Werk

Titel: Mathematische Annalen

Verlag: Springer

Jahr: 1989

Kollektion: Mathematica

Werk Id: PPN235181684_0283

PURL: http://resolver.sub.uni-goettingen.de/purl?PID=PPN235181684_0283|LOG_0054

Math. Ann. 283, 375–380 (1989)

Mathematische Annalen

On the Gauss Map of Surfaces in R^n

Sun Cun-Jin

Department of Mathematics, Suzhou University, Suzhou, Jiangsu Province, People's Republic of China

Introduction

In [1], Hoffman and Osserman gave necessary and sufficient conditions **A** (2.20, 2.21) for a map $\mathbf{\Phi}$ from a Riemann surface $\mathbf{S}_0$ to $\mathbf{C}^n \backslash \{0\}$ satisfying $\mathbf{\Phi} \cdot \mathbf{\Phi} = 0$ to represent the Gauss map $\tilde{\mathbf{G}} : \mathbf{S}_0 \to \mathbf{Q}_{n-2} \subset \mathbf{CP}^{n-1}$ of a conformal immersion $\mathbf{X} : \mathbf{S}_0 \to \mathbf{R}^n$.

In this paper we introduce necessary and sufficient conditions **B** [(6, 7) of Theorem 1], which are equivalent to **A** but are more explicit. Moreover, in the proof of Theorem 1, formula (13) shows that if **H** is the mean curvature vector of $\mathbf{X}(\mathbf{S}_0) \subset \mathbf{R}^n$, $(\log|\mathbf{H}|)_z$ can be expressed by a differential expression of order 2 in $\mathbf{\Phi}$, which generalizes a result of Kenmotsu [3, Theorem 3].

Moreover, Theorem 1 yields somewhat shorter proofs of the results of Hoffman-Osserman for the cases $n=3, 4$ [1, Sect. 3; 2].

In Theorem 2, we give a constructive proof which allows us to explicitly represent $\mathbf{X}(\mathbf{S}_0)$ by his Gauss map $\mathbf{\Phi}$, and to generalize in all dimensions the representation theorem of Kenmotsu [3, Theorem 4].

The Results

Let $\mathbf{S}_0$ be a Riemann surface and the map

$$\mathbf{X} : \mathbf{S}_0 \to \mathbf{R}^n \tag{1}$$

be a locally conformal immersion. If $z = \xi + i\eta$ is a local parameter on $\mathbf{S}_0$, and $(x_1, \ldots, x_n)$ are coordinates in $\mathbf{R}^n$, then the map defining surface **S** is given locally in the form

$$\mathbf{X}(z), \quad \mathbf{X} = (x_1, \ldots, x_n). \tag{2}$$

The Gauss map $\mathbf{G} : \mathbf{S}_0 \to \mathbf{Q}_{n-2}$ is defined by

$$\mathbf{G}(z) = \left[\frac{\partial \mathbf{X}}{\partial \bar{z}}\right], \tag{3}$$

where $\mathbf{Q}_{n-2}=\{\mathbf{Z}\in\mathbf{CP}^{n-1}|\mathbf{Z}^2=0\}$ is the complex quadric in $\mathbf{CP}^{n-1}$. We may represent $\mathbf{G}$ locally in the form $[\mathbf{\Phi}]$ such that

$$\bar{\mathbf{G}}(z)=\left[\frac{\partial\mathbf{X}}{\partial z}\right]=[\mathbf{\Phi}(z)], \tag{4}$$

where $\mathbf{\Phi}(z)=(\varphi_1,\ldots,\varphi_n)\in\mathbf{C}^n\backslash\{0\}$ satisfies

$$\mathbf{\Phi}\cdot\mathbf{\Phi}=\sum_{k=1}^{n}\varphi_k^2=0. \tag{5}$$

Theorem 1. *Let* $\mathbf{S}$ *be an oriented surface in* $\mathbf{R}^n$ *given by* (1). *Let* $\mathbf{\Phi}$ *be the Gauss map of* $\mathbf{S}$ *in the sense of* (3, 4). *If the mean curvature vector* $\mathbf{H}$ *is not zero on* $\mathbf{S}$, *then the Gauss map* $\mathbf{\Phi}$ *must satisfy*

$$\mathbf{Im}\{(\bar{\mathbf{\Phi}}_z\cdot\bar{\mathbf{\Phi}}_z)^{1/2}(\mathbf{\Phi}_{\bar z}-\eta\mathbf{\Phi})\}=0, \tag{6}$$

$$\mathbf{Im}\left\{\left(\frac{\mathbf{\Phi}_{\bar z}\cdot\mathbf{\Phi}_{z\bar z}}{\mathbf{\Phi}_{\bar z}\cdot\mathbf{\Phi}_{\bar z}}-\frac{\mathbf{\Phi}_z\cdot\bar{\mathbf{\Phi}}}{|\mathbf{\Phi}|^2}\right)_{\bar z}\right\}=0, \tag{7}$$

where η *is defined by*

$$\eta=\frac{\mathbf{\Phi}_{\bar z}\cdot\bar{\mathbf{\Phi}}}{|\mathbf{\Phi}|^2}. \tag{8}$$

Proof. The formula (4) means

$$\frac{\partial\mathbf{X}}{\partial z}=\psi\mathbf{\Phi} \tag{9}$$

for some function $\psi:\mathbf{S}_0\to\mathbf{C}$. We note that the surface is regular wherever ψ does not vanish. By (2.17, 2.18) in [1], we know

$$|\mathbf{\Phi}|^2\bar{\psi}\mathbf{H}=\mathbf{\Phi}_{\bar z}-\eta\mathbf{\Phi}. \tag{10}$$

By taking the symmetric product of (10), we get

$$|\mathbf{\Phi}|^4\bar{\psi}^2|\mathbf{H}|^2=\mathbf{\Phi}_{\bar z}\cdot\mathbf{\Phi}_{\bar z}. \tag{11}$$

From (10) and (11), we obtain

$$(\bar{\mathbf{\Phi}}_z\cdot\bar{\mathbf{\Phi}}_z)^{1/2}(\mathbf{\Phi}_{\bar z}-\eta\mathbf{\Phi})=\pm|\mathbf{\Phi}|^4|\mathbf{H}|\,|\psi|^2\mathbf{H}.$$

Thus (6) holds. This concludes the proof of the first part of the theorem.

By (2.19) in [1], we know

$$(\log\psi)_{\bar z}+\eta=0.$$

Thus we have

$$(\log\bar{\psi}^2)_z+2\bar{\eta}=0. \tag{12}$$

By (11), we know that $\mathbf{H}\neq0$ is equivalent to $\mathbf{\Phi}_{\bar z}\cdot\mathbf{\Phi}_{\bar z}\neq0$. From (11, 12), we have

$$\left(\log\frac{\mathbf{\Phi}_{\bar z}\cdot\mathbf{\Phi}_{\bar z}}{|\mathbf{\Phi}|^4|\mathbf{H}|^2}\right)_z+2\bar{\eta}=0.$$

Thus we get

$$(\log|\mathbf{H}|^2)_z=(\log(\mathbf{\Phi}_{\bar z}\cdot\mathbf{\Phi}_{\bar z}))_z-(\log|\mathbf{\Phi}|^4)_z+2\bar\eta\,.$$

Using the above formula and noting (8), we obtain

$$(\log|\mathbf{H}|)_z=\frac{\mathbf{\Phi}_{\bar z}\cdot\mathbf{\Phi}_{z\bar z}}{\mathbf{\Phi}_{\bar z}\cdot\mathbf{\Phi}_{\bar z}}-\frac{\mathbf{\Phi}_z\cdot\bar{\mathbf{\Phi}}}{|\mathbf{\Phi}|^2}\,. \tag{13}$$

Since $|\mathbf{H}|$ is real, (7) holds. This completes the proof of the theorem.

Remark 1. We can prove that conditions (2.20, 2.21) in [1] (designated by **A**) are equivalent to conditions (6, 7) (designated by **B**). In fact, the proof that **B** $\Rightarrow$ **A** is obvious, and the proof that **A** $\Rightarrow$ **B** is the proof of Theorem 1.

Remark 2. We will see that Theorem 1 yields somewhat shorter proofs of the results of Hoffman-Osserman for the cases $n=3$, 4 [1, Sect. 3; 2].

When **S** is the surface in $\mathbf{R}^4$, $\mathbf{\Phi}(z)$ is determined by (3.4) in [1]. By (3.15) in [1], we obtain

$$\mathbf{\Phi}_{\bar z}\cdot\mathbf{\Phi}_{\bar z}=\mathbf{V}\cdot\mathbf{V}=(F_1\mathbf{A}-F_2\bar{\mathbf{A}})\cdot(F_1\mathbf{A}-F_2\bar{\mathbf{A}})=-2F_1F_2|\mathbf{A}|^2\,, \tag{14}$$

where $\mathbf{V}=\mathbf{\Phi}_{\bar z}-\eta\mathbf{\Phi}$, F_i is defined by (3.5) in [1], $\mathbf{A}$ is defined by (3.14) in [1]. By (14), we know that $\mathbf{H}\neq 0$ is equivalent to $F_1F_2\neq 0$. Using (3.15) in [1] and (14), we find

$$\begin{aligned}(\bar{\mathbf{\Phi}}_z\cdot\bar{\mathbf{\Phi}}_z)^{1/2}(\mathbf{\Phi}_{\bar z}-\eta\mathbf{\Phi})&=\pm i\sqrt{2}|\mathbf{A}|(\bar F_1\bar F_2)^{1/2}(F_1\mathbf{A}-F_2\bar{\mathbf{A}})\\&=\pm i\sqrt{2}|\mathbf{A}|(|F_1|(F_1\bar F_2)^{1/2}\mathbf{A}-|F_2|(\bar F_1F_2)^{1/2}\bar{\mathbf{A}})\,.\end{aligned}$$

Thus we get

$$\mathbf{Im}\{(\bar{\mathbf{\Phi}}_z\bar{\mathbf{\Phi}}_z)^{1/2}(\mathbf{\Phi}_{\bar z}-\eta\mathbf{\Phi})\}=\pm\sqrt{2}|\mathbf{A}|(|F_1|-|F_2|)\,\mathbf{Re}\{(F_1\bar F_2)^{1/2}\mathbf{A}\}\,. \tag{15}$$

We will show that

$$\mathbf{Re}\{(F_1\bar F_2)^{1/2}\mathbf{A}\}\neq 0\,. \tag{16}$$

Assume (16) fails at some point $z_0\in\mathbf{S}_0$. Then

$$\mathbf{Re}\{(F_1(z_0)\bar F_2(z_0))^{1/2}\}\,\mathbf{Re}\{\mathbf{A}(z_0)\}-\mathbf{Im}\{(F_1(z_0)\bar F_2(z_0))^{1/2}\}\,\mathbf{Im}\{\mathbf{A}(z_0)\}=0\,,$$

but

$$(F_1(z_0)\bar F_2(z_0))^{1/2}\neq 0\,,\qquad (\text{because of } F_1(z_0)F_2(z_0)\neq 0)\,.$$

Thus we have

$$\mathbf{Re}\{\mathbf{A}(z_0)\}=\lambda\,\mathbf{Im}\{\mathbf{A}(z_0)\}\qquad(\text{or } \mathbf{Im}\{\mathbf{A}(z_0)\}=\mu\,\mathbf{Re}\{\mathbf{A}(z_0)\})\,.$$

Hence

$$\begin{gathered}\mathbf{A}(z_0)=\mathbf{Re}\{\mathbf{A}(z_0)\}+i\,\mathbf{Im}\{\mathbf{A}(z_0)\}=(\lambda+i)\,\mathbf{Im}\{\mathbf{A}(z_0)\}\\(\text{or } \mathbf{A}(z_0)=(1+i\mu)\,\mathbf{Re}\{\mathbf{A}(z_0)\})\,.\end{gathered}$$

By (3.14) in [1], we know that $\mathbf{A}^2\equiv 0$, thus $\mathbf{Im}\{\mathbf{A}(z_0)\}=\mathbf{Re}\{\mathbf{A}(z_0)\}=0$. This clearly contradicts (3.16) in [1]. Thus (16) holds. By (15, 16), we know that for a surface in $\mathbf{R}^4$ condition (6) becomes (3.7) in [1].

By (3.5, 3.16) in [1] and (14), we get

$$\Phi_{\bar z}\cdot\Phi_{\bar z}=-4(f_1)_{\bar z}(f_2)_{\bar z}, \tag{17}$$

which implies that $\mathbf{H}\neq 0$ is equivalent to $(f_1)_{\bar z}(f_2)_{\bar z}\neq 0$. Thus we find

$$\begin{aligned}\frac{\Phi_{\bar z}\cdot\Phi_{z\bar z}}{\Phi_{\bar z}\cdot\Phi_{\bar z}}&=\frac{1}{2}(\log(\Phi_{\bar z}\cdot\Phi_{\bar z}))_z=\frac{1}{2}\{(\log(f_1)_{\bar z})_z+(\log(f_2)_{\bar z})_z\}\\&=\frac{1}{2}\left\{\frac{(f_1)_{z\bar z}}{(f_1)_{\bar z}}+\frac{(f_2)_{z\bar z}}{(f_2)_{\bar z}}\right\}.\end{aligned} \tag{18}$$

Using (3.4, 3.12) in [1], we find

$$\frac{\Phi_z\cdot\bar\Phi}{|\Phi|^2}=\frac{\bar f_1(f_1)_z}{1+|f_1|^2}+\frac{\bar f_2(f_2)_z}{1+|f_2|^2}. \tag{19}$$

By (18, 19), we know that for surfaces in $\mathbf{R}^4$ condition (7) becomes (3.8) in [1].

When $\mathbf{S}$ is a surface in $\mathbf{R}^3$, $\Phi(z)$ is determined by (3.6) in [2]. Thus we find

$$\Phi_{\bar z}\cdot\Phi_{\bar z}=4(f_{\bar z})^2, \tag{20}$$

and

$$\frac{\Phi_z\cdot\bar\Phi}{|\Phi|^2}=\frac{2\bar f f_z}{1+|f|^2}. \tag{21}$$

By (3.9) in [2] and (20), we obtain

$$(\bar\Phi_z\cdot\bar\Phi_z)^{1/2}(\Phi_{\bar z}-\eta\Phi)=\pm 4|f_{\bar z}|^2\mathbf{N}. \tag{22}$$

Since $\mathbf{N}$ is a real vector, condition (6) is always satisfied for surfaces in $\mathbf{R}^3$. By (20), we know that $\mathbf{H}\neq 0$ is equivalent to $f_{\bar z}\neq 0$. Thus we have

$$\frac{\Phi_{\bar z}\cdot\Phi_{z\bar z}}{\Phi_{\bar z}\cdot\Phi_{\bar z}}=\frac{1}{2}(\log(\Phi_{\bar z}\cdot\Phi_{\bar z}))_z=\frac{f_{z\bar z}}{f_{\bar z}}. \tag{23}$$

By (21, 23), we know that for surfaces in $\mathbf{R}^3$ condition (7) becomes

$$\mathbf{Im}\left\{\left(\frac{f_{z\bar z}}{f_{\bar z}}-\frac{2\bar f f_z}{1+|f|^2}\right)_{\bar z}\right\}=0. \tag{24}$$

Remark 3. The Eq. (13) obtained in the proof of Theorem 1 generalizes Kenmotsu's result [3, Theorem 3].

By (18, 19), we know that for surfaces in $\mathbf{R}^4$ the Eq. (13) becomes

$$(\log|\mathbf{H}|^2)_z=\mathbf{S}(f_1)+\mathbf{S}(f_2), \tag{25}$$

where

$$\mathbf{S}(f_k)=\frac{(f_k)_{z\bar z}}{(f_k)_{\bar z}}-\frac{2\bar f_k(f_k)_z}{1+|f_k|^2},\qquad k=1,2. \tag{26}$$

By (21, 23), we know that for surfaces in $\mathbf{R}^3$ the Eq. (13) becomes

$$(\log|\mathbf{H}|)_z=\frac{f_{z\bar z}}{f_{\bar z}}-\frac{2\bar f f_z}{1+|f|^2}. \tag{27}$$

This is Kenmotsu's result [3, Theorem 3].

Theorem 2. *Let* $\mathbf{S}_0$ *be a simply connected Riemann surface, and* $\mathbf{G}:\mathbf{S}_0\to\mathbf{Q}_{n-2}$ *be a map into the complex quadric. Represent* $\mathbf{G}$ *locally by a map* $\mathbf{\Phi}$ *into* $\mathbf{C}^n\backslash\{0\}$ *in the sense that* $\bar{\mathbf{G}}=[\mathbf{\Phi}]$. *Define* η *in terms of* $\mathbf{\Phi}$ *by* (8). *If* $\mathbf{V}=\mathbf{\Phi}_{\bar{z}}-\eta\mathbf{\Phi}$ *is not zero on* $\mathbf{S}_0$, *then there exists a conformal immersion* $\mathbf{X}:\mathbf{S}_0\to\mathbf{R}^n$ *with Gauss map* $\mathbf{G}$ *if and only if* $\mathbf{\Phi}$ *satisfies* (6, 7).

Proof. If there exists a conformal immersion $\mathbf{X}:\mathbf{S}_0\to\mathbf{R}^n$ with Gauss map $\mathbf{G}$, we know that (6, 7) hold by Theorem 1.

Conversely, if $\mathbf{\Phi}(z)$ satisfies (6, 7), first we put

$$T(z)=\frac{\mathbf{\Phi}_{\bar{z}}\cdot\mathbf{\Phi}_{z\bar{z}}}{\mathbf{\Phi}_{\bar{z}}\cdot\mathbf{\Phi}_{\bar{z}}}-\frac{\mathbf{\Phi}_z\cdot\bar{\mathbf{\Phi}}}{|\mathbf{\Phi}|^2}, \tag{28}$$

thus we have $\mathbf{Im}\{T_{\bar{z}}\}=0$. Using Lemma 2.1 in [1], we have

$$h=\exp\int\mathbf{Re}\{2T(z)dz\}, \tag{29}$$

where h satisfies

$$(\log h)_z=T(z). \tag{30}$$

Second, we put

$$\psi(z)=\frac{(\bar{\mathbf{\Phi}}_z\cdot\bar{\mathbf{\Phi}}_z)^{1/2}}{|\mathbf{\Phi}|^2h}. \tag{31}$$

By (28), (30), and (31), we have

$$(\log\bar{\psi})_z=\frac{1}{2}(\log(\mathbf{\Phi}_{\bar{z}}\cdot\mathbf{\Phi}_{\bar{z}}))_z-(\log h)_z-(\log(\mathbf{\Phi}\cdot\bar{\mathbf{\Phi}}))_z=-\frac{\mathbf{\Phi}\cdot\bar{\mathbf{\Phi}}_z}{|\mathbf{\Phi}|^2}.$$

Thus we get

$$\frac{\psi_{\bar{z}}}{\psi}+\eta=0. \tag{32}$$

Using (31, 32) and condition (6), we obtain

$$\begin{aligned}\mathbf{Im}(\psi\mathbf{\Phi})_{\bar{z}}&=\mathbf{Im}(\psi_{\bar{z}}\mathbf{\Phi}+\psi\mathbf{\Phi}_{\bar{z}})=\mathbf{Im}\{\psi(\mathbf{\Phi}_{\bar{z}}-\eta\mathbf{\Phi})\}\\&=\frac{1}{|\mathbf{\Phi}|^2h}\,\mathbf{Im}\{(\bar{\mathbf{\Phi}}_z\cdot\bar{\mathbf{\Phi}}_z)^{1/2}(\mathbf{\Phi}_{\bar{z}}-\eta\mathbf{\Phi})\}=0.\end{aligned}$$

By the proof of Theorem 2.6 in [1], we know that there is a surface $\mathbf{X}:\mathbf{S}_0\to\mathbf{R}^n$ such that

$$\frac{\partial\mathbf{X}}{\partial z}=\psi(z)\mathbf{\Phi}(z). \tag{33}$$

Clearly, $\mathbf{X}(\mathbf{S}_0)$ is a conformal immersed surface with Gauss map $\mathbf{G}$.

Remark 4. We may write (33) in the form

$$\mathbf{X}=\int\mathbf{Re}\{2\psi\mathbf{\Phi}dz\}, \tag{34}$$

or

$$\mathbf{X} = \int \mathbf{Re} \left\{ \frac{2(\bar{\mathbf{\Phi}}_z \cdot \bar{\mathbf{\Phi}}_z)^{1/2} \mathbf{\Phi}}{|\mathbf{\Phi}|^2 h} dz \right\}, \tag{35}$$

where h is defined by (29). By (11, 31), we know that mean curvature vector $\mathbf{H}$ of $\mathbf{X}(\mathbf{S}_0)$ satisfies $|\mathbf{H}| = h$.

By Theorem 2.5 in [1], we know that if $\mathbf{X}: \mathbf{S}_0 \to \mathbf{R}^n$ is a conformal immersed surface with Gauss map $\mathbf{G}$, and the mean curvature vector $\mathbf{H}$ is not zero on $\mathbf{X}(\mathbf{S}_0)$, then $\mathbf{X}(\mathbf{S}_0)$ is determined uniquely by $\mathbf{G}$, up to translation and homothety. And $\mathbf{X}(\mathbf{S}_0)$ can be explicitly given in terms of $\mathbf{\Phi}$ by (35). Furthermore, if h is the scalar mean curvature of $\mathbf{X}(\mathbf{S}_0)$, namely, $|\mathbf{H}| = h$, then $\mathbf{X}(\mathbf{S}_0)$ can be described explicitly from h and $\mathbf{\Phi}$ by (35).

Therefore formula (35) generalizes the representation theorem of Kenmotsu [3, Theorem 4] to euclidean n-space.

References

1. Hoffman, D.A., Osserman, R.: The Gauss map of surfaces in $\mathbf{R}^n$, J. Differ. Geom. **18**, 733–754 (1983)
2. Hoffman, D.A., Osserman, R.: The Gauss map of surfaces in $\mathbf{R}^3$ and $\mathbf{R}^4$. Proc. Lond. Math. Soc. **50**, 27–56 (1985)
3. Kenmotsu, K.: Weierstrass formula for surfaces of prescribed mean curvature. Math. Ann. **245**, 89–99 (1979)
4. Kenmotsu, K.: The mean curvature vector of surfaces in $\mathbf{R}^4$. Preprint (1986)

Received June 30, 1987; in revised form February 12, 1988