{1}{\omega_n} \int_{S^n} K(x) dV$, *where* dV *and* ω_n *are the volume element and volume of standard* S^n *respectively.*
5) *There exists* $x_0 \in F_\Gamma$, *such that*

$$K(x_0) = \max_{F_\Gamma} K , \quad \textit{and} \quad \Delta K(x_0) > 0 .$$

Outline of the Proof of Theorem

Let $H_* = \left\{ u \in H^1(S^n) \,\middle|\, \int_{S^n} K(x) |u|^{\tau+1} dV > 0 \right\}$. By $K_0)$, it is easily seen that $H_* \neq \emptyset$. And obviously, H_* is an open subset in $H^1(S^n)$. Write

$$X_\Gamma = \{ u \in H^1(S^n) \,|\, u(hx) = u(x), \text{ a.e., for any } h \in \Gamma \} .$$

Define

$$J(u) = \frac{1}{2} \int_{S^n} [|\nabla u|^2 + v_n u^2] dV - \frac{1}{\tau+1} \int_{S^n} K(x) |u|^{\tau+1} dV .$$

By the symmetry condition $K_1)$, it is known that positive critical points of the functional J in $H_* \cap X_\Gamma$ are solutions of Problem $(*)$.

Define

$$M = \{ u \in H_* \cap X_\Gamma \,|\, u \neq 0, \langle J'(u), u \rangle = 0 \}$$

and

$$b = \inf_{u \in M} J(u) \tag{0.2}$$

It is not difficult to see that if $u \in M$ and $J(u) = b$, then u is a critical point of J in $H_* \cap X_\Gamma$.

Analogous to [3], one can show that $b > 0$, and there is a sequence $\{u_k\} \subset M$, such that

$$J(u_k) \to b \quad \text{and} \quad J'(u_k) \to 0 , \quad \text{as} \quad k \to \infty . \tag{0.3}$$

Then $\{u_k\}$ is bounded in $H^1(S^n)$, hence exists a subsequence (still denote by $\{u_k\}$) converging weakly to some element u_0 in $H^1(S^n)$. This leads to [3]

$$J'(u_0) = 0 \quad \text{and} \quad J(u_0) \leqq b . \tag{0.4}$$

Case 1. If u_0 is not identically equal to 0. Then it is not difficult to show that $u_0 \in H_* \cap X_\Gamma$, so by (0.4), $u_0 \in M$ and $J(u_0) = b$. Replace u_0 by $|u_0|$ if necessary, we obtain a solution of Problem $(*)$.

Case 2. If $u_0=0$. It is wellknown (e.g. cf. [1]) that there exist finite points $\{x_1,\ldots,x_s\}\subset S^n$ and a constant $c_0>0$, such that for any $\varepsilon>0$,

$$u_k\to 0 \quad \text{as} \quad k\to\infty \quad \text{in} \quad H^1\left(S^n\Big\backslash\bigcup_{i=1}^{s}\mathfrak{B}_\varepsilon(x_i)\right) \tag{0.5}$$

and

$$\int_{\mathfrak{B}_\varepsilon(x_i)}|\nabla u|^2dV\geqq c_0, \tag{0.6}$$

where $\mathfrak{B}_\varepsilon(x)$ is a geodesic ball with radius ε and centered at x on standard S^n.

Under the assumptions of our Theorem, we will show that Case 2 can never happen. Otherwise, we have the following

Lemma 1.1. *Let $x_i\in S^n$ be a point of concentration of the sequence $\{u_k\}$ as was mentioned in the Case 2, then*

$$K(x_i)>0 \quad \textit{and} \quad b\geqq\frac{1}{n}S^{n/2}\sum_{i=1}^{s}(K(x_i))^{(2-n)/2},$$

where

$$S=\inf_{\varphi\in H^1(S^n)}\frac{\int_{S^n}(|\nabla\varphi|^2+v_n\varphi^2)dV}{\left\{\int_{S^n}|\varphi|^{2n/(n-2)}dV\right\}^{(n-2)/2}}$$

is the best constant in Sobolev embedding.

While by using a symmetric Green's function estimate, we can show that

Lemma 1.2. *For every $2\leqq j\leqq m$, and for any $p_j\in F_j$, s.t.*

$$K(p_j)>0 \quad \textit{and} \quad \nabla^iK(p_j)=0, \quad \textit{for} \quad i=1,2,\ldots,n-2.$$

We have

$$b<\frac{j}{n}S^{n/2}(K(p_j))^{(2-n)/2}.$$

Now, by $\mathscr{K}_2)$ and the above two Lemmas, we must have

$$K(x_i)>0 \quad \text{and} \quad \{x_1,\ldots,x_s\}\subset F_\Gamma. \tag{0.7}$$

Then again by Lemma 1.1,

$$b\geqq\frac{1}{n}S^{n/2}\left(\max_{F_\Gamma}K\right)^{(2-n)/2}. \tag{0.8}$$

However, if K satisfies one of the conditions in the Theorem, we are able to derive contradictions with (0.7) or (0.8). Thus prove the existence of solutions for Problem (*).

1. Lemmas and the Proofs

As in Sect. 0, let M and b be defined by (0.2), $\{u_k\}$ be a sequence in M satisfying (0.3). Suppose Case 2 happen, i.e.

$$u_k\rightharpoonup 0, \quad \text{as} \quad k\to\infty; \quad \text{in} \quad H^1(S^n).$$

Then holds the following

Lemma 1.1. *$K(x_i)>0$, and*

$$b \geqq \frac{1}{n} S^{n/2} \sum_{i=1}^{s} (K(x_i))^{(2-n)/2}, \tag{1.1}$$

where $\{x_1, \dots, x_s\}$ is defined by (0.5) *and* (0.6).

Proof. 1) Since $u_k \in M$, we have

$$\int_{S^n} \{|\nabla u_k|^2 + v_n u_k^2\} dV = \int_{S^n} K(x) |u_k|^{\tau+1} dV.$$

This implies

$$J(u_k) = \frac{1}{n} \int_{S^n} K(x) |u_k|^{\tau+1} dV. \tag{1.2}$$

2) Since $J'(u_k) \to 0$, as $k \to \infty$; one has

$$-\Delta u_k + v_n u_k = K(x) |u_k|^{\tau-1} u_k + o_k(1), \tag{1.3}$$

where $o_k(1) \to 0$ as $k \to \infty$ in the dual space of $H^1(S^n)$.

Let $\eta_i \in C_0^\infty(\mathfrak{B}_\varepsilon(x_i))$; $0 \leqq \eta_i \leqq 1$, and

$$\eta_i = 1 \quad \text{for} \quad x \in \mathfrak{B}_{\varepsilon/2}(x_i).$$

Note that $u_k \rightharpoonup 0$ in $H^1(S^n)$ and consequently $u_k \to 0$ in $L^2(S^n)$, and by (0.5), one can easily verify that

$$\int_{\mathfrak{B}_\varepsilon(x_i)} |\nabla(\eta_i u_k)|^2 dV = \int_{\mathfrak{B}_\varepsilon(x_i)} K |\eta_i u_k|^{\tau+1} dV + o_k(1). \tag{1.4}$$

(1.4) and (0.6) imply, for k sufficiently large,

$$\int_{\mathfrak{B}_\varepsilon(x_i)} K |\eta_i u_k|^{\tau+1} dV \geqq c_0/2. \tag{1.5}$$

Noting that $\{u_k\}$ is bounded in $H^1(S^n)$ and consequently, bounded in $L^{\tau+1}(S^n)$, while ε is arbitrary, by the continuity of K, one can easily seen that $K(x_i)$ is positive $(i=1, \dots, s)$.

4) Since $u_k \to 0$ in $L^2(S^n)$, by (1.4) and (1.5), we have

$$\frac{\int_{\mathfrak{B}_\varepsilon(x_i)} \{|\nabla(\eta_i u_k)|^2 + v_n(\eta_i u_k)^2\} dV}{\left\{\int_{\mathfrak{B}_\varepsilon(x_i)} K |\eta_i u_k|^{\tau+1} dV\right\}^{2/(\tau+1)}} \leqq \left(\int_{\mathfrak{B}_\varepsilon(x_i)} K |\eta_i u_k|^{\tau+1} dV\right)^{2/n} + o_k(1).$$

Then by the definition of S and η_i, as well as the continuity of K,

$$\int_{\mathfrak{B}_{\varepsilon/2}(x_i)} K |u_k|^{\tau+1} dV \geqq K(x_i)^{(2-n)/2} S^{n/2} + o_k(1) + \alpha(\varepsilon),$$

where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, (1.1) follows from (1.2) easily.

Lemma 1.2. *For $2 \leqq j \leqq m$, if there exists $p_j \in F_j$, s.t.*

$$K(p_j) > 0 \quad \textit{and} \quad \nabla^i K(p_j) = 0, \quad i = 1, \dots, n-2.$$

Then

$$b < \frac{j}{n} S^{n/2}(K(p_j))^{(2-n)/2}. \tag{1.6}$$

Proof. For conciseness, we prove this for $j=2$, and the similar argument works for a general $j>2$.

1) Let q_0 be any point in F_Γ. Then obviously $-q_0 \in F_\Gamma$. Let

$$\pi: S^n \backslash \{-q_0\} \to \mathbb{R}^n$$

be the stereographic projectioon map with q_0 lying on $0 \in \mathbb{R}^n$, and $\tilde{g} = \pi^*(\delta_{ij})$ be the pullback metric on $S^n \backslash \{-q_0\}$.

For $p_2 \in F_2$, write $\{h_1 p_2, \ldots, h_m p_2\} = \{p_0, \bar{p}_0\}$. At pole p_0, the Green's function of the conformal Laplacian $\Delta_{\tilde{g}}$ under the metric $\tilde{g}$ on $S^n \backslash \{-q_0\}$ is $a|p-p_0|^{2-n}$, where $|p-p_0|$ is the distance between the 2 points under the metric $\tilde{g}$, and a is a constant.

Claim.

$$|hp - hp_0| = |p - p_0|, \quad \text{for any} \quad p \in S^n \backslash \{-q_0\}, h \in \Gamma. \tag{1.7}$$

Proof of the Claim. For any $\zeta \in S^n$, $\pi(\zeta)$ is the tangent vector at q_0 of the geodesic (on the standard S^n) linking q_0 and ζ. Since h is an isometry on S^n,

$$\angle(\pi(p), \pi(p_0)) = \angle(\pi(hp), \pi(hp_0)), \tag{1.8}$$

where $\angle(X, Y)$ stands for the angle between the 2 vectors X and Y in $\mathbb{R}^n$. Let $d(\ ,\ )$ be the geodesic distance on standard S^n. Note that $q_0 \in F_\Gamma$, we have

$$d(hp_0, q_0) = d(hp_0, hq_0) = d(p_0, q_0),$$
$$d(hp, q_0) = d(p, q_0).$$

Hence,

$$|\pi(hp_0)| = |\pi(p_0)| \quad \text{and} \quad |\pi(hp)| = |\pi(p)|. \tag{1.9}$$

Now, from (1.8) and (1.9), it is easy to see that

$$|\pi(hp) - \pi(hp_0)| = |\pi(p) - \pi(p_0)|.$$

Therefore (1.7) is true. The same equality holds for $\bar{p}_0$.

Let

$$G(x) = |x - p_0|^{2-n} + |x - \bar{p}_0|^{2-n}, \qquad x \in S^n \backslash \{-q_0\}.$$

Then

$$G(hx) = G(x), \quad \text{for any} \quad h \in \Gamma,\ x \in S^n \backslash \{-q_0\}.$$

In fact, by (1.7),

$$\begin{aligned} G(hx) &= |x - h^{-1}p_0|^{2-n} + |x - h^{-1}\bar{p}_0|^{2-n} \\ &= |x - p_0|^{2-n} + |x - \bar{p}_0|^{2-n} = G(x) \end{aligned}$$

due to the definition of $\{p_0, \bar{p}_0\}$. Where h^{-1} is the inverse of h.

2) Let

$$\lambda(x)=\left|\cos\frac{d(q_0,x)}{2}\right|^{2-n}.$$

Then

$$\lambda\in C^\infty(S^n\backslash\{-q_0\}) \quad\text{and}\quad \tilde{g}=\lambda^{\tau-1}g_0.$$

Define

$$\lambda G(-q_0)=\lim_{x\to -q_0}\lambda G(x).$$

Then it is easy to see that

$$\lambda G\in X_\Gamma. \tag{1.10}$$

Observe that the functions

$$u_{\varepsilon,p}(x)=\left(\frac{\varepsilon}{\varepsilon^2+|x-p|^2}\right)^{(n-2)/2}$$

are solutions of the equation

$$-\Delta_{\tilde{g}}u_{\varepsilon,p}=n(n-2)u^\tau_{\varepsilon,p}, \quad\text{for}\quad x\in S^n\backslash\{-q_0\},$$

while $-\Delta_{\tilde{g}}$ and $-\Delta+v_n$ are conformal, we have

$$-\Delta(\lambda u_{\varepsilon,p})+v_n\lambda u_{\varepsilon,p}=n(n-2)(\lambda u_{\varepsilon,p})^\tau, \quad\text{for}\quad x\in S^n. \tag{1.11}$$

Let $\mathfrak{B}_\varrho(p)$ be the ball centered at p with radius ϱ under the metric $\tilde{g}$. Choose ϱ_0 sufficiently small that

$$\mathfrak{B}_{\varrho_0}(p_0)\cap\mathfrak{B}_{\varrho_0}(\bar{p}_0)=\emptyset \quad\text{and}\quad -q_0\notin\mathfrak{B}_{2\varrho_0}(p_0)\cup\mathfrak{B}_{2\varrho_0}(\bar{p}_0).$$

Let $\Psi\in C_0^\infty(\mathfrak{B}_{2\varrho_0}(p_0)\cup\mathfrak{B}_{2\varrho_0}(\bar{p}_0))$, $\Psi(x)=1$ for $x\in\mathfrak{B}_{\varrho_0}(p_0)\cup\mathfrak{B}_{\varrho_0}(\bar{p}_0)$ and Ψ only depends on $|x-p_0|$ or on $|x-\bar{p}_0|$ in $\mathfrak{B}_{2\varrho_0}(p_0)$ or in $\mathfrak{B}_{2\varrho}(\bar{p}_0)$ respectively.

Define

$$\varphi(x)=\begin{cases} u_{\varepsilon,p_0}(x) & x\in\mathfrak{B}_{\varrho_0}(p_0)\\ u_{\varepsilon,\bar{p}_0}(x) & x\in\mathfrak{B}_{\varrho_0}(\bar{p}_0)\\ \varepsilon_0(G(x)-\Psi\alpha(x)) & x\in\{\mathfrak{B}_{2\varrho_0}(p_0)\backslash\mathfrak{B}_{\varrho_0}(p_0)\}\cup\{\mathfrak{B}_{2\varrho_0}(\bar{p}_0)\backslash\mathfrak{B}_{\varrho_0}(\bar{p}_0)\}\\ \varepsilon_0 G(x) & x\in S^n\backslash\{\mathfrak{B}_{2\varrho_0}(p_0)\cup\mathfrak{B}_{2\varrho_0}(\bar{p}_0), \end{cases}$$

where

$$\alpha(x)=\begin{cases}|x-\bar{p}_0|^{2-n}-A & x\in\mathfrak{B}_{2\varrho_0}(p_0)\\ |x-p_0|^{2-n}-A & x\in\mathfrak{B}_{2\varrho}(\bar{p}_0)\end{cases} \quad\text{with}\quad A=|p_0-\bar{p}_0|^{2-n}.$$

In order for φ to be continuous, we require (cf. [6])

$$\varepsilon_0(\varrho_0^{2-n}+A)=\left(\frac{\varepsilon}{\varepsilon^2+\varrho_0^2}\right)^{(n-2)/2}.$$

By (1.10), it is not difficult to verify that $\lambda\varphi \in X_\Gamma$, i.e. $\lambda\varphi(hx) = \lambda\varphi(x)$ for any $h \in \Gamma$. Moreover, by the assumption $K(p_0) = K(\bar{p}_0) > 0$, one can choose ε so small that

$$\int_{S^n} K|\lambda\varphi|^{\tau+1} dV > 0.$$

Now $\lambda\varphi \in H_* \cap X_\Gamma$.

We are going to use $\lambda\varphi$ to estimate the quotient

$$Q(u) = \frac{E(u)}{\left\{\int_{S^n} K|u|^{\tau+1} dV\right\}^{2/(\tau+1)}},$$

where

$$E(u) = \int_{S^n} (|\nabla u|^2 + v_n u^2) dV.$$

The following approach is somewhat standard (cf. [6, 4]).

By (1.11), we have

$$\begin{aligned}
&\int_{\mathfrak{B}_{\varrho_0}(p_0)} \{|\nabla(\lambda u_{\varepsilon,p_0})|^2 + v_n(\lambda u_{\varepsilon,p_0})^2\} dV \\
&= n(n-2) \int_{\mathfrak{B}_{\varrho_0}(p_0)} (\lambda u_{\varepsilon,p_0})^{\tau+1} dV + \int_{\partial\mathfrak{B}_{\varrho_0}(p_0)} \lambda u_{\varepsilon,p_0} \frac{\partial}{\partial v}(\lambda u_{\varepsilon,p_0}) dS \\
&\leqq S\left\{\int_{\mathfrak{B}_{\varrho_0}(p_0)} (\lambda u_{\varepsilon,p_0})^{\tau+1} dV\right\}^{(n-2)/n} + \int_{\partial\mathfrak{B}_{\varrho_0}(p_0)} \lambda u_{\varepsilon,p_0} \frac{\partial}{\partial v}(\lambda u_{\varepsilon,p_0}) dS. \qquad (1.12)
\end{aligned}$$

The same inequality holds for $\mathfrak{B}_{\varrho_0}(\bar{p}_0)$ due to the symmetry of $\lambda\varphi$. Here we have used the fact that

$$S = n(n-2)\left\{\int_{\mathbb{R}^n} u_\varepsilon^{\tau+1} dx\right\}^{2/n}. \qquad (1.13)$$

Note that

$$-\Delta(\lambda G) + v_n(\lambda G) = 0 \quad \text{for} \quad x \in S^n \backslash \{\mathfrak{B}_{\varrho_0}(p_0) \cup \mathfrak{B}_{\varrho_0}(\bar{p}_0)\},$$

and due to the symmetry of $\lambda\varphi$, we have

$$\begin{aligned}
E(\lambda\varphi) \leqq\ & 2S\left\{\int_{\mathfrak{B}_{\varrho_0}(p_0)} |\lambda\varphi|^{\tau+1} dV\right\}^{(n-2)/2} \\
& + 2\int_{\partial\mathfrak{B}_{\varrho_0}(p_0)} \left\{\lambda u_{\varepsilon,p_0} \frac{\partial}{\partial v}(\lambda u_{\varepsilon,p_0}) - \varepsilon_0^2 \lambda G \frac{\partial}{\partial v}(\lambda G)\right\} dS + c\varrho_0\varepsilon_0^2.
\end{aligned}$$

Since $\lambda > 0$, and $\lambda \in C^\infty(S^n \backslash \{-q_0\})$, similar to [6], we obtain

$$E(\lambda\varphi) \leqq 2S\left\{\int_{\mathfrak{B}_{\varrho_0}(p_0)} |\lambda\varphi|^{\tau+1} dV\right\}^{(n-2)/n} - a_0 A \varepsilon_0^2 + c\varrho_0^{-n} \varepsilon_0^{\tau+1} + c\varrho_0 \varepsilon_0^2$$

with $a_0 > 0$.

Now, due to K_1) and the flatness assumption

$$\nabla^i K(p_0) = 0, \quad i = 1, \ldots, n-2$$

analogous to [4], we arrive at

$$
\begin{aligned}
E(\lambda\varphi) &\leqq 2^{2/n} K(p_0)^{(2-n)/n} S \left\{ \int_{\mathfrak{B}_{\varrho_0}(p_0) \cup \mathfrak{B}_{\varrho_0}(p_0)} K(x) |\lambda\varphi|^{\tau+1} dV \right\}^{(n-2)/n} \\
&\quad - a_0 A \varepsilon_0^2 + c(\varepsilon^{n-1} + \varrho_0^{-n} \varepsilon_0^{\tau+1} + \varrho_0 \varepsilon_0^2) \\
&\leqq 2^{2/n} K(p_0)^{(2-n)/n} S \left\{ \int_{S^n} K(x) |\lambda\varphi|^{\tau+1} dV \right\}^{(n-2)/n} \\
&\quad - a_0 A \varepsilon_0^2 + c(\varrho_0^{-2n} \varepsilon_0^{\tau+1} + \varrho_0 \varepsilon_0^2) .
\end{aligned}
$$

Because $a_0 A > 0$, we can choose ϱ_0 small and ε_0 smaller to verify

$$E(\lambda\varphi) < 2^{2/n} K(p_0)^{(2-n)/n} S \left\{ \int_{S^n} K(x) |\lambda\varphi|^{\tau+1} dV \right\}^{(n-2)/n} .$$

That is

$$Q(\lambda\varphi) < 2^{2/n} K(p_0)^{(2-n)/n} S . \tag{1.14}$$

Choose a constant t such that $\langle J'(t\lambda\varphi), \lambda\varphi \rangle = 0$, then

$$t\lambda\varphi \in M , \quad \text{and} \quad J(t\lambda\varphi) = \frac{1}{n} Q(\lambda\varphi)^{n/2} .$$

Therefore $b \leqq J(t\lambda\varphi)$, and (1.6) ($j=2$) follows from (1.14). This completes the proof.

Lemma 1.3. *The flatness condition $\mathscr{K}_2$) is always satisfied for $n=3$.*

Proof. It suffice to show that for any $2 \leqq j \leqq m$, at maximal points of K on D_j,

$$\nabla K = 0 .$$

Consider the case $j = m-1$. Suppose that $p_0 \in D_{m-1}$,

$$K(p_0) = \max_{D_{m-1}} K , \quad \text{but} \quad \nabla K(p_0) \neq 0 . \tag{1.15}$$

Then by the definition of D_{m-1}, there is at least one $h \in \Gamma$, such that $hp_0 = p_0$. By the linearity of h and because of $K(hx) = K(x)$ for any $x \in S^3$, we have

$$\nabla K(x)|_{x=p_0} = \nabla K(hx)|_{x=p_0} = h \nabla K(x)|_{x=hp_0} = h \nabla K(x)|_{x=p_0} . \tag{1.16}$$

Let Π be the plane in $\mathbb{R}^4$ spanned by the vectors $\nabla K(p_0)$ and p_0, let $S^1 = \Pi \cap S^3$. Then due to $hp_0 = p_0$ and (1.16), and the linearity of h, it is easy to see that

$$hx = x \quad \forall x \in S^1$$

which implies $S^1 \subset D_{m-1}$. Hence

$$K(p_0) = \max_{x \in S^1} K .$$

This leads to $\nabla K(p_0) = 0$, a contradiction with (1.15). Therefore we must have

$$\nabla K(p_0) = 0 .$$

Similarly, one can prove that for any $2 \leqq j \leqq m$, at maximal points of K on D_j, holds $\nabla K = 0$.

Remark 1.1. In fact, the conclusion of Lemma 1.3 is true on S^n for any n and for any $j = 1, 2, \ldots, m$.

2. Existence Theorems and the Proofs

Throughout this section, we assume that $K(x)$ satisfies the conditions K_0), K_1), and $\mathscr{K}_2$).

Let's still consider the sequence $\{u_k\} \subset M$, such that

$$J(u_k) \to b, \quad \text{and} \quad J'(u_k) \to 0, \quad \text{as} \quad k \to \infty.$$

As we argued in Sect. 0, in order to find a solution of Problem (*), one only needs to show that Case 2 can't happen. We argue indirectly. Suppose Case 2 occurs, i.e.

$$u_k \rightharpoonup 0 \quad \text{as} \quad k \to \infty, \quad \text{in} \quad H^1(S^n).$$

Then by Lemma 1.1, one must have

$$K(x_i) > 0 \quad \text{and} \quad b \geqq \frac{1}{n} S^{n/2} \sum_{i=1}^{s} K(x_i)^{(2-n)/2}, \tag{2.1}$$

where $\{x_1, \ldots, x_s\}$ is defined by (0.5) and (0.6).

Under the assumptions of the following theorems, we will derive contradictions with (2.1), therefore prove the existence of solutions for Problem (*).

Theorem 2.1. *If $F_\Gamma = \emptyset$ or if $\max_{F_\Gamma} K \leqq 0$. Then Problem (*) has a solution.*

Proof. Since $K(x_i) > 0$, under either one of the assumption of the Theorem, we have

$$\{x_1, \ldots, x_s\} \cap F_\Gamma = \emptyset.$$

Consequently, there exists $x_i \in F_j$ with $2 \leqq j \leqq m$, $1 \leqq i \leqq s$. By the symmetry of K, there are j points in $\{x_1, \ldots, x_s\}$ belonging to D_j, hence by (2.1),

$$b \geqq \frac{j}{n} S^{n/2} \left[\max_{D_j} K\right]^{(2-n)/2}. \tag{2.2}$$

while on the other hand, by $\mathscr{K}_2$) and the definition of D_j, there exists $p \in F_k$ with $2 \leqq k \leqq j$, such that

$$K(p) = \max_{D_j} K \quad \text{and} \quad \nabla^i K(p) = 0, \quad i = 1, \ldots, n-2.$$

Hence by Lemma 1.2,

$$b < \frac{k}{n} S^{n/2} \left[\max_{D_j} K\right]^{(2-n)/2}.$$

An obvious contradiction with (2.2). This completes the proof.

The following theorems deal with the situation that $\max_{F_\Gamma} K > 0$.

Theorem 2.2. *If there is j_0, $2 \leqq j_0 \leqq m$, s.t.*

$$j_0^{2/(n-2)} \max_{F_\Gamma} K \leqq \max_{D_{j_0}} K. \tag{2.3}$$

Then Problem (*) has a solution.

Proof. First case, $\{x_1, \ldots, x_s\} \cap F_\Gamma = \emptyset$, the proof is then the same as the one in Theorem 2.1.

Second case, $\{x_1, \ldots, x_s\} \cap F_\Gamma \neq \emptyset$, then

$$b \geqq \frac{1}{n} S^{n/2} \left[\max_{F_\Gamma} K\right]^{(2-n)/2}. \tag{2.4}$$

while by $\mathscr{K}_2$) and Lemma 1.2,

$$b < \frac{j}{n} S^{n/2} \left[\max_{D_j} K\right]^{(2-n)/2} \quad \forall j = 2, \ldots, m.$$

Consequently,

$$j\left[\max_{D_j} K\right]^{(2-n)/2} > \left[\max_{F_\Gamma} K\right]^{(2-n)/2} \quad \forall j = 2, \ldots, m.$$

This contradicts with (2.3). The proof is completed.

Now, from the proofs of Theorem 2.1 and Theorem 2.2, we see that in order to find a solution of Problem (*), it suffice to derive a contradiction with (2.4), i.e. to verify the following inequality

$$b < \frac{1}{n} S^{n/2} \left[\max_{F_\Gamma} K\right]^{(2-n)/2}. \tag{2.5}$$

Theorem 2.3. *If*

$$0 < \max_{F_\Gamma} K < \frac{1}{\omega_n} \int_{S^n} K(x) dV. \tag{2.6}$$

Then Problem (*) *has a solution.*

Proof. We are going to verify (2.5). By the assumption $0 < \int_{S^n} K(x) dV$, we see that any constant $c \in H_*$. Choose a proper c such that $\langle J'(c), c \rangle = 0$, then $c \in M$. Hence

$$b \leqq J(c) = \frac{1}{n} S^{n/2} \left\{\omega_n \Big/ \int_{S^n} K(x) dV\right\}^{(n-2)/2} \tag{2.7}$$

because $S = \frac{n(n-2)}{4} \omega_n^{2/n}$. Now it is easily seen that (2.6) and (2.7) imply (2.5). This completes the proof.

Theorem 2.4. *If there exists* $x_0 \in F_\Gamma$, *such that*

$$K(x_0) = \max_{F_\Gamma} K > 0 \quad \text{and} \quad \Delta K(x_0) > 0.$$

Then Problem (*) *admits a solution.*

Proof. Let

$$w_\varepsilon(x) = (2\varepsilon)^{(n-2)/2} \{\varepsilon^2 + 4 + (\varepsilon^2 - 4)\cos d(x_0, x)\}^{(2-n)/2}.$$

Then it is not difficult to verify that

$$-\Delta w_\varepsilon + \nu_n w_\varepsilon = n(n-2) w_\varepsilon^\tau, \qquad \forall x \in S^n, \tag{2.8}$$

Note that

$$hx_0 = x_0 \quad \text{and} \quad d(hx_0, hx) = d(x_0, x) \quad \forall h \in \Gamma,\ x \in S^n,$$

it is easily seen that

$$w_\varepsilon(hx) = w_\varepsilon(x) \quad \forall h \in \Gamma,\ x \in S^n.$$

That is

$$w_\varepsilon \in X_\Gamma.$$

Since $K(x_0) > 0$, and

$$w_\varepsilon(x) \to \begin{cases} 0 & x \neq x_0 \\ +\infty & x = x_0 \end{cases} \quad \text{as} \quad \varepsilon \to 0.$$

We see that for sufficiently small ε, $w_\varepsilon \in H_*$. Choose a suitable constant t_ε such that $\langle J'(t_\varepsilon w_\varepsilon), w_\varepsilon \rangle = 0$, then

$$t_\varepsilon w_\varepsilon \in M \quad \text{and} \quad b \leqq J(t_\varepsilon w_\varepsilon) = \frac{1}{n}[Q(w_\varepsilon)]^{n/2}. \tag{2.9}$$

Now, let's estimate the quotient $Q(w_\varepsilon)$. Again let π be the stereographic projection from $S^n \setminus \{-x_0\}$ to $\mathbb{R}^n$, with x_0 lying on the origin of $\mathbb{R}^n$. Let $B_\varrho(0)$ be the ball of radius ϱ and centered at origin in $\mathbb{R}^n$. Obviously, $B_\varrho(0) = \pi(\mathfrak{B}_\varrho(x_0))$, where $\mathfrak{B}_\varrho(x)$ is defined in Sect. 1. Taking into account of the fact that

$$\int_D w_\varepsilon^{\tau+1} dV = \int_{\pi(D)} u_\varepsilon^{\tau+1} dx \quad \forall D \subset S^n \tag{2.10}$$

by (1.13), (2.8) and through a direct calculation, we obtain

$$\begin{aligned} \int_{S^n} \{|\nabla w_\varepsilon|^2 + v_n w_\varepsilon^2\} dV \leqq S \left\{ \int_{\mathfrak{B}_\varrho(x_0)} w_\varepsilon^{\tau+1} dV \right\}^{(n-2)/n} \\ + c \int_{\mathbb{R}^n \setminus B_\varrho(0)} u_\varepsilon^{\tau+1} dx. \end{aligned} \tag{2.11}$$

Using the second order Taylor expansion of the function $K(x)$ at point x_0, taking into account that

$$\int_{B_\varrho(0)} y_i y_j u_\varepsilon(y)^{\tau+1} dy = 0 \quad \text{for} \quad i \neq j,\ 1 \leqq i, j \leqq n$$

due to the symmetry of u_ε (here $y = (y_1, \ldots, y_n)$); we arrive at

$$\int_{\mathfrak{B}_\varrho(x_0)} w_\varepsilon^{\tau+1} dV \leqq \frac{1}{K(x_0)} \int_{\mathfrak{B}_\varrho(x_0)} [K(x) - \tfrac{1}{4} \Delta K(x_0) d^2(x, x_0)] w_\varepsilon^{\tau+1} dV \tag{2.12}$$

for ϱ sufficiently small.

By the assumption that $\Delta K(x_0) > 0$, boundedness of K and (2.10), (2.11), and (2.12), we have

$$\begin{aligned} \int_{S^n} \{|\nabla w_\varepsilon|^2 + v_n w_\varepsilon^2\} dV \leqq S[K(x_0)]^{(2-n)/n} \left\{ \int_{S^n} K(x) w_\varepsilon^{\tau+1} dV \right\}^{(n-2)/n} \\ - c_1 \int_{B_\varrho(0)} |x|^2 u_\varepsilon^{\tau+1} dx + c \int_{\mathbb{R}^n \setminus B_\varrho(0)} u_\varepsilon^{\tau+1} dx \end{aligned}$$

with the constant $c_1 > 0$. Now, by an elementary calculus, it is not difficult to verify that, for ϱ small and ε much smaller,

$$Q(w_\varepsilon) < S[K(x_0)]^{(2-n)/n}.$$

This verifies (2.5) due to (2.9), and the proof is completed.

Remark 2.1. By Theorem 2.1, if the fixed points set on S^n under the action of Γ is empty, and if K satisfies K_0), K_1), and $\mathscr{K}_2$), the Problem (*) has a solution, which implies this part of results in paper [4].

Remark 2.2. In paper [7], the sufficient conditions for Problem (*) to have a solution are, in the language of our paper

1) K satisfies K_0) and K_1)
2) for all $x \in S^n$,

$$b < \frac{j(x)}{n} S^{n/2}[K(x)]^{(2-n)/2} \tag{2.13}$$

where $j(x) = j$ for $x \in F_j$, $j = i, 2, \ldots, m$.

However, in our paper, due to Lemma 1.1, we only require condition (2.13) be satisfied on at most m points. Moreover, if K satisfies the flatness condition $\mathscr{K}_2$) (it is satisfied automatically for $n = 3$) then we only require (2.13) be satisfied at one point on S^n, that is [cf. (2.5)], if there is $x_1 \in F_\Gamma$, such that

$$K(x_1) = \max_{F_\Gamma} K \quad \text{and} \quad b < \frac{1}{n} S^{n/2}[K(x_1)]^{(2-n)/2}. \tag{2.14}$$

Furthermore, we provide some direct and verifiable conditions on K (cf. Theorems 2.1–2.4) so that (2.14) can be satisfied. Now one can see that our results are much stronger than that in paper [7].

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