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Finite Groups and Hecke Operators

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1. Introduction

One of the residual mysteries of the classification of the finite simple group concerns the connections between the Monster group M and certain genus zero function fields associated to elliptic modular functions. These relationships were developed by Conway-Norton, Thompson et al. at the end of the last decade, and concerned something which is now called a *Thompson series*; thus it is conjectured that there is a sequence $\gamma_{-1}=1_M, \gamma_1, \gamma_2, \ldots$ of characters of the group M such that the formal power series

$$(1.1) \qquad \Gamma_M=\sum\gamma_n q^n$$

has (among others) the following properties: (i) if q is interpreted as $e^{2\pi iz}$ ($z\in\mathfrak{h}$ = upper halfplane) then $\sum\gamma_n(1)q^n$ is the modular function j-744 (here 1 is of course the identity of M); (ii) for each $g\in M$ the q-expansion $\sum\gamma_n(g)q^n$ is that of a hauptfunktion of some level $N=N(g)$ divisible by the order $o(g)$ of g. More precisely if $f=f_g=\sum\gamma_n(g)q^n$ then it is conjectured that the invariance group G of f in $SL_2(R)$ contains $\Gamma_0(N)$ as a normal subgroup, that the compactified Riemann surface $(\mathfrak{h}/G)^*$ is a sphere, and that f generates the field of modular functions on $(\mathfrak{h}/G)^*$).

Despite the overwhelming evidence for the truth of this conjecture, we seem as far today from understanding it as ever. One of the difficulties which makes the conjecture so mysterious and compelling is the apparent disparity of data which must be reconciled. Almost as remarkable is that the only real progress to date has been achieved by Frenkel et al. [FLM] using methods of Kac-Moody Lie algebras, adding another ingredient to the brew.

The point of view which we adopt in the present paper is foundational. We attempt to develop the beginnings of a theory of q-expansions of the type in (1.1) for

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an *arbitrary* finite group. To paraphrase a remark of Langlands made elsewhere but which applies equally well here, if the problem does not fall to a series of vigorous assaults then we must prepare for a long siege.

As soon as a more general point of view is adopted, it becomes natural to consider formal q-expansions (1.1) where not only are the γ_n allowed to be *generalized* (or virtual) characters of the finite group G, but also if $g \in G$ then the q-expansion $\sum \gamma_n(g)q^n$ is the Fourier expansion of a modular form of some weight k, level N and character ε and depending on g (more precisely on the conjugacy class of G determined by g). Although there is no need to so restrict ourselves, we assume throughout this paper that the following additional conditions hold: that for each $g \in G$ the q-expansion $\sum \gamma_n(g)q^n$ is a modular form on $\Gamma_0(N)$ for some N depending on g and that this q-expansion has rational integer coefficients. Thus we arrive at the

Definition. Let G be a finite group. The formal q-expansion

$$\Gamma = \sum \gamma_n q^n$$

is called a *Thompson series* (for G) if each γ_n is a rational-valued, generalized character of G and if for each $g \in G$ the q-expansion

$$\Gamma_g = \sum \gamma_n(g) q^n$$

is that of a modular form on $\Gamma_0(N)$ for some $N = N(g)$, integral weight $k(g)$ and character ε_g.

Generally, we will be concerned with Thompson series where either all $k(g) = 0$ (the original situation pertaining to the Monster), or at least each Γ_g is meromorphic. There are also important examples where each Γ_g is holomorphic, etc., but we will usually not dwell on the various situations that may arise concerning the analytic properties of the individual forms.

One can now ask an apparently naive question: can one extend (some of) the elementary theory elliptic modular forms to the context of Thompson series? In this paper we are concerned in particular with the possibility of defining Hecke operators for Thompson series. That the answer, in at least some situations, is affirmative is part of the theory for the sporadic group M_{24} as developed in [M3]. There we constructed a Thompson series for M_{24} whose corresponding L-series $\sum_{n \geq 1} \gamma_n/n^s$ has an Euler product. In this paper we show that the existence of this Thompson series is just part of a general theory of Hecke operators for Thompson series.

What is perhaps surprising is that the Hecke operators we construct are ultimately related to the theory of the (oriented) Bott cannibalistic class, regarded in this instance as a certain virtual character of a spin group. It is clear that this circumstance is just part of a much wider topological context, which however we will not pursue here.*

A brief summary of the paper is as follows: in Sect. 2 we define some important Thompson series and present certain spaces $\mathfrak{M}(\varrho)$ of Thompson series which will be our analogues of spaces of modular forms with a given weight and character, and

* Such a context seems to be provided by elliptic cohomology

on which our generalized Hecke operators will operate. In Sect. 3 we study the Bott cannabilistic class, and in Sect. 4 show how to define the Hecke operators themselves. In Sects. 5 and 6 we give some illustrations of how one can use the Hecke operators to construct (i) Thompson series which are simultaneous eigenforms and which include the M_{24} example alluded to above as a special case; (ii) Thompson series analogues of Eisenstein series; (ii) Thompson series for which the identity Γ_1 is of the form $j(q)+\text{constant}$. In an appendix we list the η-functions associated to the Conway group $\cdot O$ and which play a rôle in several places.

It remains only to record thanks to those several individuals who have contributed, in one way or another, to the results contained herein. It was Oliver Atkin who computed the q-expansions of the forms listed in the appendix and thereby provoked the author into thinking about Hecke operators, while Marvin Knopp pointed out the existence of the paper [Ba] and Michel Broué supplied a copy of his work [Br]. Finally, it is a pleasure to thank Professor F. Hirzebruch and the Max Planck Institut in Bonn for their hospitality during 1983 and for giving me a chance to think about modular forms for so many uninterrupted hours.

2. The Spaces $\mathfrak{M}(\varrho)$

In this section we establish some notation, recall some earlier results of relevance, and introduce the spaces $\mathfrak{M}(\varrho)$ of Thompson series on which our Hecke operators will act.

Our results will depend on a rational representation of the finite group G:

(2.1) V is an even-dimensional Q-vector space and

$$\varrho : G \to SL(V)$$

a representation of G by unimodular matrices.

In this situation we denote the characteristic polynomial of $\varrho(g)$ by $\chi^g(t)$. As ϱ is a rational representation one knows (cf. [M2]) that

$$\chi^g(t) = \prod_{i \geqq 1} (t^i - 1)^{k(i)} \tag{2.2}$$

for certain integers $k(i)$ which are zero unless i divides the order $o(g)$ of $\varrho(g)$.

Given (2.2), we also set

$$\begin{aligned} d(g) &= \prod i^{k(i)}, \\ k(g) &= \tfrac{1}{2} \sum k(i), \\ \Delta(g) &= (-1)^{k(g)} d(g). \end{aligned} \tag{2.3}$$

Note that $k(g) = \frac{1}{2} \dim V^g$ is an integer where V^g is the subspace of g-invariants of V, so that we can define the Dirichlet character ε_g via

$$\varepsilon_g(p) = \left(\frac{\Delta(g)}{p} \right) \tag{2.4}$$

for odd primes p.

As usual we denote by η the Dedekind eta-function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q=e^{2\pi iz}$, z in the upper half-plane. If g is as in (2.2), set

$$\eta_g(z)=\prod_{i\geqq 1}\eta(iz)^{k(i)} \tag{2.5}$$

and define functions ω_n on G, $n=1,2,\dots$ by

$$\begin{aligned}\Omega_G&=\sum_{n\geqq 1}\omega_n q^n,\\ \Omega_g&=\sum_{n\geqq 1}\omega_n(g)q^n=\eta_g(z).\end{aligned} \tag{2.6}$$

From the results of [M2], for example, we know that Ω_G is a Thompson series for G. Furthermore, one knows that $\eta_g(z)$ has weight $k(g)$ and Dirichlet character induced from ε_g. For this and a discussion of the level, see Sect. 4 of [M5] or Proposition 3.2 of [Br] for example.

An important situation with which we will be interested concerns the case where G is a suitable group of isometries of an even-dimensional even lattice $\mathbb{L}$. Thus $\mathbb{L}$ is a free abelian group with a positive-definite, G-invariant, symmetric, integral inner produce $(\,,)$, which is *even*, i.e., $(x,x)\in 2\mathbb{Z}$ for $x\in\mathbb{L}$. This situation was first investigated by Thompson [T], where the following Thompson series was introduced:

$$\begin{aligned}\Theta_G&=\sum_{n\geqq 0}\alpha_n q^n,\qquad \alpha_n\in RG,\\ \Theta_g&=\sum_{n\geqq 0}\alpha_n(g)q^n=\theta_{\mathbb{L}^g}(z),\end{aligned} \tag{2.7}$$

where $\theta_{\mathbb{L}^g}(z)$ is the theta-function of the lattice $\mathbb{L}^g$ of g-invariants in $\mathbb{L}$. In [M5] it is shown that

(2.8) If $\mathbb{L}$ is unimodular and $g\in G$ acts on $\mathbb{L}$ with determinant 1, and on $\mathbb{L}\otimes_{\mathbb{Z}}Q$ with spinor norm 1, then Θ_g has Dirichlet character ε_g.

Now fix a representation ϱ of G as in (2.1). We define

(2.9) $\mathfrak{M}(\varrho)=$ complex vector space spanned by those Thompson series Γ_G of G such that for $g\in G$, $\Gamma_g(z)$ is a form of weight $k(g)$ and Dirichlet character induced from ε_g.

Thus we do not specify the precise level of the forms Γ_g. In any case from the preceding we have

Theorem 2. *$\Omega_G\in\mathcal{M}(\varrho)$. Moreover if $\mathbb{L}$ is a unimodular lattice and ϱ is a representation of G as isometries of $\mathbb{L}$ satisfying* (2.8) *then $\Theta_G\in\mathfrak{M}(\varrho)$.*

There are variations on $\mathfrak{M}(\varrho)$; one that we make use of is $\mathfrak{S}(\varrho)$ – those Thompson series Γ_G in $\mathfrak{M}(\varrho)$ for which each $\Gamma_g(z)$ is holomorphic and a cusp-form.

We remark finally that whereas each of the theta-series Θ_g of (2.7) are holomorphic forms, the same is not necessarily the case with the forms η_g. From Sects. 4, 5 of [Ba] we have

(2.10) If $\eta(z)$ is given by (2.5) and has level N, its order at the cusp r/s is

$$\frac{N}{24(N,s^2)}\sum_{i\geqq 1}\frac{k(i)}{i}(s,i)^2.$$

3. On the Bott Cannibalistic Class

As preparation for our construction of Hecke operators in Sect. 4, we study in this section certain generalized characters of the real orthogonal groups $O(n, R)$ and their universal covering groups $\mathrm{Spin}(n, R)$.

Recall first that the appropriate Clifford algebras and their representations afford so-called spin modules for the spin groups. Using the notation of Chap. 13 of [Hu], $\mathrm{Spin}(2r+1)$ has a (complex) spin module $\Delta(r)=\Delta$ of dimension 2^r, whilst the corresponding module for $\mathrm{Spin}(2r)$ decomposes into the sum of the two half-spin modules $\Delta^+(r)$, $\Delta^-(r)$ each of dimension 2^{r-1}. Moreover, with respect to the canonical embedding $\mathrm{Spin}\,(2r-1)\to\mathrm{Spin}(2r)$, the restriction of both $\Delta^+(r)$ and $\Delta^-(r)$ to $\mathrm{Spin}(2r-1)$ are isomorphic to the spin module $\Delta(r-1)$.

There is a rather more recondite analogue of these constructions where, roughly speaking, 2 is replaced by some odd prime p, or more generally by an odd integer k. This involves the so-called Bott cannibalistic classes. We will not be concerned in this paper with any of the topological aspects of these characteristic classes, only with some of their formal properties. We refer the reader to the appropriate sections of [Hu] or [Bo] for the general theory; the more formal theory may be found, for example, in [Td] or [AT].

Thus first let k be any positive integer. Then the Bott cannibalistic class θ_k operates on special λ-rings, in particular if E is the natural n-dimensional module for $O(n, R)$ then $\theta_k(E)$ is a certain (generalized) module for $O(n, R)$ of dimension k^n. If k is odd then there is a refinement: namely there is the so-called oriented Bott class θ_k^{or} which operates on oriented λ-rings. In particular $\theta_k^{\mathrm{or}}(E)$ exists if $\dim E$ is even and we have

(3.1) $$\theta_k^{\mathrm{or}}(E)^2=\theta_k(E), \qquad k \text{ odd}.$$

We note here that $\theta_k^{\mathrm{or}}(E)$ is a (generalized) module for $O(2n, R)$ rather than its covering group.

(3.2) **Lemma** *There is a generalized module F for $O(2n-1, R)$ such that restriction yields an isomorphism of $O(2n-1, R)$-modules*

$$\theta_k^{\mathrm{or}}(E)\cong kF.$$

Proof. Write $E=E_0\oplus T$ where $\dim T=2$. Then the fact that θ_k^{or} is an exponential map (on spaces of even dimension) yields an isomorphism

$$\theta_k^{\mathrm{or}}(E)=\theta_k^{\mathrm{or}}(E_0)\theta_k^{\mathrm{or}}(T)$$

of $O(2n-2, R)$-modules. Moreover $\theta_k^{\mathrm{or}}(T)$ is the trivial module of dimension k, so that

(3.3) $$\theta_k^{\mathrm{or}}(E)=k\theta_k^{\mathrm{or}}(E_0)$$

as $O(2n-2, R)$-modules.

Now restriction induces maps of character rings

$$R\,\mathrm{Spin}(2n)\xrightarrow{f} R\,\mathrm{Spin}(2n-1)\xrightarrow{g} R\,\mathrm{Spin}(2n-2)$$

and g is an injection (Proposition 13.13 of [Hu]). Moreover $R\,\mathrm{Spin}(2n-2)$ is a free $R\,\mathrm{Spin}(2n-1)$ module with generators 1, $\Delta^+(n-1)$ (loc. cit.). But it is clear from (3.3) that $k\theta_k^{\mathrm{or}}(E_0)$ belongs to $\operatorname{im} g$, so the freeness of $R\,\mathrm{Spin}(2n-2)$ means that

$\theta_k^{\text{or}}(E)$ is itself the restriction of a generalized Spin$(2n-1)$-module, call it F. The lemma follows immediately.

Notation

1. β_k is the generalized character of $O(n, R)$ afforded by $\theta_k(E)$. It has degree k^n.

2. ψ_2 is the character of Spin$(2r, R)$ afforded by the half-spin module $\Delta^+(r)$, while for odd k we let ψ_k be the generalized character of $O(2r-1, R)$ afforded by the module F of Lemma 3.2. We sometimes also use ψ_2 for the restriction to Spin$(2r-1, R)$; ψ_k has degree k^{r-1}.

3. β_2^{or} is the character of Spin(n, R) afforded by the spin module $\Delta(r)$, $n=2r+1$ or $2r$; for k odd, β_k^{or} is the generalized character of $O(2r, R)$ afforded by $\theta_k^{\text{or}}(E)$. So β_k^{or} has degree k^r.

By (3.1), Lemma 3.2 we get

(3.3) (a) $k\psi_k=\beta_k^{\text{or}}$ *as generalized characters of* Spin$(2r-1, R)$.

(b) $$\beta_k^{\text{or}})^2=\begin{cases}\beta_k & \text{if } \dim E=2r,\\ 2\beta_k & \text{if } \dim E=2r+1 \text{ (and } k=2).\end{cases}$$

There are explicit formulae for these characters. Thus we have

$$\beta_k(g)=\prod(1+\mu+\ldots+\mu^{k-1}), \tag{3.4}$$

where the product runs over the eigenvalues μ of g (with multiplicity). Furthermore for odd k we have

$$\beta_k^{\text{or}}(g)=\prod_{*}(\mu^{k-1/2}+\mu^{k-3/2}+\ldots+1+\ldots+\mu^{-(k-1)/2)}). \tag{3.5}$$

Here, $*$ means that the product runs over a set U of eigenvalues of g defined as follows: pair the eigenvalues of g into 2-element sets $\{\mu, \mu^{-1}\}$, and choose U to contain exactly one element of each pair.

There is an alternate description as follows. Since g lies in a maximal torus of $O(n, R)$, $n=2r+1$ or $2r$, then there are real numbers $t_1, t_2, \ldots, t_r$, $0 \leqq t_i < 1$, such that g is conjugate to the matrix

$$\operatorname{diag}\left(\ldots, \begin{pmatrix}\cos 2\pi t_j, & -\sin 2\pi t_j\\ \sin 2\pi t_j, & \cos 2\pi t_j\end{pmatrix}, \ldots\right). \tag{3.6}$$

Then from (3.3) we deduce that for odd k,

$$\beta_k^{\text{or}}(g)=k^{1/2\dim(E^g)}\prod_{t_j>0}\frac{\sin k\pi t_j}{\sin \pi t_j}. \tag{3.7}$$

Here, the product runs over those t_j greater than 0, while E^g = the subspace of g-invariants of E, which has dimension equal to twice the number of t_j equal to 0.

The analogous formula for $k=2$ essentially involves replacing k by 2 and replacing the odd function $\sin x$ by the even function $\cos x$. Hence

(3.8) Let $g \in$ Spin(n, R), $n=2r+1$ or $2r$. Then

$$\beta_2^{\text{or}}(g)=2^r\prod_{j=1}^{r}\cos 2\pi t_j.$$

Some further explanation of this is in order. We have regarded g as an element $(t_1, \ldots, t_r)$ of the abstract torus $T^r \cong (R/Z)^r$, and to interpret (3.8) one must map T^r onto a maximal torus of $\mathrm{Spin}(n, R)$; see Sect. 8 of Chap. 13 of [Hu]. For our purposes we do not need to be too careful with the various tori involved in this situation. Note that for n even, $\Delta(r) = \Delta^+(r) + \Delta^-(r)$.

The following definition will be relevant.

Definition. G is a group, ϱ a representation of G on some (possibly generalized) module, and k is an integer $\geqq 2$. Let $g \in G$ have finite order.

Call g *strongly k-singular* (with respect to ϱ) if some eigenvalue of $\varrho(g)$ is a primitive d-th root of unity for some $1 \neq d \mid k$. Call g k-regular (w.r.t. ϱ) if all eigenvalues of $\varrho(g)$ are l-th roots of unity for l coprime to k, i.e., $\varrho(g)$ has order coprime to k. Call g *weakly k-singular* (w.r.t. ϱ) if g is neither strongly k-singular nor k-regular.

We now choose a *rational* element $g \in SO(n, R)$, i.e., an element g whose characteristic polynomial with respect to the natural representation ϱ on E is given by (2.2). We then let $d(g)$, $k(g)$, and $\Delta(g)$ be as in (2.3).

At last we may state the main results of the present section.

Proposition 3.9. *The following holds for $k \geqq 2$:*

$$\beta_k(g) = \begin{cases} k^{2k(g)}, & \text{if} \quad g \text{ is } k\text{-regular (with respect to } \varrho) \\ 0, & \text{if} \quad g \text{ is strongly } k\text{-singular} \end{cases}$$

Moreover if k is a prime and g is weakly k-singular of order f then

$$\beta_k(g) = k^{2k(g)} \Big(\prod_{1 < p^\alpha \mid f/k} p^{e(p^\alpha k)} \Big)^{k-1}$$

where the product runs over non-trivial prime power divisors p^α of f/k and for $d \mid f$ we have set

$$e(d) = \sum_{d \mid i} k(i) \quad (cf.\ (2.2)).$$

Remarks. There is an analogous formula for $\beta_k(g)$ for composite k, but it is more complicated to state and in any case we do not need it. Note also that if g is weakly k-singular for k a prime then k necessarily divides the order f of g.

Theorem 3. *Let $k = 2$ or k be odd and let* $\dim E = 2r$. *Assume that g is not weakly k-singular and define g_0 as follows: $g_0 = g$ if k is odd; if $k = 2$ then g_0 is a pre-image of g under the natural map* $\mathrm{Spin}(2r, R) \to O(2r, R)$ *chosen arbitrarily if g is (strongly) 2-singular and chosen to have odd order if g is 2-regular. Then*

$$\beta_k^{\mathrm{or}}(g_0) = \left(\frac{(-1)^r \Delta(g)}{k} \right) k^{k(g)}.$$

Remarks. 1. If $k = 2$ and g is 2-regular then g_0 is, of course, uniquely determined.

2. If g is strongly k-singular then k divides $d(g)$ and so $\beta_k^{\mathrm{or}}(g_0) = 0$ by the theorem, as it must be to be consistent with (3.3b) and Proposition 3.9.

3. One verifies that if g has odd order then $(-1)^r\, \Delta(g) \equiv 1 (\mathrm{mod}\, 4)$, so that the Kronecker symbol $\left(\frac{(-1)^r \Delta(g)}{k} \right)$ makes sense in the case $k = 2$, g 2-regular. Thus the

case $k=2$ of Theorem 3 can be reformulated as follows (and in this form it also holds if $\dim E = 2r+1$):

(3.10) **Proposition.** *Let g and g_0 be as in Theorem 3. Then*

$$\beta_2^{\mathrm{or}}(g_0) = \left(\frac{2}{d(g)}\right) 2^{[k(g)]}.$$

We begin with a proof of Proposition 3.10 which is based on Eisenstein's proof of the law of quadratic reciprocity, as exposited in Serre's text [Se] on p. 10. Thus we have the identity

(3.11) For m a positive odd integer,

$$\frac{\sin mx}{\sin x} = (-4)^{(m-1)/2} \prod_{j=1}^{(m-1)/2} \left(\sin^2 x - \sin^2 \frac{2\pi j}{m}\right),$$

(3.12) For m a positive odd integer,

$$\prod_{j=1}^{(m-1)/2} \cos \frac{2\pi j}{m} = \left(\frac{2}{m}\right) 2^{-(m-1)/2}.$$

Proof. Set $x=\pi/2$ in (3.11), noting that $\sin m\pi/2 = (-1)^{(m-1)/2}$. We then obtain the equation

$$2^{m-1} \prod_{j=1}^{(m-1)/2} \cos^2 \frac{2\pi j}{m} = 1$$

and it is sufficient to verify that the *sign* involved in the statement of (3.12) is correct. Thus we are looking for the number of integers j which satisfy $m/4 < j \leqq (m-1)/2$, for these are the values of j for which $\cos 2\pi j/m$ is negative. One readily verifies that the number of such j, say N, is even if, and only if, $m \equiv \pm 1 \pmod 8$. The result follows.

We can now complete the proof of Proposition 3.10. We may assume that g has odd order (cf. the second remark following the statement of Theorem 3), in which case $\beta_2^{\mathrm{or}}(g_0)$ is given by (3.8) where we may take t_j to be the form j/m for m odd and $0 \leqq j \leqq (m-1)/2$. In effect, then, (3.12) gives us the "contribution" of a single cycle of length m to the value of $\beta_2^{\mathrm{or}}(g_0)$, so if g has characteristic polynomial as in (2.2) then we get

$$\begin{aligned} \beta_2^{\mathrm{or}}(g_0) &= 2^r \prod_{i\,\mathrm{odd}} \left(\frac{2}{i}\right)^{m(i)} 2^{-(i-1)k(i)/2} \\ &= \left(\frac{2}{d(g)}\right) 2^f, \end{aligned}$$

with

$$\begin{aligned} f &= r - \textstyle\sum (i-1)k(i)/2 \\ &= r - n/2 + k(g). \end{aligned}$$

The proposition is now proved.

We turn next to the proof of Proposition 3.9. It is clear from (3.4) that if g is strongly k-singular, i.e., if g has an eigenvalue μ which is a primitive d-th root of unity for some $d \geqq 2$ which divides k, then $\beta_k(g)=0$; the converse is also true. So assume from now on that g is not strongly k-singular. Then (3.4) yields

$$\beta_k(g)=k^{2k(g)} \prod_{\mu \neq 1} \frac{\mu^k-1}{\mu-1}. \tag{3.13}$$

Now if g is k-regular then μ^k ranges over the non-trivial eigenvalues of g as μ does (since g is rational), so in this case $\beta_k(g)=\beta^{2k(g)}$ as required. It remains to treat the case that g is weakly k-singular and k is a prime. We only sketch the proof of the desired formula as it will not be crucial in what follows. Let f be the order of g and recall that $k \mid f$.

In the following we let $\Phi_d(t)$ denote the d-th cyclotomic polynomial:

$$\Phi_d(t)=\prod_i (t-\mu_i)=\sum_{a \mid d} (t^a-1)^{\mu(d/a)} \tag{3.14}$$

where $\mu(d/a)$ is the Möbius function and μ_i ranges over the primitive d-th roots of unity. Set also

$$b(d)=\prod_i (\mu_i-1)=(-1)^{\varnothing(d)}\Phi_d(1) \tag{3.15}$$

We have

$$\Phi_d(1)=\begin{cases} 0, & d=1 \\ p, & d=p^\alpha>1 \text{ is a prime power.} \\ 1, & \text{otherwise} \end{cases} \tag{3.16}$$

To see this, use (3.14) to see that we may take d to be square-free, say $d=p_1p_2 \dots p_s$ with p_i distinct primes. The result is clear if $s=0$, and if $s \geqq 1$ we see that

$$\Phi_d(t)=\frac{\prod(t^u-1)}{\prod(t^v-1)}=\frac{\prod(1+t+\dots+t^{u-1})}{\prod(1+t+\dots+t^{v-1})}$$

where u, v jointly range over all divisors of d such that d/u resp. d/v has an even resp. odd number of prime factors. Then $\Phi_d(1)=U/V$ where U resp. V is the product of all u resp. v.

Finally, a given prime p_i is a divisor of exactly $\sum_{j \geqq 0} \binom{s-1}{2j}$ divisors of d with an odd number of prime factors, and exactly $\sum_{j \geqq 0} \binom{s-1}{2j+1}$ divisors of d with an even number of prime factors. If $s \geqq 2$ these numbers are equal, whence $\Phi_d(t)=1$ in this case. If $s=1$ the result is clear.

Now if we set

$$a_k(d)=\prod_k \frac{\mu_i^k-1}{\mu_i-1}, \quad d \geqq 2 \tag{3.17}$$

where μ_i ranges over the primitive d-th roots of unity, the theory of cyclic groups yields

$$a_k(d)=\begin{cases} b(d/k)^k/(b/d) & \text{if } k^2 \mid d \\ b(d/k)^{k-1}/b(d) & \text{if } k^2 \nmid d \end{cases} \tag{3.18}$$

Putting (3.15)–(3.18) together we get

$$a_k(d)=\begin{cases}0, & k=d,\\ p^{k-1}, & d=kp^{\alpha}, \alpha\geqq 1, p \text{ prime},\\ 1, & \text{otherwise}.\end{cases} \tag{3.19}$$

Finally, $a_k(d)$ gives the contribution to $\beta_k(g)$ in (3.13) which derives from a given primitive d-th root of unity. The multiplity of such an eigenvalue is given by the integer $e(d)$ of Proposition 3.9, which is now a consequence of (3.19).

We now turn to the principal result of this section – the proof of Theorem 3. There are several approaches to this result; after Proposition 3.9 and (3.3b) we know that

$$\beta_k^{\mathrm{or}}(g)=\varepsilon(g)k^{k(g)} \tag{3.20}$$

for some sign $\varepsilon(g)=\pm 1$ (g is assumed to be k-regular). Now since β_k^{or} restricts to a generalized character of $\langle g\rangle$, it is a triviality that (3.20) uniquely determines what the sign $\varepsilon(g)$ must be *in case g has odd order*, and of course Theorem 3 implicitly says just what it is. Equally trivially, $\varepsilon(g)$ is *not* determined for g of even order by (3.20) if one assumes only that β_k^{or} is a generalized character, so in this case it is necessary to go back to the Definition (3.7).

We will use a mixture of these approaches. Consider the following result:

(3.21) **Proposition.** *Let* $k\geqq 3$ *be an odd integer,* $m\geqq 2$ *an integer and assume* $(m,k)=1$.

Define an integer $N=N(m,k)$ *as follows: N is the number of integers a satisfying* (i) $1\leqq a\leqq m/2$; (ii) $(a,m)=1$; (iii) $ak/2m-[ak/2m]>\frac{1}{2}$. *Then the following holds:*

$$(-1)^N=\begin{cases}\left(\frac{k}{p}\right), & m=p^e>1,\ p \text{ an odd prime}\\ \left(\frac{k}{p}\right)\left(\frac{p}{k}\right), & m=2p^e>2,\ p \text{ an odd prime}\\ \left(\frac{2}{k}\right), & \mathrm{m}=2^e\geqq 8\\ \left(\frac{-2}{k}\right), & m=4\\ \left(\frac{-1}{k}\right), & m=2\\ 1, & \text{otherwise}\end{cases}$$

We will show that Proposition 3.21 is equivalent to Theorem 3. Then we give a direct proof of Proposition 3.21 in case m is even. If m is odd (which essentially corresponds to g having odd order) we give a proof based only on the fact that β_k^{or} is a generalized character, which will also provide a second proof of Proposition 3.10.

Now after the remarks following the statement of Theorem 3 we may restrict our attention to the case in which k is odd and g is k-regular. Because of (3.3, 3.7) and Proposition 3.9 we are reduced to determining the sign of the product

$$\prod_{t_j>0} \frac{\sin k\pi t_j}{\sin \pi t_j} \tag{3.22}$$

in the notation of (3.6, 3.7). Now the eigenvalues of g are given by $\exp(\pm 2\pi i t_j)$ for $0 \leqq t_j < 1$, so as g is rational we may take each t_j to lie in the interval $[0, \frac{1}{2}]$. With this choice, the denominator of (3.22) is positive.

We will now show that Proposition 3.21 implies Theorem 3. An eigenvalue of g corresponds to some value a/m of t_j, and for the purposes of studying (3.22) we may take $(a, m)=1$, $1 \leqq a \leqq m/2$. Moreover, the condition $ak/2m-[ak/2m]>\frac{1}{2}$ says exactly that $\sin k\pi t_j = \sin k\pi a/m$ is negative. In effect, then, for a given m the value of $(-1)^N$ gives the contribution to the sign of (3.22) which accrues from a single primitive m-th root of unity together with its Galois conjugates.

Now assume that g has characteristic polynomial (2.2) and consider the sign contribution to (3.22) from a single cycle of length m. If $m=\prod p_i^{e_i}$ is the prime power decomposition then the relevant eigenvalues are all the m-th roots of unity, primitive or otherwise.

Assume first that m is odd. Then the only eigenvalues which may contribute are, from 3.21, the prime powers dividing m, and the total contribution is exactly

$$\prod_i \left(\frac{k}{p_i}\right)^{e_i} = \left(\frac{k}{m}\right). \tag{3.23}$$

Assume next that $m \equiv 2 \pmod 4$, $m>2$, with $p_1^{e_1}=2$, say. Then Proposition 3.21 tells us that the contribution from eigenvalues *distinct from* -1 is exactly

$$\prod_{i \geqq 2} \left[\left(\frac{k}{p_i}\right)\left(\frac{p_i}{k}\right)\right]^{e_i} \prod_{i \geqq 2} \left(\frac{k}{p_i}\right)^{e_i},$$

which is simply

$$\left(\frac{2m}{k}\right), \quad (m>2). \tag{3.24}$$

Similarly, the contribution from eigenvalues *distinct from* -1 in case $m \equiv 0 \pmod 4$ is seen to be

$$\left(\frac{-2m}{k}\right). \tag{3.25}$$

We excluded the eigenvalue -1 from the above considerations since it is the (only!) eigenvalue distinct from 1 which coincides with its Galois conjugates. After the discussion of (3.6) and (3.7), one should not calculate the contribution of -1 to a single cycle, but to the whole expression; after we remember that only one-half of these eigenvalues contribute to (3.22), we obtain from (3.21) that

$$\left(\frac{-1}{k}\right)^{1/2 \sum_{m \text{ even}} k(m)} \tag{3.26}$$

is the *total* contribution from -1's. (Here and below, $k(n)$ is as in (2.2).) We can now at last make explicit the sign of (3.22); from (3.23–3.26) it is

$$(3.27)\qquad \prod_{m\,\mathrm{odd}}\left(\frac{k}{m}\right)^{k(m)}\prod_{\substack{m\equiv 2(4)\\ m>2}}\left(\frac{2m}{k}\right)^{k(m)}\cdot\prod_{m\equiv 0(4)}\left(\frac{-2m}{k}\right)^{k(m)}\cdot\left(\frac{-1}{k}\right)^{1/2\sum_{m\,\mathrm{even}}k(m)}$$

Since $\det g=1$ then $\sum_{m\,\mathrm{even}} k(m)\equiv 0(\mathrm{mod}\,2)$. If we further let $c=\prod_{m\,\mathrm{odd}} m^{k(m)}$ then quadratic reciprocity allows us to write the first product in (3.27) as $\left(\frac{c}{k}\right)\left(\frac{-1}{k}\right)^{(c-1)/2}$. Then (3.27) can be re-written as

$$(3.28)\qquad \begin{aligned}&\left(\frac{c}{k}\right)\left(\frac{-1}{k}\right)^{(c-1)/2}\prod_{m\,\mathrm{even}}\left(\frac{m}{k}\right)^{k(m)}\left(\frac{2}{k}\right)^{\sum_{m\,\mathrm{even}}k(m)}\left(\frac{-1}{k}\right)^{h}\\ &=\left(\frac{d(g)}{k}\right)\left(\frac{-1}{k}\right)^{h+(c-1)/2}\end{aligned}$$

where $h=\sum_{m\equiv 0(4)} k(m)+\frac{1}{2}\sum_{m\,\mathrm{even}} k(m)$. Comparing this with the statement of Theorem 3, we must show that

$$(3.29)\qquad h+\frac{c-1}{2}\equiv k(g)+r\ (\mathrm{mod}\,2).$$

From the definition of c we easily find that $c-1\equiv 2\sum_{m\equiv 3(4)} k(m)\ (\mathrm{mod}\,4)$, so we can write (3.29) in the form

$$2\sum_{m\equiv 2(4)} k(m)+\sum_{m\,\mathrm{even}} k(m)+2\sum_{m\equiv 3(4)} k(m)\equiv\sum_{\mathrm{all}\,m} k(m)+\sum_{\mathrm{all}\,m} mk(m)\ (\mathrm{mod}\,4).$$

Finally this is easily seen to be equivalent to the assertion

$$2\Big(\sum_{m\equiv 0(4)} k(m)+\sum_{m\equiv 3(4)} k(m)\Big)\equiv 2\Big(\sum_{m\equiv 1(4)} k(m)+\sum_{m\equiv 2(4)} k(m)\Big)\ (\mathrm{mod}\,4)$$

which is true since $2k(g)=\sum_{\mathrm{all}\,m} k(m)\equiv 0(\mathrm{mod}\,2)$.

This completes the proof that Proposition 3.21 implies Theorem 3. The converse follows in a similar way, for by the above discussion we only have to determine the sign of (3.22) in case t_j runs over the rational numbers a/m for $(a,m)=1$ and $1\leqq a\leqq m/2$ (at least if $m\geqq 3$), which corresponds to evaluating $\left(\frac{(-1)^r\Delta(g)}{k}\right)$ in the case the eigenvalues of g are just the primitive m-th roots of unity. Then $k(g)=0$, $r=\varnothing(m)/2$, $d(g)=1$ unless m is a prime power, and Proposition 3.21 is readily deduced.

We will now prove Proposition 3.21 in case m is even, although some of the arguments hold for all m. Of course Proposition 3.2 bears much more than a mere superficial resemblance to the Gauss lemma; our proof reflects this. The case $m=2$ is trivial, so for convenience we may assume $m\geqq 3$. Set

$$F(i)=\{a\in N\,|\,(i)\ 1\leqq a\leqq m/2,\ (ii)\ (a,m)=1,\ (iii)\ (i-1)m/2<(ak/2m)<im/2\}$$

for $i=1,2,3,4$, with

$$f(i)=|F(i)|.$$

In the notation of Proposition 3.21 we have

$$N=f(3)+f(4). \tag{3.30}$$

Recall next that the Gauss lemma itself tells us how many residues $ak(\bmod m)$ lie in the interval $(\frac{1}{2}m, m)$, or at least gives the parity of the number of such residues. In present terms it tells us that

$$(-1)^{f(2)+f(4)} \equiv k^{\varnothing(m)/2} \pmod m. \tag{3.31}$$

The next two results follow from the structure of the group $(Z/mZ)^*$.

$$k^{\varnothing(m)/2} \equiv \begin{cases} \left(\dfrac{k}{p}\right), & m=p^e \text{ or } 2p^e,\ p^e>1 \text{ an odd prime power,} \\ \left(\dfrac{-1}{k}\right), & m=4 \\ 1, & \text{otherwise,} \end{cases} \tag{3.32}$$

the congruence being $(\bmod m)$.

(3.33) Suppose that m is neither a prime power nor twice a prime power. Then both $(Z/mZ)^*$ and $(Z/2mZ)^*$ have exponent dividing $\varnothing(m)/2$.

After these preparations we turn to the proof of Proposition 3.21 itself for m even. Let a range over the interval $1 \leqq a \leqq m/2$ with $(a,m)=1$, and consider which of the sets $F(i)$ contains the least positive residue of $ak\,(\bmod 2m)$. Let r denote a typical such residue. We have

$$k^{\varnothing(m)/2} \prod_{a\in F(1)} a \equiv \prod_{i=1}^{4} \prod_{r\in F(i)} r \equiv (-1)^{f(2)+f(4)} \prod_{r\in F(1)} r \prod_{r\in F(2)} (-r) \prod_{r\in F(3)} r \prod_{r\in F(4)} (2m-r), \tag{3.34}$$

the congruences being $(\bmod 2m)$. Next we claim

(3.35) For m even, $f(2)+f(3)$ is even if, and only if,

$$k^{\varnothing(m)/2} \equiv (-1)^{f(2)+f(4)} (\bmod 2m).$$

Proof. The point is that if r_i, s_i, $i=2,3$, are typical residues in $F(i)$ then $(m-r_2)(m-s_2)$, $(r_3-m)(s_3-m)$ and $(m-r_2)(r_3-m)$ are congruent to $(-r_2)(-s_2)$, $(r_3)(s_3)$, $(-r_2)(r_3)\,(\bmod 2m)$ respectively, since each r_i, s_i is odd (being coprime to m). But then $f(2)+f(3)$ is even, if and only if,

$$\prod_{r\in F(2)} (-r) \prod_{r\in F(3)} r \equiv \prod_{r\in F(2)} (m-r) \prod_{r\in F(3)} (r-m) \pmod{2m}.$$

But clearly

$$\prod_{a\in F(1)} a \equiv \prod_{r\in F(1)} r \prod_{r\in F(2)} (m-r) \prod_{r\in F(3)} (r-m) \prod_{r\in F(4)} (2m-r)$$

so (3.34) yields that $f(2)+f(3)$ is even if, and only if, $k^{\varnothing(m)/2} \equiv (-1)^{f(2)+f(4)} (\bmod 2m)$ as claimed.

Now assume that m is neither a power of 2 nor twice an odd prime power. Then (3.33) yields $k^{\varnothing(m)/2} \equiv 1 (\bmod 2m)$, whence $f(2)+f(4)$ is even by (3.31). Then (3.35) yields that $f(2)+f(3)$ is even, whence also $f(3)+f(4)$ is even. So Proposition 3.21 holds for such m.

Next assume that $m=2^e \geqq 8$. Then $k^{\varnothing(m)/2} \equiv 1 (\bmod m)$ by (3.32), so $f(2)+f(4)$ is even by (3.31). Then by (3.35) we get $f(3)+f(4)$ even if, and only if $f(2)+f(3)$ is even; if, and only if, $k^{\varnothing(m)/2} \equiv 1 (\bmod 2m)$. But this hold if, and only if, $k \equiv \pm 1 (\bmod 8)$, i.e., $\left(\frac{2}{k}\right)=1$, so again Proposition 3.21 holds.

Now assume $m=2p^e$, $p^e>1$ an odd prime power. By (3.32) we get $k^{\varnothing(m)/2} \equiv \left(\frac{k}{p}\right)$ $(\bmod m)$, so (3.31) yields $(-1)^{f(2)+f(4)} = \left(\frac{k}{p}\right)$. Also, by (3.35) we see that $f(2)+f(3)$ is even if, and only if, $k^{\varnothing(m)/2} \equiv \left(\frac{k}{p}\right) (\bmod 2m)$, and since $k^{\varnothing(m)/2} \equiv \left(\frac{k}{p}\right) (\bmod m)$ then this is equivalent to $k^{\varnothing(m)/2} \equiv \left(\frac{k}{p}\right) (\bmod 4)$. Moreoever $k^{\varnothing(m)/2} \equiv \left(\frac{-1}{k}\right)^{p-1/2}$ $\equiv \left(\frac{k}{p}\right)\left(\frac{p}{k}\right) (\bmod 4)$, so we conclude that $(-1)^{f(2)+f(3)} = \left(\frac{p}{k}\right)$. Finally, we now get

$$(-1)^{f(3)+f(4)} = (-1)^{f(2)+f(3)}(-1)^{f(2)+f(4)} = \left(\frac{p}{k}\right)\left(\frac{k}{p}\right)$$

as required by Proposition 3.21, which is now established whenever m is even (we leave the case $m=4$ to the reader).

Finally, we prove Proposition 3.21 for m odd. By a previous discussion this amounts to the following: take g of odd order such that the eigenvalues of g are just the primitive m-th roots of unity. Then $k(g)=0$ and $\beta=\beta_k^{\mathrm{or}}$ is a generalized character satisfying

$$\beta(g)=\varepsilon(g), \quad \beta(1)=k^{\varnothing(m)/2} \tag{3.36}$$

and β is exponential. We must show that in the notation of Proposition 3.21, $\varepsilon(g)$ $=(-1)^N = \left(\frac{k}{p}\right)$ or 1 according as m is a prime power $p^e>1$ or not.

We proceed by induction on the order of g, the result being trivial if $g=1$. Assume that $m=p^e>1$ is a prime power. For each $1 \neq d \mid m$ we have $\beta(g^{m/d})$ $=\varepsilon(g^{m/d})^{\varnothing(m)/\varnothing(d)} = \varepsilon(g^{m/d}) = \left(\frac{k}{p}\right)$ for $d \neq m$ by induction. Using (3.36) and the integrality condition implicit in the equation

$$(\beta, 1)_{\langle g\rangle} \in Z \tag{3.37}$$

we get

$$\varnothing(m)\varepsilon(g) + \sum_{\substack{d \mid m \\ 1 \neq d \neq m}} \varnothing(d)\left(\frac{k}{p}\right) + k^{\varnothing(m)/2} \equiv 0 (\bmod m). \tag{3.38}$$

Now $k^{\varnothing(m)/2} \equiv \left(\frac{k}{p}\right) \pmod{m}$, so (3.38) becomes

$$\varnothing(m)\left(\varepsilon(g)-\left(\frac{k}{p}\right)\right)+\sum_{d|m}\varnothing(d)\left(\frac{k}{p}\right)=\varnothing(m)\left(\varepsilon(g)-\left(\frac{k}{p}\right)\right)+m\left(\frac{k}{p}\right)\equiv 0 \pmod{m},$$

whence $\varepsilon(g)=\left(\frac{k}{p}\right)$ is immediate.

Finally, if m is not a prime power then $k^{\varnothing(m)/2}\equiv 1 \pmod{m}$, $\beta(g^{m/d})^{\varnothing(m)/\varnothing(d)}=1$ since $\varnothing(m)/\varnothing(d)$ is even (for $1 \neq d \neq m$), so in this case (3.37) reads

$$\varnothing(m)\varepsilon(g)+\sum_{\substack{d|m\\ 1\neq d\neq m}}\varnothing(d)+1\equiv 0 \pmod{m}$$

and as $\sum_{d|m}\varnothing(d)=m$ then $\varepsilon(g)=1$ follows, as required.

4. Hecke Operators

After the contortions of the last section we are ready to construct our Hecke operators, which will act on the spaces $\mathfrak{M}(\varrho)$ of Sect. 2. Our approach will be naive, i.e., purely formal; we define our operators via their action on q-expansions, as in Sect. 3, Chap. VII of [L], for example.

Thus let A be any commutative ring with identity 1. We can define operators U_d and V_d on $A[[q]]$, $d \in N$, as follows:

$$U_d:\sum \gamma_n q^n \to \sum_{d|n} \gamma_n q^{n/d}, \qquad V_d:\sum \gamma_n q^n \to \sum \gamma_n q^{dn}. \tag{4.1}$$

For each prime p choose $\psi_p \in A$ and set

$$\begin{aligned}&\psi_n=\psi_{p_1}^{e_1}\ldots\psi_{p_r}^{e_r} \quad \text{if} \quad n=p_i^{e_i}\ldots p_r^{e_r},\ p_i \text{ distinct primes,}\\ &\psi_1=1,\end{aligned} \tag{4.2}$$

and identify ψ_n with the operator which acts on $A[\![q]\!]$ by component-wise multiplication by ψ_n.

The Hecke operator T_n (with respec to the ψ_p) is defined by

$$T_n=\sum_{d|n}\psi_d V_d \circ U_{n/d}, \tag{4.3}$$

for example for a prime p we have

$$T_p=U_p+\psi_p V_p. \tag{4.4}$$

We then formally derive multiplicativity:

$$T_m T_n = T_{mn} \quad \text{if} \quad (m,n)=1, \tag{4.5}$$

and the Euler product representation:

$$\sum_{n\geq 1}\frac{T_n}{n^s}=\prod_p\left(1-\frac{T_p}{p^s}+\frac{\psi_p}{p^{2s}}\right)^{-1}. \tag{4.6}$$

Our principal examples (aside from the original Hecke operators themselves!) arise by taking for A a suitable ring of generalized characters. Let $n=2r$ be an integer divisible by 4 in the following.

Example 1. $A=R\,\mathrm{Spin}(n,R)$ with ψ_2 the half-spin character of $\mathrm{Spin}(n,R)$. This allows us to define T_2.

Example 2. $A=R\,\mathrm{Spin}(n-1,R)$ with ψ_p for p an odd prime the generalized character of $\mathrm{Spin}(n-1,R)$ defined following the proof of Lemma 3.2. We emphasize that in this section ψ_n for composite n is defined by (4.2); generally this will *not* be the same as the ψ_n of Sect. 3.

We deduce from Theorem 3 the following consequence (cf. (3.3a)):

(4.7) Let $g\in SO(n-1,R)$ be rational and assume that g and g_0 are as in Theorem 3 with $k=p$ prime. Then

$$\psi_p(g_0)=\left(\frac{\Delta(g)}{p}\right)p^{k(g)-1}=\varepsilon_g(p)p^{k(g)-1}.$$

Example 3. By restricting the ψ_p above to appropriate subgroups G we obtain operators T_p for $A=RG$.

Now let G, V satisfy hypothesis 2.1 with $n=\dim V=2r$ divisible by 4. By extending scalars we get a containment $G\leqq SO(n,R)$ and we can pull back G along $\mathrm{Spin}(n,R)\to SO(n,R)$ to a group $\tilde{G}$. We make use of the notation of Sect. 2, in particular we have the space $\mathfrak{M}(\varrho)$. With the generalized character $\psi_p\in RG$ for p an odd prime available, we can ask if T_p preserves $\mathfrak{M}(\varrho)$. If $\Gamma\in\mathfrak{M}(\varrho)$ with $\Gamma=\sum\gamma_n q^n$ then $T_p\Gamma\in\mathfrak{M}(\varrho)$ precisely when for each $g\in G$ the q-expansion

$$(U_p+\psi_p(g)V_p)\left(\sum\gamma_n(g)q^n\right) \tag{4.8}$$

is again a modular form of weight $k(g)$ and character ε_g. But after (4.2), (4.8) is just the action of the usual Hecke operator, so that (4.8) indeed has the required properties. Of course we must qualify this assertion with the remark that (4.8) only holds if g is not weakly p-singular and if $g\in SO(n-1,r)\subseteqq SO(n,R)$.

When considering the case $p=2$ we have already seen that it is necessary to pass to the group $\tilde{G}:\psi_2$ will generally not induce a character of G. In favorable situations the extension

$$Z_2\to\tilde{G}\to G \tag{4.9}$$

will split, so that $\tilde{G}\cong Z_2\times G$, and we may indeed think of ψ_2 as a character of G itself. On the other hand if $g\in G$ is not weakly 2-singular and $\pm g_0$ are the preimages of g with g_0 as in Theorem 3 then we have

$$\psi_2(-g_0)=-\varepsilon_g(2)2^{k(g)-1}.$$

Now $\varepsilon_g(2)=0$ if $o(g)$ is even and there is no problem; if $o(g)$ is odd, say N, then ε_g is defined modulo N (cf. Sect. 2) and $\Gamma_g(z)=\sum\gamma_n(g)q^n$ is a form on $\Gamma_0(N)$. Then $V_2(\Gamma_g(z))$ is a form on $\Gamma_0(2N)$ with the same weight and character ε_g, so that also $(U_2+\psi_2(-g_0)V_2)(\Gamma_g(z))$ is on $\Gamma_0(2N)$. In this case, then, one should perhaps think of T_2 as a map

$$T_2;\mathfrak{M}(\varrho)\to\tilde{\mathfrak{M}}(\varrho)$$

where $\tilde{\mathfrak{M}}(\varrho)$ is a space of Thompson series for $\tilde{G}$ satisfying the same sort of conditions as $\mathfrak{M}(\varrho)$ (cf. (2.10)). Of course the canonical embedding $RG \to R\tilde{G}$ induces a map $\mathfrak{M}(\varrho) \to \tilde{\mathfrak{M}}(\varrho)$, but in any case T_2 does not generally act as the usual Hecke operator on $\Gamma_g(z)$ if g has odd order. Nevertheless we will find it profitable to use the operator T_2 later.

We gather some of the preceding in the next result, which epitomizes the "nicest" situation. We use the

Definition. Given G as above, we say that T_p *exists* if T_p defined by (4.4) satisfies

$$T_p : \mathfrak{M}(\varrho) \to \mathfrak{M}(\varrho)$$

and if for each $g \in G$, $\Gamma \in \mathfrak{M}(\varrho)$, the operator $T_p(g) := U_p + \psi_p(g) V_p$ acts on $\Gamma_g(z)$ as the "usual" Hecke operator.

Theorem 4. *Let the notation and assumptions be as above.*

(i) *Assume that $G \leqq SO(n-1, R)$, and let p be an odd prime such that G contains no weakly p-singular elements. Then the Hecke operator T_p exists.*

(ii) *Assume that the extension* (6.9) *splits and that G contains no weakly 2-singular elements. Then T_2 exists.*

Remark that G will certainly contain no weakly p-singular elements if $|G|$ is not divisible by p, but these are by no means sufficient conditions. If G is represented on V by permutation matrices, for example, then it certainly contains no weakly p-singular elements for any prime p.

Three final comments are appropriate. First, although the weakly p-singular elements do not fit the formalism of Hecke operators, nevertheless they are not entirely without interest; we will encounter some in connection with the Leech lattice in Sect. 5. Secondly, if we are willing to deal with the operator pT_p rather than T_p itself then from (3.3) we see that

$$pT_p = pU_p + \beta_p^{\mathrm{or}} V_p,$$

in other words we can deal with Spin(n, R) and its subgroups other than having to stay inside Spin$(n-1, R)$ (we can in any case do this if $p=2$, so these comments mainly apply to odd p).

Finally, as soon as we know that T_p exists, we can use the results of Theorem 2 to produce new (and often interesting) Thompson series $T_p\Omega_G$ and $T_p\Theta_G$.

5. The Leech Lattice and Some Euler Products

Having constructed the operators T_p, at least under certain assumptions, it is natural to ask about the existence of Thompson series Γ_G which are *eigenforms*, i.e., satisfy

$$T_p \Gamma_G = \alpha(p) \Gamma_G \tag{5.1}$$

for some set of primes p. If $\Gamma_G = \sum_{n \geqq 0} \gamma_n q^n$ is holomorphic one knows by the usual formalism (cf. (4.6) and [L]) that (5.1) holds for the prime p if, and only if, the formal Dirichlet series associated Γ_G has an Euler factor for the prime p. In particular if

(5.1) holds for all p and if $\gamma_1 = 1_G$ then

$$\sum_{n \geqq 1} \frac{\gamma_n}{n^s} = \prod_p \left(1_G - \frac{\gamma_p}{p^s} + \frac{\psi_p}{p^{2s}}\right)^{-1} \tag{5.2}$$

We will use the symbol Λ for the Leech lattice, characterized as the unique 24-dimensional even, unimodular, integral lattice with no vectors of length 2 (see Conway's article in [C] for this and other facts we use below about Λ). The group of isometries of Λ is the Conway group, hereby denoted by Co. If ϱ is the rational representation of Co afforded by Λ we will study the Thompson series Ω_{Co} of Sect. 2. By Theorem 2 we have $\Omega_{\mathrm{Co}} \in \mathfrak{M}(\varrho)$, and we will look more closely at the forms $\eta_g(z)$ defined by (2.2) for $g \in \mathrm{Co}$. In Appendix 2 the reader will find a tabulation of these forms and various facts about them; almost all of this material has been supplied by A.O.L. Atkin.

The following definition is useful: call g of *permutation-type* if the characteristic polynomial $\chi^g(t)$ of $\varrho(g)$ – (2.2) – is such that each $k(i)$ is non-negative. In the following we let $\eta_g(z)$ be as in (2.4) with $N(g)$ the corresponding level; thus $\eta_g(z)$ is a form on $\Gamma_0(N(g))$ of weight $k(g)$ and Dirichlet character ε_g.

The following results, which I still find remarkable, summarize some facts about the forms $\eta_g(z)$ for $g \in \mathrm{Co}$.

(5.3) Precisely one of the following holds:

(a) $k(g) = 0$, that is $\eta_g(z)$ has weight zero or equivalently g fixes no non-zero vectors in Λ.

(b) $k(g) > 0$ and g is of permutation-type. In this case $\eta_g(z)$ is a (holomorphic) cusp-form and is the unique normalized cusp-form of level $N(g)$ and weight $k(g)$.

(c) $k(g) > 0$ and g is not of permutation-type. In this case there are no non-zero cusp-forms of level $N(g)$ and weight $k(g)$ and $\eta_g(z)$ is a holomorphic Eisenstein series.

(5.4) Suppose that $k(g) > 0$. Then the Dirichlet series associated to $\eta_g(z)$ has an Euler-factor for the prime p, if, and only if, g is not weakly p-singular. In particular this holds for all $p \geqq 5$. Moreover no g is weakly p-singular for both $p = 2$ and 3, so in any case the coefficients of $\eta_g(z)$ are multiplicative.

The assertions of (5.3b) are considered in more detail in [K], [M3] and [KM]; more precisely these papers classify all eta-products of permutation-type which are multiplicative. There appear to be just three such eta-products which do *not* appear in Co, namely

$$18 \cdot 6, \quad 16 \cdot 8, \quad 3^2 \cdot 9^2.$$

As for (5.3c), one can check using Lemma 2.10 that even if g is not of permutation-type then $\eta_g(z)$ is holomorphic (we pointed out in Sect. 2 that this is definitely not the case in general).

Concerning the assertions of (5.4), if g is of permutative-type then the fact that $\eta_g(z)$ is an eigenform follows from (5.3b). As for the Eisenstein series, the multiplicativity of the coefficients still resides in the realms of the miraculous.

Now let

$$\Omega_{\mathrm{Co}} = \Omega = \sum_{n=1}^{\infty} \omega_n q^n \tag{5.5}$$

be the Thompson series for Co given by (2.6). We also introduce the corresponding Dirichlet series

$$L_{\mathrm{Co}}=L=\sum_{n=1}^{\infty}\frac{\omega_n}{n^s}. \tag{5.6}$$

After (5.4) we know that if $g\in\mathrm{Co}$ satisfies $k(g)>0$ then the "usual" Dirichlet series given by

$$L_g(s)=\sum_{n=1}^{\infty}\frac{\omega_n(g)}{n^s} \tag{5.7}$$

has an Euler factor for all but perhaps one prime and thus in any case has multiplicative coefficients. Let us denote the "p-part" of L by

$$L_p=\sum_{n=0}^{\infty}\frac{\omega_{p^n}}{p^{ns}},$$

and that of $L_g(s)$ by

$$L_{p,g}(s)=\sum_{n=0}^{\infty}\frac{\omega_{p^n}(g)}{p^{ns}}.$$

Thus

$$L_g(s)=\prod_p L_{p,g}(s)\,(k(g)>0).$$

In order to clarify the nature of the series L_p we introduce the half-spin character ψ_2 of $\mathrm{Spin}(24,R)$ and the generalized character ψ_p of $SO(23,R)$ for odd p as in Examples 1 and 2 of the previous section, regarding ψ_2 as a character of $\mathrm{Spin}(23,R)$ by restriction when convenient. The next result follows from the preceding discussion and the results of Sect. 4:

(5.8) **Lemma.** *Suppose that $k(g)>0$. If g is not weakly p-singular then*

$$L_{p,g}(s)=\left(1-\frac{\omega_p(g)}{p^s}+\frac{\psi_p(g)}{p^{2s}}\right)^{-1};$$

in particular this holds if $p\geqq 5$.

The factors $L_{p,g}(s)$ for g weakly p-singular seem to be quite interesting; the data from Appendix 2 indicates the following result:

(5.9) If g is weakly p-singular and $k(g)>0$ then

$$L_{p,g}(s)=\left(1-\frac{\omega_p(g)}{p^s}+\sum_{n=2}^{\infty}\psi_p(g)\frac{p^{k(n-2)}}{p^{ns}}\right)^{-1}.$$

This can also be written in the form

$$L_{p,g}(s)=\left(1-\frac{\omega_p(g)}{p^s}+\frac{\psi_p(g)}{p^s(p^s-p^k)}\right)^{-1}.$$

We take this opportunity to say something more about this result. In Sect. 3 we avoided the use of the values of the characters ψ_p on weakly p-singular elements, but so far as Co is concerned we may proceed as follows: first, the extension (4.9) splits (since Co has a trivial Schur multiplier [G]) so we may regard Co $\leqq \mathrm{Spin}(24, R)$. It is readily verified that the following holds:

(5.10) As Co-modules we have

$$\Delta^{+}(12) \cong \Lambda + \lambda^{3}(\Lambda)$$

(here we identify Λ with the complex Co-module $\Lambda \otimes C$ and $\lambda^{3}(\Lambda)$ is its third exterior power). On the other hand it will follow easily from the next section that the following holds:

(5.11)
$$\begin{aligned} \omega_2 &= -\Lambda \\ \omega_4 &= -(\lambda^{3}(\Lambda) - \Lambda \otimes \Lambda + \Lambda) \end{aligned}$$

(here we identify a module with its character). Comparing (5.10) and (5.11) we see that

(5.12) As Co-module, $\Delta^{+}(12)$ affords the character $\omega_2^2 - \omega_4$, i.e., $\psi_2 = \omega_2^2 - \omega_4$.

This shows that the term corresponding to $n = 2$ in (5.9) is correct ($p = 2$). There is only one class of weakly 3-singular elements g in Co, and the value of $\psi_3(g)$ is easily verified to be consistent with (5.9). We recall that if f is a normalized modular form, say $f = \sum_{n \geqq 1} a(n) q^n$, and if f is invariant under the Hecke operator T_2, then one has $a(2)^2 - a(4) = \varepsilon(2) 2^{k-1}$ where k is the weight and ε the Dirichlet character of f. The result (5.12) is, of course, the Co-analogue of this fact.

Finally, we state some specific results for subgroups of Co which fix a non-zero vector of Λ.

Theorem 5. *Let $G \leqq$ Co fix a non-zero vector of Λ. Then the Hecke operator T_p exists for G whenever G has no weakly p-singular elements (in particular if $p \geqq 5$). In this case we have*

$$T_p \Omega = \omega_p \Omega,$$

equivalently

$$L_p = \left(1 - \frac{\omega_p}{p^s} + \frac{\psi_p}{p^{2s}}\right)^{-1}.$$

Corollary. *Let $M_{24} \leqq$ Co be the usual subgroup permuting an orthonormal frame of Λ. Then we have*

$$T_p \Omega = \omega_p \Omega, \quad \textit{all primes } p,$$

and

$$\sum_{n=1}^{\infty} \frac{\omega_n}{n^s} = \prod_p \left(1 - \frac{\omega_p}{p^s} + \frac{\psi_p}{p^{2s}}\right)^{-1}.$$

(In these two results we have identified the characters ω_n and ψ_p with their restriction to G.)

Proof. Since G fixes a non-zero vector in Λ we get an embedding $G \leqq SO(23, R)$, and even $G \leqq \mathrm{Spin}(23, R)$ coming from Co $\leqq \mathrm{Spin}(24, R)$. Now Theorem 2, Theorem 4, the theory of Hecke operators and the results of this section yield the theorem. As M_{24} is a permutation group on a basis of $\Lambda \otimes Q$ the corollary follows from Theorem 5 and the remarks following Theorem 4.

In Table 5 of [C] the reader will find listed many vector-stabilizers within Co to which Theorem 5 applies, among them several sporadic groups such as Co_2, Co_3 (the two smallest Conway groups), Mc (McLaughlin), HS (Higman-Sims), as well as M_{24} and several classical groups.

Of course there is an analogue of the corollary for each of these groups, but since they will generally contain weakly p-singular elements we have to use (5.9) and consequently the factor L_p may be more complicated.

Finally we mention another sort of Euler product that we receive gratis from the results of Sects. 3 and 4. For simplicity we restrict our attention to the groups $\mathrm{Spin}(2r-1, R)$ and $O(2r-1, R)$ with r even, and let ψ_p for p a prime be as in Examples 1 and 2 of Sect. 4. Consider the formal expansion

(5.13) $$\prod_p \left(1 - \frac{\psi_p}{p^s}\right)^{-1}.$$

"Evaluating" (5.13) at a rational element g which is not weakly p-singular for any prime p yields the product (cf. (4.7))

(5.14) $$\prod_p \left(1 - \frac{\varepsilon_g(p) p^{k(g)-1}}{p^s}\right)^{-1},$$

and (5.14) is just the Euler product for the Dirichlet series of the "twisted" Eisenstein series $E_{k(g),\varepsilon}(s)$ defined by

(5.15) $$E_{k,\varepsilon}(s) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1,\varepsilon}(n)}{n^s}$$

where

(5.16) $$\sigma_{k-1,\varepsilon}(n) = \sum_{d|n} \varepsilon(d) d^{k-1}.$$

Thus if we define the generalized characters σ_n of $\mathrm{Spin}(2r-1, R)$ via

$$\sigma_n = \sum_{d|n} \psi_d$$

then we obtain the equivariant analogue of (5.14–5.16). Thus in particular we have

(5.17) **Lemma.** *Let* $G \leqq \mathrm{Spin}(2r-1, R)$ *be a finite group containing no weakly p-singular elements for any prime p. Then the formal identity*

$$\sum_{n=1}^{\infty} \frac{\sigma_n}{n^s} = \prod_p \left(1 - \frac{\psi_p}{p^s}\right)^{-1}$$

reduces to (5.14–5.16) *at each element* $g \in G$.

We remark that we have been careful to avoid the Thompson series associated to the Dirichlet series $\sum \sigma_n n^{-s}$ of Lemma 5.17, for that entails studying the constant term of the Eisenstein series associated to (5.15).

6. Some Explicit Modules

Suppose that $\Gamma_G = \sum \gamma_n q^n$ is a Thompson series for the group G with V_n the (generalized) G-module affording the character γ_n. Then the graded G-module

$$\sum_n \oplus V_n \tag{6.1}$$

affords Γ_G in an obvious sense; conversely gives a graded G-module (6.1) we can form the series

$$\sum_n V_n q^n = \sum_n \gamma_n q^n \tag{6.2}$$

(identifying a module with the character it affords). In this section we will be concerned with the following issues: to describe genuine graded G-modules, i.e., each V_n is a genuine G-module, such that it affords a Thompson series Γ_G with the property that for each $g \in G$, $\Gamma_G(z) = \sum_n \gamma_n(g) q^n$ is a modular function, i.e., has weight 0.

This problem is inspired, of course, by the questions raised in [CN] concerning the Monster. Furthermore anyone familiar with the work of [FLM] will see some familiar objects below, though perhaps from a different perspective.

First let L be an even G-lattice (hypothesis 2.5) with $\mathbb{C}L$ the group algebra of L and $L_n = \{x \in L \mid (x,x) = 2n\}$. Then there is a natural G-grading attached to $\mathbb{C}L$:

$$\mathbb{C}L = \sum_{n \geqq 0} \oplus L_n \tag{6.3}$$

and of course $\mathbb{C}L$ affords the Thompson series Θ_G of (2.7).

Next, let V be some G-module, and consider the graded module

$$M = M(V) = \sum_{i \geqq 1} \oplus V_i \tag{6.4}$$

where each $V_i \cong V$. We let $S(M)$ and $\Lambda(M)$ be the symmetric and exterior algebras of M, each of which carries a natural G-grading. Specifically we have in terms of q-expansions,

$$\begin{aligned} S(M) &= \prod_{k=1}^{\infty} \sum_{r=0}^{\infty} S^r(V) q^{kr}, \\ \Lambda(M) &= \prod_{k=1}^{\infty} \sum_{r=0}^{\infty} \lambda^r(V) q^{kr}, \end{aligned} \tag{6.5}$$

where $S^r(V)$, $\lambda^r(V)$ are the r-th symmetric and exterior powers of V (cf. Sect. 4 of [M2] or Sect. 2.2 of [Br]).

We will also have cause to consider a variant of these modules, namely let

$$M_{1/2} = \sum_{n \geqq 1} \oplus V_{n-1/2}$$

be graded by half-integers, where again $V_{n-1/2} \cong V$; then define $\Lambda(M_{1/2})$ to be the exterior algebra on $M_{1/2}$ and let

$$\Lambda^o(M_{1/2})$$

be the submodule consisting elements of integral degree. In terms of q-expansions we may write

$$(6.6) \qquad \Lambda^\circ(M_{1/2}) = \left[\prod_{k=1}^{\infty} \sum_{r=0}^{\infty} \lambda^r(V) q^{r(k-1/2)}\right]_Z$$

where the subscript Z means that we take only the integral powers of q.

(6.7) **Lemma.** *Let ϱ be a rational representation of the finite group G by unimodular matrices of degree $24d$, $d \in Z$, on the space V. Then with M as in (6.4), the Thompson series Ω_G^{-1} is afforded by $S(M)$ with the grading decreased by d. In terms of q-expansions, Ω_G^{-1} arises from*

$$q^{-d} S(M) = q^{-d} \prod_{k=1}^{\infty} \sum_{r=0}^{\infty} S^r(V) q^{kr}.$$

Proof. See [M2], where it is also shown that Ω_G itself corresponds to

$$q^d \prod_{k=1}^{\infty} \sum_{r=0}^{\infty} (-1)^r \lambda^r(V) q^{kr}.$$

(Equation (5.11) follows from this.)

Theorem 6.1 *Let L be an even, unimodular G-lattice with $V = V_L$ the corresponding rational G-module. Assume that $G \leqq \Theta(V_L)$ and that* $\dim L = 24d$, *$d \in Z$. Let X be the graded G-module*

$$q^{-d} \mathbb{C}L \otimes S(M).$$

Then the following hold:

(i) *X affords the Thompson series $\theta_{L,G}/\Omega_G = J_G$, say.*

(ii) *If $g \in G$ then $J_g(z) = \theta_{Lg}(z)/\eta_g(z)$ is a modular function.*

(iii) *$J_g(z)$ has level $N(g)$ dividing $24o(g)$.*

Remark. Part (iii) should be compared to part (4) of Thompson's conjecture in [T]. Note that X is a genuine G-module, moreover for any given G we can find an L satisfying the hypothesis of Theorem 6.1.

Proof. See [M5].

We note that if $g = 1$ in Theorem 6.1 then $J_1(z)$ (the Poincaré series of X) is given by $\theta_L(z)/\eta(z)^{24d}$ and by (iii) is a modular function of level 1. The q-expansion is of the form q^{-d} + higher terms, in particular if $\dim L = 24$ then $J_1(z)$ must differ from the modular function $j(z)$ by a constant. Let us set

$$J(q) = q^{-1} + 196884q + \ldots$$

so that $J(q) = j(z) - 744$. We may ask for those groups G with the property that there is a Thompson series $\Gamma_G = \sum_{n \geqq 1} \gamma_n q^n$ such that

(a) $\Gamma_g(z) = \sum \gamma_n(g) q^n$ is a modular function for each $g \in G$.

(b) $\Gamma_1(z) = J(q) + \text{constant}$.

Of course we can take each γ_n to be a sum of trivial G-modules – hardly an interesting situation – we are interested in non-trivial examples. Theorem 6.1

provides some examples whenever G acts on an even, unimodular lattice of dimension 24. We give another example which is in some sense "48-dimensional."

Theorem 6.2. *Let V be the standard 48-dimensional module for $SO(48, R)$, let M be as in (6.4), and let X be the graded* Spin$(48, R)$ *module given by*

$$q^{-1}\Lambda^{o}(M_{1/2})\oplus q^{2}\Delta^{+}(24)\Lambda(M).$$

Then the following hold:

(i) *The Poincaré series of X is $J(q)+1128$.*

(ii) *Let $G\leqq SO(48, R)$ be represented rationally on V, and assume that G contains no weakly 2-singular elements and also that the pull-back of G into* Spin$(48, R)$ *splits. Then X affords the Thompson series*

$$J_{G}=T_{2}\Omega_{G}/\Omega_{G}$$

of G where T_2 is the Hecke operator of Theorem 4.

Remarks. A discussion of the case in which G contains weakly 2-singular elements or does not split with lifted to Spin$(48, R)$ is also possible, although we will ignore it here.

Proof. Let us compute the q-expansion of the level 1 form $T_2\eta(z)^{48}/\eta(z)^{48}$, which is

$$\begin{aligned}
\frac{(U_2+2^{23}V_2)\left(q^2\prod\limits_{n\geqq 1}(1-q^n)^{48}\right)}{q^2\prod\limits_{n\geqq 1}(1-q^n)^{48}} &= q^{-1}U_2\left\{\frac{\prod\limits_{n\geqq 1}(1-q^n)^{48}}{\prod\limits_{n\geqq 1}(1-q^{2n-1})^{48}}\right\} \\
&\quad +2^{23}q^2\frac{\prod\limits_{n\geqq 1}(1-q^{2n})^{48}}{\prod\limits_{n\geqq 1}(1-q^n)^{48}} \\
&= q^{-1}U_2\left(\prod_{n\geqq 1}(1-q^{2n-1})^{48}\right)+2^{23}q^2\prod_{n\geqq 1}(1+q^n)^{48} \\
&= q^{-1}U_2\left(\prod_{n\geqq 1}(1+q^{2n-1}\right)^{48}+2^{23}q^2\prod_{n\geqq 1}(1+q^n)^{48} \\
&= q^{-1}\left[\prod_{n\geqq 1}(1+q^{n-1/2})^{48}\right]_{Z}+2^{23}q^2\prod_{n\geqq 1}(1+q^n)^{48}
\end{aligned}$$

where the subscript "Z" indicates that we take only the integral powers of q.

Now with V as in the theorem, the formalism of Sect. 2 of [M2] shows that we obtain the graded Spin$(48, R)$-module whose Poincaré series is this last q-expansion by replacing $(1+q^i)^{48}$ by$\sum\lambda^r(V)q^{ri}$ and replacing 2^{23} by the half-spin module $\Delta^+(24)$. A comparison with (6.5, 6.6) shows that we obtain the module X of the theorem.

Since $T_2\eta(z)^{48}/\eta(z)^{48}=q^{-1}+1128+\dots$ is a modular function of level 1 then (i) holds, and more generally Sect. 2 of [M2] shows that (ii) also holds as long as the "equivariant" operator T_2 is well-behaved, which is guaranteed by Theorem 4.1(ii).

The first few terms of the q-expansion afforded by X are as follows:

$$q^{-1}+\lambda^2V+(\lambda^4V+V^2)q+(\lambda^6V+V\lambda^3V+\lambda^2V+V^2+\Delta^+(23))q^2+\dots$$

Let us consider some finite subgroups G of $SO(48, R)$ to which Theorem 6.2 applies. Certainly if G acts on 48-dimensional lattice then X affords a Thompson series for the pullback $\tilde{G}$ of G into Spin$(48, R)$. For example if we take $G \cong \mathrm{Co} \times \mathrm{Co}$ (acting on the sum of two copies of the Leech lattice) we get a Thompson series for G, moreover by taking either one of the direct factors or a "diagonal" copy of Co we get distinct Thompson series.

Another interesting example is obtained by taking $G \cong A_{48}$ acting on its natural permutation module. Then the graded module X of Theorem 6.2 affords a Thompson series for the 2-fold covering group $\hat{A}_{48}$ and part (ii) of the theorem applies to any subgroup of A_{48} which splits over the center of $\hat{A}_{48}$. For example we may take $L_2(47)$ (acting on the projective line over $GF(47)$) of M_{24}.

One can check that in the $L_2(47)$ case each of the series $J_g(z) = T_2\eta_g(z)/\eta_g(z)$ coincides (up to a constant) with the Thompson series of an element of the same order in the Monster as in [CN], suggesting that perhaps $L_2(47) \leq M$ (although in fact it is not *). If we take $M_{24} \leq \hat{A}_{48}$ such that it acts with two orbits of length 24 on the 48 letter permuted by A_{48} then the Thompson series $T_2\Omega_{M_{24}}/\Omega_{M_{24}}$ appears to coincide with that of Theorem 6.1 (with L = Leech lattice) up to a constant.

Appendix

Eta-Products for Co

We list information about the forms $\eta_g(z)$ arising from the action of the Conway group Co on the Leech lattice, as studied in Sect. 5. More precisely in Table 1 we give the Dirichlet series corresponding to $\eta_g(z)$ for those $g \in \mathrm{Co}$ with $k(g) > 0$, g not of permutation-type, and g not weakly p-singular for any prime p (cf. (5.4, 5.3c)); Table 2 gives similar information for g weakly p-singular, but with the first few terms of the p-part of the Dirichlet series given explicitly; Table 3 gives the coefficient of q^p (for the first few primes p) in the q-expansion of $\eta_g(z)$ for g of permutation type (cf. (5.3b)).

We use the following notation: for a Dirichlet character χ and integer $k \geq 1$ set

$$E_k(s, \chi) = \sum_{n=1}^{\infty} a(n) n^{-s}, \qquad a(n) = \sum_{d|n} \chi\left(\frac{n}{d}\right) d^{k-1}.$$

For an integer m we also let $E_k(s, \chi)^{(m)}$ be the Dirichlet series $E_k(s, \chi)$ with the Euler p-factors removed for each prime $p|m$. In the tables we index the elements $g \in \mathrm{Co}$ by $a \cdot b$, a being the order of g and b being used to differentiate between non-conjugate elements of the same order. The shape column gives the integers $k(i)$ in the characteristic polynomial (2.2), the weight and level being that of the corresponding eta-product $\eta_g(z)$. The column "char" gives the conductor of the primitive character which induces the Dirichlet character of $\eta_g(z)$. In Tables 1 and 2 we write E_k and $E_k^{(m)}$ for the corresponding Dirichlet series since χ is uniquely determined by the level and conductor.

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* I thank Simon Norton for pointing this out to me

Table 1. "Eisensteins"

Element	Shape	Wt.	Lvl.	Char.	D. Series
2.2	$2^{16}1^{-8}$	4	2	1	E_4
3.2	$3^{9}1^{-3}$	3	3	− 3	E_3
4.2	$2^{6}4^{4}1^{-4}$	3	4	− 4	E_3
4.3	$4^{8}2^{-4}$	2	4	1	$E_2^{(2)}$
5.2	$5^{5}1^{-1}$	2	5	5	E_2
6.3	$2^{5}3^{4}6.1^{-4}$	3	6	− 3	E_3
6.5	$2^{4}6^{4}1^{-2}3^{-2}$	2	6	1	$(1-3^{-s})^{-1}E_2^{(3)}$
6.6	$3^{3}6^{3}1^{-1}2^{-1}$	2	6	1	$(1-2^{-s})^{-1}E_2^{(2)}$
6.10	$1.6^{6}2^{-2}3^{-3}$	1	6	− 3	$(1+2^{-s})^{-1}E_1^{(2)}$
8.2	$2^{3}48^{2}1^{-1}$	2	8	8	E_2
8.3	$8^{4}4^{-2}$	1	8	− 4	$E_1^{(2)}$
9.2	$9^{3}3^{-1}$	1	9	− 3	$E_1^{(3)}$
10.4	$2^{3}5^{2}.10.1^{-1}$	2	10	5	E_2
12.3	$2^{2}3^{2}4.12.1^{-2}$	2	12	12	E_2
12.6	$2^{3}6.12^{2}.1^{-1}3^{-1}4^{-2}$	1	12	− 4	$(1+3^{-s})^{-1}E_1^{(3)}$
12.10	$4^{2}12^{2}2^{1}6^{-1}$	1	12	− 3	$E_1^{(2)}$
12.14	$12^{2}3.12^{2}.4^{-2}$	2	12	12	$\prod_p\left[1-\left(\left(\frac{-4}{p}\right)p+\left(\frac{-3}{p}\right)\right)p^{-s}\right]^{-1}$
12.15	$1^{2}4.6^{2}.12.3^{-2}$	2	12	12	$\prod_p\left[\left(\left(\frac{-3}{p}\right)p+\left(\frac{-4}{p}\right)\right)p^{-s}\right]^{-1}$
14.2	$2^{2}.14^{2}.1^{-1}7^{-1}$	1	14	− 7	E_1
16.2	$2^{2}.16^{2}.4^{-1}8^{-1}$	1	32	− 8	$\prod_p\left[1-\left(\left(\frac{8}{p}\right)+\left(\frac{-4}{p}\right)\right)p^{-s}\right]^{-1}$
18.4	$2^{2}9.18.1^{-1}6^{-1}$	1	18	− 3	$E_1^{(3)}$
20.2	$2^{2}5.20.1^{-1}4^{-1}$	1	20	− 4	E_1
20.6	$1.2.10.20.4^{-1}5^{-1}$	1	20	−20	$\prod_p\left[1-\left(\left(\frac{5}{p}\right)+\left(\frac{-4}{p}\right)\right)p^{-s}\right]^{-1}$
24.2	$2.3.4.24.1^{-1}8^{-1}$	1	24	− 8	E_1
24.8	$2.6.8.24.4^{-1}12^{-1}$	1	48	− 3	$\prod_p\left[1-\left(\left(\frac{-4}{p}\right)+\left(\frac{12}{p}\right)\right)p^{-s}\right]^{-1}$
24.9	$1.4.6.24.3^{-1}8^{-1}$	1	24	−24	$\prod_p\left[1-\left(\left(\frac{-3}{p}\right)+\left(\frac{8}{p}\right)\right)p^{-s}\right]^{-1}$
30.4	$2.3.5.30.1^{-1}15^{-1}$	1	30	−15	E_1
30.10	$1.6.10.15.3^{-1}.5^{-1}$	1	30	−15	$(1+2^{-s})^{-1}\prod_{p\neq 2}\left[1-\left(\left(\frac{-3}{p}\right)+\left(\frac{5}{p}\right)\right)p^{-s}\right]^{-1}$

Table 2. Weakly p-singular elements

Element	Shape	Wt.	Lvl.	Char.	p'-part of D. series	p	p-part of D. series
4.8	$1^8 4^8 2^{-8}$	4	4	1	$E_4^{(2)}$	2	$1-8.2^{-s}-64.4^{-s}-512.8^{-s}-4096.16^{-s}\ldots$
6.15	$1^5 3.6^4 2^{-4}$	3	6	-3	$E_3^{(2)}$	2	$1-5.2^{-s}-11.4^{-s}-53.8^{-s}-203.16^{-s}\ldots$
8.9	$1^4 8^4 2^{-2} 4^{-2}$	2	8	1	$E_2^{(2)}$	2	$1-4.2^{-s}+0.4^{-s}+0.8^3+0.16^{-s}\ldots$
10.9	$1^3.5.10^2.2^{-2}$	2	10	5	$E_2^{(2)}$	2	$1-3.2^{-s}-1.4^{-s}-7.8^{-s}-9.16^{-s}\ldots$
12.5	$2.3^3.12^3.1^{-1}4^{-1}6^{-3}$	1	12	-3	$E_1^{(2)}$	2	$1+1.2^{-s}-1.4^{-s}+1.8^{-s}-1.16^{-s}\ldots$
12.16	$1^2 3^2 4^2.12^2.2^{-2} 6^{-2}$	2	12	1	$E_2^{(2)}$	2	$1-2.2^{-s}-4.4^{-s}-8.8^{-s}-16.6^{-s}\ldots$
12.17	$1^3.12^3 2^{-1} 3^{-1} 4^{-1} 6^{-1}$	1	12	-3	$E_1^{(2)}$	2	$1-3.2^{-s}+3.4^{-s}-3.8^{-s}+3.16^{-s}\ldots$
16.3	$1^2.16^2 2^{-1} 8^{-1}$	1	16	-4	$E_1^{(2)}$	2	$1-2.2^{-2}+0.4^{-s}+0.8^{-s}+0.16^{-s}\ldots$
18.7	$1.2.18^2 6^{-1} 9^{-1}$	1	18	-3	$(1+2^{-s})^{-1}E_1^{(6)}$	3	$1-2.3^{-s}-2.9^{-s}-2.27^{-s}\ldots$
18.8	$1^2.9.18.2^{-1}.3^{-1}$	1	18	-3	$E_1^{(6)}$	2	$1-2.2^{-s}+1.4^{-s}-2.8^{-s}+1.16^{-s}\ldots$
28.3	$1.4.7.28.2^{-1}14^{-1}$	1	28	-7	$E_1^{(2)}$	2	$1-1.2^{-s}-1.4^{-s}-1.8^{-s}-1.16^{-s}\ldots$
30.7	$2.3.5.30.6^{-1}10^{-1}$	1	30	-15	$\prod_{p\neq 2}\left[1-\left(\left(\frac{-3}{p}\right)+\left(\frac{5}{p}\right)\right)p^{-s}\right]^{-1}$	2	$1+0.2^{-s}-1.4^{-s}+2.8^{-s}-3.16^{-s}\ldots$

Table 3. The cusp-forms

Element	Shape	Wt.	Lvl.	Char.	2	3	5	7	11	13	17
1.1	1^{24}	12	1	1	−24	252	4830	−16744	534612	−577738	−6905934
2.3	2^{12}	6	4	1	0	−12	54	−88	540	−418	594
2.4	$1^8 2^8$	8	2	1	−8	12	−210	1016	1092	1382	14706
3.3	3^8	4	9	1	0	0	0	20	0	−70	0
3.4	$1^6 3^6$	6	3	1	−6	9	6	−40	−564	638	882
5.3	$1^4 5^4$	4	5	1	−4	2	−5	6	32	−38	26
4.4	4^6	3	16	−4	0	0	−6	0	0	10	−30
4.5	$2^4 4^4$	4	8	1	0	−4	−2	24	−44	22	50
4.7	$1^4 2^4 4^4$	5	4	−4	−4	0	−14	0	0	−238	322
6.7	$2^3 6^3$	3	12	−3	0	−3	0	2	0	−22	0
6.9	6^4	2	36	1	0	0	0	−4	0	2	0
6.11	$1^2 2^2 3^2 6^2$	4	6	1	−2	−3	6	−16	12	38	−126
7.2	$1^3 7^3$	3	7	−7	−3	0	0	−7	−6	0	0
8.4	$4^2 8^2$	2	32	1	0	0	−2	0	0	6	2
8.8	$1^2 2 4 8^2$	3	8	−8	−2	−2	0	0	14	0	2
10.5	$2^2 10^2$	2	20	1	0	−2	−1	2	0	2	−6
11.1	$1^2 11^3$	2	11	1	−2	−1	1	−2	1	4	−2
12.11	12^2	1	144	−4	0	0	0	0	0	−2	0
12.12	2.4.6.12	2	24	1	0	−1	−2	0	4	−2	2
14.3	1.2.7.14	2	14	1	−1	−2	0	1	0	−4	6
15.3	1.3.5.15	2	15	1	−1	−1	1	0	−4	−2	2
20.4	4.20	1	80	−20	0	0	−1	0	0	0	0
21.2	3.21	1	63	−7	0	0	0	−1	0	0	0
22.2	2.22	1	44	−11	0	−1	−1	0	1	0	0
23.1	1.23	1	23	−23	−1	−1	0	0	0	−1	0

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