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Bielliptic Abelian Surfaces

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0. Introduction

In this paper we consider polarized abelian surfaces (A, H_A) where H_A is a polarization of type $(1, p)$. By polarization we mean as usual a Riemann form on a lattice defining A or equivalently a line bundle on A modulo translation. We shall assume $p \geqq 5$ and prime. The latter assumption is not essential for our methods but helps to keep the computations short. Our aim is to prove a criterion for H_A to be very ample which can be formulated in terms of period matrices (Theorems 0.8 and 0.9). Note that all the results of this paper except for 0.1, 0.2, 0.8, 0.9, 0.10 and 4.4 (iii) are also true for $p=3$.

Our starting point is a result of Ramanan [R] which gives a geometric characterisation of such surfaces. In order to formulate Ramanan's result, recall that one can find a cyclic covering

$$\pi : A \to A/\mathbb{Z}_p =: B$$

onto a principally polarized abelian surface B such that H_A is the pullback of the principal polarization H_B on B. Since (B, H_B) is a principally polarized abelian surface it is necessarily a Jacobian and one of the two following possibilities occurs:

(i) $B = \operatorname{Jac} C$ where C is a smooth genus 2 curve

(ii) $B = E_1 \times E_2$ where E_1 and E_2 are elliptic curves and the principal polarization is given by the reducible curve $C = E_1 \times \{0\} \cup \{0\} \times E_2$.

The polarization H_A on A is then given by $D = \pi^{-1}(C)$. In case (i) D is a smooth curve of genus $p+1$. This case was considered in [R].

Proposition 0.1 (Ramanan). *H_A is very ample unless D and C admit elliptic involutions which commute with the Galois action of the covering $A \to B$, i.e. if and only if there is a cartesian diagram*

$$\begin{array}{ccc} D & \xrightarrow{2:1} & E \\ \pi \downarrow & & \downarrow p:1 \\ C & \xrightarrow{2:1} & E' \end{array}$$

where E and E' are elliptic curves.

Proof. [R].

The remaining case (ii) was treated in [HL].

Proposition 0.2 (Hulek-Lange). *Let $\alpha=(\alpha_1, \alpha_2)\in E_1\times E_2$ be a p-torsion point which corresponds to the covering $A\to B$. Then H_A is very ample unless one of the following two cases occurs:*

(i) *$\alpha_1=0$ or $\alpha_2=0$. Then (A, H_A) splits as a polarized abelian surface.*

(ii) *There exists an isomorphism $\phi: E_1\to E_2$ with $\phi(\alpha_1)=\alpha_2$.*

Proof. [HL].

Remark 0.3. (i) Condition (ii) of Proposition 0.2 just means that the reducible curves C, resp. D admit $2:1$ covers onto elliptic curves which commute with the Galois covering π. This corresponds to Ramanan's result.

(ii) An independent proof can be given using Reider's result [Be].

As the reader may already have noticed, and as will become even clearer later, many of the arguments in this paper (or in this subject) use the cyclic covering $\pi: A\to B$. (Such a covering always exists but is not unique.) It is worthwhile to formalize this notion.

Definition 0.4. Let (A, H_A) be an abelian surface with a polarization of type $(1,p)$. A *root* of (A, H_A) is a (necessarily p-fold) cyclic covering map $\pi:(A, H_A)\to(B, H_B)$ with (B, H_B) a principally polarized abelian surface, i.e. a p-fold cover $\pi: A\to B$ with H_A the pullback of the principal polarization H_B via π. (A, H_A) together with a root will be called a *rooted* polarized abelian surface.

In order to formulate our result we consider the *Siegel upper half space* of *degree* 2:

$$\mathfrak{S}_2=\{\tau\in M(2\times 2, \mathbb{C}),\ \tau={}^t\tau,\ \operatorname{Im}\tau>0\} .$$

On $\mathbb{R}^4$ we fix the standard symplectic form

$$J=\begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix}$$

and denote the corresponding symplectic group over $\mathbb{Q}$ by

$$Sp(4, \mathbb{Q})=\{X\in GL(4, \mathbb{Q}),\ XJ{}^tX=J\} .$$

Note that vectors in $\mathbb{R}^4$ will be considered as row vectors with the group $GL(4, \mathbb{Q})$ operating by multiplication from the right. The group $Sp(4, \mathbb{Q})$ operates on $\mathfrak{S}_2$ by

$$\tau\mapsto(A\tau+B)(C\tau+D)^{-1}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(4, \mathbb{Q})$ and A, B, C and D are 2×2 matrices. Consider the groups

$$\Gamma'=Sp(4, \mathbb{Z})$$

$$\Gamma^0_{1,p}=\left\{X\in Sp(4, \mathbb{Q}),\ X\in\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ \frac{1}{p}\mathbb{Z} & \frac{1}{p}\mathbb{Z} & \frac{1}{p}\mathbb{Z} & \mathbb{Z} \end{pmatrix}\right\}$$

$$\Gamma_{1,p}=\left\{X\in\Gamma^0_{1,p},\ X-\mathbb{1}_4\in\begin{pmatrix}\mathbb{Z}&\mathbb{Z}&\mathbb{Z}&p\mathbb{Z}\\ p\mathbb{Z}&p\mathbb{Z}&p\mathbb{Z}&p^2\mathbb{Z}\\ \mathbb{Z}&\mathbb{Z}&\mathbb{Z}&p\mathbb{Z}\\ \mathbb{Z}&\mathbb{Z}&\mathbb{Z}&p\mathbb{Z}\end{pmatrix}\right\}$$

$${}_0\Gamma_{1,p}=\Gamma^0_{1,p}\cap\Gamma' .$$

As we shall see shortly the quotients of $\mathfrak{S}_2$ by these groups are moduli spaces. But first we need a lemma on the relation of the groups themselves.

Lemma 0.5. (i) *There is a tower of groups with indices as shown:*

$$\begin{tikzcd}
\Gamma^0_{1,p} \arrow[dr, dash, "p+1"'] & & \Gamma'=Sp(4,\mathbb{Z}) \arrow[dl, dash, "(p+1)(p^2+1)"] \\
& \Gamma^0_{1,p}\cap\Gamma'={}_0\Gamma_{1,p} \arrow[d, dash, "p(p-1)"] & \\
& \Gamma_{1,p} &
\end{tikzcd}$$

(ii) $\Gamma_{1,p}\lhd\Gamma^0_{1,p}$ *with quotient* $SL(2,\mathbb{Z}_p)$

$\Gamma_{1,p}\lhd{}_0\Gamma_{1,p}$ *with quotient* $\left\{\begin{pmatrix}a&b\\0&d\end{pmatrix}\in SL(2,\mathbb{Z}_p)\right\}$

Proof. It is convenient to transform the problem so that we are working with integer matrices. To this end let F be the diagonal matrix $F=\mathrm{Diag}(1,1,1,p)$, let $\tilde{\Gamma}^0_{1,p}=F\Gamma^0_{1,p}F^{-1}$, and similarly for the other groups. Then $\tilde{\Gamma}^0_{1,p}=\tilde{S}p(4,\mathbb{Z})$, the group of integer matrices symplectic with respect to the matrix FJ^tF. This matrix gives the inner product

$$\langle(v_1,v_2,v_3,v_4),\ (w_1,w_2,w_3,w_4)\rangle=\det\begin{pmatrix}v_1&v_3\\w_1&w_3\end{pmatrix}+p\cdot\det\begin{pmatrix}v_2&v_4\\w_2&w_4\end{pmatrix} .$$

Let $X=(x_{ij})$ be an element of $\tilde{S}p(4,\mathbb{Z})$. Observe that $x_{21}\equiv x_{23}\equiv x_{41}\equiv x_{43}\equiv 0$ $(\mathrm{mod}\,p)$. To see this let x_i denote row i of X. Then X in $\tilde{S}p(4,\mathbb{Z})$ implies

$$\langle x_1,x_3\rangle=1\equiv\det\begin{pmatrix}x_{11}&x_{13}\\x_{31}&x_{33}\end{pmatrix}(\mathrm{mod}\,p)$$

and

$$\langle x_i,x_j\rangle=0\equiv\det\begin{pmatrix}x_{i1}&x_{i3}\\x_{j1}&x_{j3}\end{pmatrix}(\mathrm{mod}\,p)\quad\text{for}\quad i=1,3,\ j=2,4$$

The first of these equations shows that (x_{11},x_{13}) and (x_{31},x_{33}), regarded as vectors in $(\mathbb{Z}_p)^2$, are linearly independent, and then the second shows that (x_{21},x_{23}) and (x_{41},x_{43}), regarded as vectors in $(\mathbb{Z}_p)^2$, are both zero, as claimed.

Thus for $X \in \tilde{\Gamma}^0_{1,p}$,

$$X \equiv \begin{pmatrix} * & * & * & * \\ 0 & a & 0 & b \\ * & * & * & * \\ 0 & c & 0 & d \end{pmatrix} \pmod{p} ,$$

where an entry marked $*$ is allowed to be arbitrary. Then direct calculation shows

$$\tilde{\Gamma}_{1,p} = \left\{ X \in \tilde{\Gamma}^0_{1,p},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p} \right\}$$

and

$${}_0\tilde{\Gamma}_{1,p} = \{ X \in \tilde{\Gamma}^0_{1,p},\ c \equiv 0 \pmod{p} \} .$$

The map

$$\tilde{\Gamma}^0_{1,p} \to SL(2, \mathbb{Z}_p)$$

given by

$$M \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \pmod{p}$$

is easily checked to be a homomorphism, so its kernel, $\tilde{\Gamma}_{1,p}$, is a normal subgroup. Since there is an inclusion $SL(2,\mathbb{Z}) \to \tilde{\Gamma}^0_{1,p}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix}$$

it is then immediate that the quotient $\tilde{\Gamma}^0_{1,p}/\tilde{\Gamma}_{1,p}$ is isomorphic to $SL(2,\mathbb{Z}_p)$, of order $p(p^2-1)$. By inspection ${}_0\tilde{\Gamma}_{1,p}/\tilde{\Gamma}_{1,p}$ is then seen to be as claimed, and this group has order $p(p-1)$, giving the indices as claimed.

We are left with computing $[\Gamma' : {}_0\Gamma_{1,p}]$. We see

$$\Gamma' \cap \Gamma^0_{1,p} = \left\{ X \in Sp(4,\mathbb{Z}),\ X \equiv \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & d \end{pmatrix} \pmod{p} \right\}$$

where of course d is not congruent to zero mod p. This group obviously has the corresponding subgroup where $d \equiv 1 \pmod{p}$ as a subgroup of index $p-1$. On the other hand, $Sp(4,\mathbb{Z})$ acts transitively on the p^4-1 non-zero elements of $(\mathbb{Z}_p)^4$, and this latter subgroup is the stabilizer of one of them, hence has index p^4-1, and so $[\Gamma' : \Gamma' \cap \Gamma^0_{1,p}] = (p^4-1)/(p-1) = (p+1)(p^2+1)$.

We then obtain a diagram

$$\begin{array}{ccc} \mathscr{A}^0_{1,p}=\mathfrak{S}_2/\Gamma^0_{1,p} & & \mathscr{A}'=\mathfrak{S}_2/\Gamma' \\ \nwarrow & & \nearrow \\ & {}_0\mathscr{A}_{1,p}=\mathfrak{S}_2/{}_0\Gamma_{1,p} & \\ & \uparrow & \\ & \mathscr{A}_{1,p}=\mathfrak{S}_2/\Gamma_{1,p} & \end{array}$$

in which the arrows indicate branched covering maps. The covers $\mathscr{A}_{1,p}\to\mathscr{A}^0_{1,p}$ and $\mathscr{A}_{1,p}\to{}_0\mathscr{A}_{1,p}$ are Galois coverings with groups given by Lemma 0.5 (ii). Note, however, that the center $\{\pm\mathbb{1}_4\}$ of $Sp(4,\mathbb{Q})$ acts trivially on $\mathfrak{S}_2$. The non-trivial element of the center is in ${}_0\Gamma_{1,p}$ but not $\Gamma_{1,p}$. Hence the degree of the cover $\mathscr{A}_{1,p}\to{}_0\mathscr{A}_{1,p}$ is $p(p-1)/2$, and similarly for the other covers, and the groups which act as the effective Galois groups are the quotients of the above-mentioned groups by their centers of order 2.

Expressed differently, the action of $Sp(4,\mathbb{Q})$ on $\mathfrak{S}_2$ factors through the projective group $PSp(4,\mathbb{Q})$. We prefer to work with$Sp(4,\mathbb{Q})$, so that we may write down matrices unambiguously, but this preference accounts for many of the $\pm$ signs in the sequel.

These spaces are moduli spaces and have the following interpretations:

$$\mathscr{A}'=\{(B,H_B),\ B \text{ is an abelian surface; } H_B \text{ is a principal polarization}\}$$

$$\mathscr{A}^0_{1,p}=\{(A,H_A),\ A \text{ is an abelian surface; } H_A \text{ is a polarization of type } (1,p)\}$$

$$\mathscr{A}_{1,p}=\{(A,H_A,\alpha),\ \alpha \text{ is a level-}p\text{ structure}\}$$

$${}_0\mathscr{A}_{1,p}=\{(\mathrm{A},\mathrm{H}_A,\pi),\ \pi \text{ is a root}\}.$$

The first three of these are standard (see [I1] or [H]); we prove the fourth.

Proposition 0.6. *${}_0\mathscr{A}_{1,p}$ is the moduli space of rooted abelian surfaces with polarization of type $(1,p)$.*

Proof. Given a point t in $\mathscr{A}_{1,p}$, we obtain a rooted abelian surface as follows: The point t corresponds to (A,H_A,α) with α an isomorphism $\alpha: L^\vee/L\to(\mathbb{Z}_p)^2$: Here L is a lattice defining A and $L^\vee$ is its dual. Let K be the subspace $\{(0,k)\}\subset(\mathbb{Z}_p)^2$. Then $\alpha^{-1}(K)$ is a subgroup of $A^{(p)}$, the group of p-torsion points of A. Set $B=A/\alpha^{-1}(K)$ with the obvious projection π and polarization H_B. Thus we have a map

$$\mathscr{A}_{1,p}\to\mathscr{M}$$

where $\mathscr{M}$ is the desired moduli space. Note that t_1 and t_2 in $\mathscr{A}_{1,p}$ will yield the same rooted abelian surface if and only if they define the same polarized abelian surface (A,H_A) and the corresponding level structures α_1 and α_2 satisfy $\alpha_1^{-1}(K)=\alpha_2^{-1}(K)$, i.e. $\alpha_2\alpha_1^{-1}(K)=K$.

Now $\alpha_2\alpha_1^{-1}\in SL(2,\mathbb{Z}_p)$, and the stabilizer of K in $SL(2,\mathbb{Z}_p)$ is exactly the group $\left\{\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\right\}$, so $\mathscr{M}$ is the quotient of $\mathscr{A}_{1,p}$ by the action of this group,

$$\mathscr{M}=\mathscr{A}_{1,p}\Big/\left\{\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\in SL(2,\mathbb{Z}_p)\right\}={}_0\mathscr{A}_{1,p}$$

by Lemma 0.5.

Remark 0.7. A point t in ${}_0\mathscr{A}_{1,p}$ corresponds to $\pi:(A,H_A)\to(B,H_B)$. The maps ${}_0\mathscr{A}_{1,p}\to\mathscr{A}^0_{1,p}$ and ${}_0\mathscr{A}_{1,p}\to\mathscr{A}'$ are the obvious forgetful maps taking this pair to (A,H_A) and (B,H_B) respectively.

The degree of the cover ${}_0\mathscr{A}_{1,p}\to\mathscr{A}^0_{1,p}$ being $(p+1)$ corresponds to the fact that $(\mathbb{Z}_p)^2$ has $(p+1)$ subspaces isomorphic to $\mathbb{Z}_p$.

For fixed p we define the following surfaces in $\mathfrak{S}_2$:

$$\mathfrak{H}_1=\left\{\tau=\begin{pmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_3\end{pmatrix}\in\mathfrak{S}_2,\quad \tau_2=0\right\}$$

$$\mathfrak{H}_2=\left\{\tau=\begin{pmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_3\end{pmatrix}\in\mathfrak{S}_2,\quad p\tau_1-2\tau_2=0\right\}.$$

These are Humbert surfaces in the sense of [F], [vG]. Their discriminant is $\Delta(\mathfrak{H}_1)=1$, resp. $\Delta(\mathfrak{H}_2)=4$. We denote their images under the natural projections from $\mathfrak{S}_2$ to $\mathscr{A}_{1,p}$, ${}_0\mathscr{A}_{1,p}$, $\mathscr{A}^0_{1,p}$, and $\mathscr{A}'$ by $\mathscr{H}_i$, ${}_0\mathscr{H}_i$, $\mathscr{H}^0_i$, and $\mathscr{H}_i'$ respectively ($i=1,2$). While $\mathscr{H}_2'$ in $\mathscr{A}'$ may appear to depend on p, this is in fact not the case; under the action of the element

$$\begin{pmatrix}\frac{1-p}{2} & 1 & & \\ -\frac{1+p}{2} & 1 & & \\ & & 1 & \frac{1+p}{2}\\ & & -1 & \frac{1-p}{2}\end{pmatrix}$$

of $\Gamma'=Sp(4,\mathbb{Z})$ the space $\mathfrak{H}_2$ is taken to

$$\mathfrak{H}_3=\left\{\tau=\begin{pmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_3\end{pmatrix}\in\mathfrak{S}_2,\quad \tau_1-\tau_3=0\right\}$$

which is evidently independent of p.

We can now formulate our main result as follows:

Theorem 0.8. *Let*

$$\tilde{\mathscr{A}}_{1,p}:=\mathscr{A}_{1,p}\backslash(\mathscr{H}_1\cup\mathscr{H}_2).$$

Then

$$\tilde{\mathscr{A}}_{1,p}=\{(A,H_A,\alpha)\in\mathscr{A}_{1,p},\ H_A \text{ is very ample}\}$$

i.e. $\tilde{\mathscr{A}}_{1,p}$ *is the moduli space of abelian surfaces with a very ample polarization of type* $(1,p)$ *and level-p structure.*

As an immediate consequence we get

Corollary 0.9. *Let*

$$\tilde{\mathscr{A}}^0_{1,p}:=\mathscr{A}^0_{1,p}\backslash(\mathscr{H}^0_1\cup\mathscr{H}^0_2).$$

Then

$$\widetilde{\mathscr{A}}^0_{1,p}=\{(A,H_A)\in\mathscr{A}^0_{1,p},\ H_A \text{ is very ample}\}$$

i.e. $\widetilde{\mathscr{A}}^0_{1,p}$ *is the moduli space of abelian surfaces with a very ample polarization of type* $(1,p)$.

Remark 0.10. The analogue for ${}_0\widetilde{\mathscr{A}}_{1,p}={}_0\mathscr{A}_{1,p}\backslash({}_0\mathscr{H}_1\cup{}_0\mathscr{H}_2)$ obviously holds as well.

Remark 0.11. We shall see that $\mathscr{H}_1^0$ parametrizes the abelian surfaces which split as polarized abelian surfaces, whereas $\mathscr{H}_2^0$ parametrizes the abelian surfaces described in Proposition 0.1 and Proposition 0.2 (ii).

1. Bielliptic Abelian Surfaces

We first consider principally polarized abelian surfaces (B,H_B) and write them in the form

$$B=\mathbb{C}^2/L_\tau$$

where τ is an element in the Siegel space $\mathfrak{S}_2$ and the lattice L_τ is generated by the rows of the matrix $\begin{pmatrix}\tau\\ \mathbb{1}_2\end{pmatrix}$ such that the polarization H_B with respect to this basis is given by the matrix J. An involution of the principally polarized abelian surface (B,H_B) is an involution $j:B\to B$ with $j^*(H_B)=H_B$. Every principally polarized abelian surface admits the involution $\iota:x\mapsto -x$. Now assume that (B,H_B) admits an additional involution $j\neq\iota$. Then j induces a symplectic automorphism of the lattice L_τ, i.e. it defines an element $X(j)\in Sp(4,\mathbb{Z})$. The involutions in $Sp(4,\mathbb{Z})$ are well known and it follows from [U, p. 198] that up to conjugation we are in one of the following two cases:

$$\pm X(j)=T:=\begin{pmatrix}-1&&&\\&1&&\\&&-1&\\&&&1\end{pmatrix} \tag{1}$$

and

$$\tau\in\operatorname{Fix}T=\left\{\tau=\begin{pmatrix}\tau_1&\tau_2\\ \tau_2&\tau_3\end{pmatrix}\in\mathfrak{S}_2,\quad \tau_2=0\right\}=\mathfrak{H}_1 .$$

Remark 1.1. In this case (B,H_B) splits and the involution on $B=E_1\times E_2$ is given by $j=\pm(-id_{E_1},id_{E_2})$.

$$\pm X(j)=U:=\begin{pmatrix}0&1&&\\1&0&&\\&&0&1\\&&1&0\end{pmatrix} \tag{2}$$

and

$$\tau\in\operatorname{Fix}U=\left\{\tau=\begin{pmatrix}\tau_1&\tau_2\\ \tau_2&\tau_3\end{pmatrix}\in\mathfrak{S}_2,\quad \tau_1-\tau_3=0\right\}=\mathfrak{H}_3 .$$

Definition 1.2. An involution of type (2) will be called an *elliptic involution*. If (B, H_B) admits an elliptic involution we call it a *bielliptic* principally polarized abelian surface.

Remark 1.3. Bolza [B] and Igusa [I2] studied genus 2 curves with non-trivial automorphisms. In this context they found the above surfaces. They are precisely the Jacobians of genus 2 curves C which admit an elliptic involution, i.e. a 2:1 cover onto an elliptic curve E. For a simple proof of this see Proposition 4.2. If C is reducible this means the $B=E\times E$ and $C=E\times\{0\}\cup\{0\}\times E$. The elliptic involution on C is then just given by the obvious map to E. These are the abelian surfaces corresponding to the intersection $\mathfrak{H}_1\cap\mathfrak{H}_3$.

We shall now consider polarized abelian surfaces (A, H_A) where H_A is a polarization of type $(1,p)$. Here p is an odd prime. As before we can write

$$A=\mathbb{C}^2/L_\tau$$

with $\tau\in\mathfrak{S}_2$ and L_τ the lattice spanned by the rows of the matrix $\begin{pmatrix} & \tau & \\ 1 & & 0 \\ 0 & & p\end{pmatrix}$ such that H_A with respect to this basis is given by the matrix

$$\begin{pmatrix} & & 1 & 0\\ & & 0 & p\\ -1 & 0 & & \\ 0 & -p & & \end{pmatrix}$$

The dual lattice $L_\tau^\vee$ is given by

$$L_\tau^\vee=\{x, H_A(x,y)\in\mathbb{Z} \text{ for all } y\in L_\tau\} .$$

Definition 1.4. Involutions j_A of (A, H_A) and j_B of (B, H_B) (with $j_A\neq\iota, j_B\neq\iota$) form an *involutory pair* (j_A, j_B) if there is a root $\pi:(A,H_A)\to(B,H_B)$ of (A,H_A) making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{j_A} & A\\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi}\\ B & \xrightarrow{j_B} & B\end{array} \tag{D}$$

Lemma 1.5: (i) *Let $j_A\neq\iota$ be an involution of (A,H_A), A an abelian surface with polarization H_A of type $(1,p)$. Then j_A is contained in an involutory pair (j_A, j_B).*

(ii) *Let $j_B\neq\iota$ be an involution of (B,H_B), B an abelian surface with principal polarization H_B. Then j_B is contained in an involutory pair (j_A, j_B).*

Proof. (i) The involution j_A defines an involution on L_τ which we shall also denote by j_A. Since j_A leaves the form H_A on L_τ invariant it maps $L_\tau^\vee$ into itself. Hence j_A induces an involution $\bar{j}_A$ on $L_\tau^\vee/L_\tau$ which is symplectic with respect to the form induced on this quotient. Choosing a sympletic basis of $L_\tau^\vee/L_\tau$ we can consider $\bar{j}_A$ as an element in $Sp(2,\mathbb{Z}_p)=SL(2,\mathbb{Z}_p)$. The only involutions in $SL(2,\mathbb{Z}_p)$ are $\pm id$. Hence $\bar{j}_A$ leaves every subgroup $\mathbb{Z}_p\subset L_\tau^\vee/L_\tau$ invariant and dividing out by such a subgroup gives the desired diagram (D).

(ii) Let $B^{(p)}$ be the group of p-torsion points of B. The involution j_B defines a decomposition $B^{(p)}=K_+\oplus K_-$, where $K_\pm$ are the eigenspaces for ± 1. Note that

$K_+ \simeq K_- \simeq (\mathbb{Z}_p)^2$. Every $x \in K_+$ or K_- defines a covering $A \to B$. Let H_A be the pullback of H_B to A. Then j_B lifts to an involution j_A on A over j_B with $j^*(H_A) = H_A$.

Remark 1.6. Note that we have shown that if (A, H_A) admits an involution and $\pi: (A, H_A) \to (B, H_B)$ is *any* root of (A, H_A) so does (B, H_B). On the other hand we have seen that beginning with (B, H_B) admitting an involution exactly $2(p+1)$ of the $(p+1)(p^2+1)$ choices of (A, H_A) with $\pi: (A, H_A) \to (B, H_B)$ admit such an involution.

From what we have said above there are now the following two possibilities:

(1) j_B is an involution of type (1). Then (B, H_B) splits, i.e. $B = E_1 \times E_2$ and $j_B = \pm(-\mathrm{id}_{E_1}, \mathrm{id}_{E_2})$. The covering $A \to B$ corresponds to a p-torsion point $\alpha = (\alpha_1, \alpha_2) \in E_1 \times E_2$. Since j_B lifts to the involution j_A it follows that $\alpha_1 = 0$ or $\alpha_2 = 0$. Then (A, H_A) splits as a polarized abelian surface.

(2) j_B is an elliptic involution.

Remark 1.7. Whether case (1) or (2) occurs does not depend on the choice of the map $\pi: A \to B$. This follows e.g. from the fact that in case (2) the linear system $|H_A|$ is base point free [R], [HL].

Definition 1.8. If a diagram (D) exists such that j_B is an elliptic involution we shall also call j_A an *elliptic involution*. If (A, H_A) admits an elliptic involution we shall call it a *bielliptic* polarized abelian surface.

Remark 1.9. By Remark 1.3 the bielliptic polarized abelian surfaces are just the surfaces described in Proposition 0.1 and Proposition 0.2 (ii).

2. Computations

We recall the Humbert surface

$$\mathfrak{H}_1 = \mathrm{Fix}\, T = \{\tau, \tau_2 = 0\} .$$

We now want to introduce the involution

$$S = \begin{pmatrix} -1 & 0 & & \\ -p & 1 & & \\ & & -1 & -p \\ & & 0 & 1 \end{pmatrix}$$

Straightforward calculation shows

$$\mathrm{Fix}\, S = \{\tau \in \mathfrak{S}_2, p\tau_1 - 2\tau_2\} = \mathfrak{H}_2$$

and we have seen that $\mathfrak{H}_2$ and $\mathfrak{H}_3$ are equivalent under $Sp(4, \mathbb{Z})$ (although not under $\Gamma^0_{1,p}$).

In this section we want to compute the stabilizer subgroups

$$P^0_i := \{g \in \Gamma^0_{1,p}, g(\mathfrak{H}_i) = \mathfrak{H}_i\}$$

resp.

$$P_i = P^0_i \cap \Gamma_{1,p} = \{g \in \Gamma_{1,p}, g(\mathfrak{H}_i) = \mathfrak{H}_i\} \ (i = 1, 2) .$$

Proposition 2.1. *$P_i^0/P_i \simeq SL(2, \mathbb{Z}_p)$ for $i=1,2$.*

Proof. We first treat the case $i=1$. Let

$$P_1^{\mathbb{Q}} := \{g \in Sp(4, \mathbb{Q}),\ g(\mathfrak{H}_1) = \mathfrak{H}_1\} \ .$$

It follows e.g. from Franke [F, Lemma 3.2.6] that

$$P_1^{\mathbb{Q}} = \left\{ g \in Sp(4, \mathbb{Q}),\ g = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix} \quad \text{or} \quad g = \begin{pmatrix} 0 & a & 0 & b \\ a' & 0 & b' & 0 \\ 0 & c & 0 & d \\ c' & 0 & d' & 0 \end{pmatrix} \right.$$

$$\left. \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2, \mathbb{Q}) \right\} .$$

An element of the form

$$\begin{pmatrix} 0 & a & 0 & b \\ a' & 0 & b' & 0 \\ 0 & c & 0 & d \\ c' & 0 & d' & 0 \end{pmatrix}$$

cannot be in $\Gamma_{1,p}^0$ since this would imply $a, c \in \mathbb{Z}$, $b, d \in p\mathbb{Z}$ a contradiction to $ad - bc = 1$. Hence the inclusion

$$\phi_1 : SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \to \Gamma_{1,p}^0$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & pb' \\ c & 0 & d & 0 \\ 0 & \frac{c'}{p} & 0 & d' \end{pmatrix}$$

gives an isomorphism

$$P_1^0 \simeq SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \ .$$

An element $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right)$ gives rise to an element in $\Gamma_{1,p}$ if and only if

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_1(p) = \{M \in SL(2, \mathbb{Z}),\ M \equiv \mathbb{1} \bmod p\} \ .$$

Hence

$$P_1^0/P_1 \simeq SL(2, \mathbb{Z})/\Gamma_1(p) \simeq SL(2, \mathbb{Z}_p) \ .$$

We now treat the case $i=2$. Consider the matrix

$$g_0=\begin{pmatrix} 1 & 0 & & \\ \frac{p}{2} & 1 & & \\ & & 1 & -\frac{p}{2} \\ & & 0 & 1 \end{pmatrix} \in Sp(4,\mathbb{Q}) .$$

Then

$$g_0 T g_0^{-1} = S$$

and hence

$$P_2^{\mathbb{Q}} = \{g \in Sp(4,\mathbb{Q}),\ g(\mathfrak{H}_2)=\mathfrak{H}_2\} = g_0 P_1^{\mathbb{Q}} g_0^{-1} .$$

For an element

$$g=\begin{pmatrix} 0 & a & 0 & b \\ a' & 0 & b' & 0 \\ 0 & c & 0 & d \\ c' & 0 & d' & 0 \end{pmatrix} \in P_1^{\mathbb{Q}}$$

we have

$$g_0 g g_0^{-1} = \begin{pmatrix} -\frac{p}{2}a & a & 0 & b \\ a'-\frac{p^2}{4}a & \frac{p}{2}a & b' & \frac{p}{2}(b+b') \\ -\frac{p}{2}(c+c') & c & -\frac{p}{2}d' & d-\frac{p^2}{4}d' \\ c' & 0 & d' & \frac{p}{2}d' \end{pmatrix}$$

Such an element cannot be in $\Gamma_{1,p}^0$ since this would imply $a, c \in \mathbb{Z}$; $b \in p\mathbb{Z}$ and $d' \in \frac{2}{p}\mathbb{Z}$ hence $d \in \frac{p}{2}\mathbb{Z}$ which contradicts $ad-bc=1$.

In order to determine P_2^0 and P_2 we now consider the inclusion

$$\phi_2 : SL(2,\mathbb{Q}) \times SL(2,\mathbb{Q}) \to P_1^{\mathbb{Q}}$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) \mapsto \begin{pmatrix} a & 0 & 2b & 0 \\ 0 & a' & 0 & \frac{p}{2}b' \\ \frac{c}{2} & 0 & d & 0 \\ 0 & \frac{2}{p}c' & 0 & d' \end{pmatrix}$$

We then have

$$g_0\begin{pmatrix} a & 0 & 2b & 0 \\ 0 & a' & 0 & \frac{p}{2}b' \\ \frac{c}{2} & 0 & d & 0 \\ 0 & \frac{2}{p}c' & 0 & d' \end{pmatrix} g_0^{-1} = \begin{pmatrix} a & 0 & 2b & pb \\ \frac{p}{2}(a-a') & a' & pb & \frac{p}{2}(pb+b') \\ \frac{1}{2}(pc'+c) & -c' & d & \frac{p}{2}(d-d') \\ -c' & \frac{2}{p}c' & 0 & d' \end{pmatrix}$$

This matrix is in $\Gamma_{1,p}^0$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2,\mathbb{Z}) ,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \bmod 2 .$$

Hence

$$P_2^0 \simeq \{(M,N) \in SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z}),\ M \equiv N \bmod 2\} .$$

Moreover a pair (M,N) gives rise to an element in $\Gamma_{1,p}$ if and only if $N \in \Gamma_1(p)$. This shows that

$$P_2 \simeq \{(M,N) \in SL(2,\mathbb{Z}) \times \Gamma_1(p),\ M \equiv N \bmod 2\}$$

and hence

$$P_2^0/P_2 \simeq SL(2,\mathbb{Z})/\Gamma_1(p) \simeq SL(2,\mathbb{Z}_p) .$$

Now let X be a set and assume that the group G^0 operates transitively on X. Let G be a normal subgroup of G^0 of finite index. For $x_0 \in X$ we consider the stabilizer subgroups

$$P^0 = \{g \in G^0,\ g(x_0) = x_0\}$$

resp.

$$P = P^0 \cap G = \{g \in G,\ g(x_0) = x_0\} .$$

Lemma 2.2. *If* $[G^0 : G] = [P^0 : P]$ *then* G *acts transitively on* X.

Proof. Since G^0 acts transitively on X we have an identification

$$X = G^0/P^0 .$$

Since G is normal we have

$$G\backslash X = G\backslash(G^0/P^0) = (G^0/P^0)/G = G^0/P^0G .$$

On the other hand

$$[G^0 : P^0G]\,[P^0G : G] = [G^0 : G] = [P^0 : P] .$$

Since

$$[P^0G : G] \geqq [P^0 : P^0 \cap G] = [P^0 : P]$$

this implies $[G^0 : P^0G] = 1$, hence $\#(G\backslash X) = 1$, i.e. G acts transitively.

We now return to the quotient map

$$\sigma : \mathscr{A}_{1,p} \to \mathscr{A}^0_{1,p}$$

which is given by the natural action of the group

$$\Gamma^0_{1,p}/\Gamma_{1,p} \simeq SL(2,\mathbb{Z}_p) \ .$$

Proposition 2.3. *$\sigma^{-1}(\mathscr{H}_i^0) = \mathscr{H}_i$ for $i=1,2$.*

Proof. Clearly $\mathscr{H}_i \subset \sigma^{-1}(\mathscr{H}_i^0)$. What we have to see is that $\sigma^{-1}(\mathscr{H}_i^0)$ consists of only one component. In order to see this we consider for fixed i the following set of Humbert surfaces

$$X = \{g(\mathfrak{H}_i),\ g \in \Gamma^0_{1,p}\} \ .$$

We have to show that $\Gamma_{1,p}$ operates transitively on X. But this is now an immediate consequence of Proposition 2.1 and Lemma 2.2.

Let ${}_0\sigma : \mathscr{A}_{1,p} \to \mathscr{A}^0_{1,p}$ be the quotient map.

Corollary 2.4. *${}_0\sigma^{-1}(\mathscr{H}_i^0) = {}_0\mathscr{H}_i$ for $i=1,2$.*

Proof. ${}_0\mathscr{H}_i$ is the image of $\mathscr{H}_i$ under the quotient map $\mathscr{A}_{1,p} \to {}_0\mathscr{A}_{1,p}$. Since $\mathscr{H}_i$ has only one component, the same holds for ${}_0\mathscr{H}_i$ $(i=1,2)$.

Remark 2.5. If ${}_0P_i = P_i^0 \cap {}_0\Gamma_{1,p}$, then

$${}_0P_i/P_i \simeq \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL(2,\mathbb{Z}_p)\right\} \quad \text{for} \quad i=1,2 \ .$$

A similar computation to 2.1 shows

Proposition 2.6. *Let P_i' be the stabilizer of $\mathfrak{H}_i$ in $Sp(4,\mathbb{Z})$. Then $[P_i' : {}_0P_i] = 2(p+1)$ for $i=1,2$.*

Proof. In the formulas for ϕ_1 and ϕ_2 in the proof of 2.1 set $p=1$ to obtain new maps ψ_1 and ψ_2, and use ψ_i in place of ϕ_i. If we let $Q = \{X = (x_{ij}) \in Sp(4,\mathbb{Q}),\ x_{ij} = 0$ for $i+j$ odd$\}$, and $P_1'' = P_1' \cap Q$, then obviously $[P_1' : P_1''] = 2$, and we have

$$P_1' = P_1'' \cup h_0 P_1'' \ \text{ for } \ h_0 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$

Now

$$\begin{aligned} P_2' &= g_0(P_1^{\mathbb{Q}})g_0^{-1} \cap Sp(4,\mathbb{Z}) \\ &= g_0(Q \cup h_0 Q)g_0^{-1} \cap Sp(4,\mathbb{Z}) \\ &= (g_0 Q g_0^{-1} \cap Sp(4,\mathbb{Z})) \cup (h_1(g_0 Q g_0^{-1}) \cap Sp(4,\mathbb{Z})) \end{aligned}$$

where $h_1 = g_0 h_0 g_0^{-1}$. Defining $P_2'' = g_0 Q g_0^{-1} \cap Sp(4,\mathbb{Z})$, we see that, since $h_1 \in Sp(4,\mathbb{Z})$, we also have $[P_2' : P_2''] = 2$.

Now, following the proof of 2.1, we see that ${}_0P_i \subset P_i''$, $i=1,2$. Then the rest of the proof goes through unchanged, except at the end we find, for $i=1,2$,

$$P_i''/{}_0P_i \simeq SL(2,\mathbb{Z})/\left\{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2,\mathbb{Z}),\ b' \equiv 0 \ (\operatorname{mod} p)\right\}$$

and this subgroup has index $p+1$ in $SL(2,\mathbb{Z})$.

Thus $[P_i' : {}_0P_i] = [P_i' : P_i''][P_i'' : {}_0P_i] = 2(p+1)$ as claimed.

3. Proof of the Main Result

In this section we want to prove Theorem 0.8. The main problem is to study the bielliptic polarized abelian surfaces. Recall from Sects. 1 and 2 that every bielliptic principally polarized abelian surface can be given by a period matrix of the form $\begin{pmatrix} \tau \\ \mathbb{1}_2 \end{pmatrix}$ where $\tau \in \mathfrak{H}_2$.

Proposition 3.1 *The bielliptic abelian surfaces with a polarization of type* $(1,p)$ *and level-p structure form an irreducible* 2*-dimensional family. They are parametrized by the surface* $\mathscr{H}_2 \subset \mathscr{A}_{1,p}$.

In view of Proposition 2.3 it will be enough to prove this result without level-p structure:

Proposition 3.1. *The bielliptic abelian surfaces with a polarization of type* $(1,p)$ *form an irreducible* 2*-dimensional family. They are parametrized by the surface* $\mathscr{H}_2^0 \subset \mathscr{A}_{1,p}^0$.

Proof. Let $\tau \in \mathfrak{H}_2$. Then the polarized abelian surface A defined by the period matrix $\begin{pmatrix} & \tau & \\ 1 & & 0 \\ 0 & & p \end{pmatrix}$ admits an elliptic involution. To see this note that since $\tau \in \mathfrak{H}_2$ it is of the form $\begin{pmatrix} 2\tau_1 & p\tau_1 \\ p\tau_1 & \tau_3 \end{pmatrix}$. The assertion then follows from the equality

$$\begin{pmatrix} -1 & 0 & & \\ -p & 1 & & \\ & & 1 & -1 \\ & & 0 & 1 \end{pmatrix} \begin{pmatrix} 2\tau_1 & p\tau_1 \\ p\tau_1 & \tau_3 \\ 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 2\tau_1 & p\tau_1 \\ p\tau_1 & \tau_3 \\ 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix}$$

which shows that $\begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix}$ induces an involution on A which lies over the elliptic involution defined by S on the corresponding principally polarized abelian surface.

We now have to show conversely that every bielliptic abelian surface with a polarization of type $(1,p)$ can be given by a period matrix of the form

$$\Omega_\tau = \begin{pmatrix} & \tau & \\ 1 & & 0 \\ 0 & & p \end{pmatrix}, \quad \tau \in \mathfrak{H}_2\ .$$

In order to do this we first observe that ${}_0\mathscr{H}_2$ parameterizes rooted abelian surfaces with an elliptic involution forming part of an involutory pair. This follows as ${}_0\mathscr{H}_2$ is the inverse image of $\mathscr{H}_2^0$ (from 2.4), so that for every (A, H_A, π) in ${}_0\mathscr{H}_2$, $\pi:(A, H_A)\to(B, H_B)$, the surface (A, H_A) admits an elliptic involution j_A (as we have just shown) and then Lemma 1.5 yields j_B. Conversely, given (B, H_B) with an elliptic involution j_B, by Lemma 1.5 and Remark 1.6 there are generically $2(p+1)$ choices for a rooted (A, H_A, π), with $\pi:(A, H_A)\to(B, H_B)$, admitting an elliptic involution. We shall show below that $2(p+1)$ is the degree of the cover ${}_0\mathscr{H}_2\to\mathscr{H}_2'$, and so all of these (A, H_A, π) must be parameterized by points in ${}_0\mathscr{H}_2$.

Now let $t_0\in\mathscr{A}_{1,p}^0$ parameterize an elliptic surface (A, H_A) with polarization of type $(1,p)$ admitting an elliptic involution j_A. Then by Lemma 1.5 there is a point ${}_0t\in{}_0\mathscr{A}_{1,p}$ parameterizing (A, H_A, π) with an elliptic involution forming part of an involutory pair. Thus ${}_0t\in{}_0\mathscr{H}_2$, so $t_0\in\mathscr{H}_2^0$ as claimed.

It remains to compute the degree of the cover ${}_0\mathscr{H}_2\to\mathscr{H}_2'$. By [$U$, p. 198] the set of points $Z=\{z\in\mathfrak{S}_2\}$ whose isotropy group in $\Gamma'=Sp(4,\mathbb{Z})$ is precisely $\{\pm\mathrm{id}, \pm S\}$ forms a Zariski open set of $\mathfrak{H}_2$, which is obviously P_2'-invariant, so we may compute the degree of the cover from this set. By [F, Satz 3.3.6], if $g\in\Gamma'$, $g\notin P_2'$ then $g(Z)\cap Z=\emptyset$. On the other hand, $P_2'/\{\pm\mathrm{id}, \pm S\}$ acts freely on Z, and $\{\pm\mathrm{id}, \pm S\}\subset{}_0P_2$, so the degree of the cover is $[P_2'/\{\pm\mathrm{id}, \pm S\}:{}_0P_2/\{\pm\mathrm{id}, \pm S\}]$ $=[P_2':{}_0P_2]=2(p+1)$ by 2.6.

Proof of Theorem 0.8. This is now straightforward. It remains to show that the polarized abelian surfaces with level-p structure which split are parameterized by $\mathscr{H}_1$. In view of Proposition 2.3 it is enough to prove this without level-p structure. But there the statement is obvious.

4. Geometric Properties of Bielliptic Abelian Surfaces

In this section we shall study some geometric properties of bielliptic abelian surfaces.

Proposition 4.1. *Let (A, H_A) be a bielliptic abelian surface which is either principally polarized or has a polarization of type $(1,p)$. Then A is isogenous to a product; more precisely there exist elliptic curves E and F such that $A=E\times F/\mathbb{Z}_2\times\mathbb{Z}_2$ and such that j_A is induced by* $(\mathrm{id}_E, -\mathrm{id}_F)$.

Proof. We first assume the case of a principal polarization and in order to remain consistent with our previous notation we denote the abelian surface by (B, H_B) and the elliptic involution by j_B. As we know from Sect. 1 we can assume that j_B is induced by the symplectic matrix

$$\pm U=\pm\begin{pmatrix}0&1&&\\1&0&&\\&&0&1\\&&1&0\end{pmatrix}$$

and that B is given by a period matrix of the form $\begin{pmatrix}\tau\\ \mathbb{1}_2\end{pmatrix}$ with $\tau\in\mathfrak{H}_3=\operatorname{Fix} U$, i.e. τ is of the form

$$\tau=\begin{pmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_1\end{pmatrix}.$$

On $\mathbb{C}^2$ the involution is given by the linear map $\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ operating on $\mathbb{C}^2$ by multiplication from the right. We shall restrict ourselves to the plus sign. Otherwise in what follows the roles of E_1 and E_2 will have to be interchanged.

Since $\tau\in\mathfrak{H}_3$ we have $\operatorname{Im}(\tau_1\pm\tau_2)\neq 0$. We consider the elliptic curves

$$E:=\mathbb{C}/\mathbb{Z}(\tau_1+\tau_2)+\mathbb{Z}\ ,\qquad F:=\mathbb{C}/\mathbb{Z}(\tau_1-\tau_2)+\mathbb{Z}$$

The maps

$$\mathbb{C}\to\mathbb{C}^2\ ,\qquad z\mapsto(z,z)\ \text{resp.}$$

$$\mathbb{C}\to\mathbb{C}^2\ ,\qquad z\mapsto(z,-z)$$

define embeddings of E and F into B such that

$$j_B|E=\mathrm{id}_E,\ j_B|F=-\mathrm{id}_F\ .$$

E and F intersect transversally in their respective 2-torsion points. From this it follows immediately that

$$B=E\times F/\mathbb{Z}_2\times\mathbb{Z}_2$$

and that j_B is induced by $(\mathrm{id}_E,\ -\mathrm{id}_F)$.

Now let (A,H_A) be a bielliptic surface with a polarization of type $(1,p)$. Then there exists a diagram

$$\begin{array}{ccc} A & \xrightarrow{j_A} & A\\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi}\\ B & \xrightarrow{j_B} & B\end{array}$$

where (B,H_B) is as above and where the covering π is given by a p-torsion point $x\in B$ with $j_B(x)=\pm x$. We shall assume that $j_B(x)=x$. The other case can be reduced to this by replacing j_B by $-j_B$. Then $x\in E$. Let $E'\to E$ be the corresponding covering. We then have a commutative diagram

$$\begin{array}{ccc} E'\times F & \longrightarrow & E\times F\\ \downarrow & & \downarrow\\ A & \longrightarrow & B\end{array}$$

such that $A=E'\times F/\mathbb{Z}_2\times\mathbb{Z}_2$ and that j_A is induced by $(\mathrm{id}_{E'},\ -\mathrm{id}_F)$.

In what follows it is useful to treat the case of principally polarized abelian surfaces first. So let (B,H_B) be a bielliptic principally polarized abelian surface with elliptic involution j_B and let E and F be as in Proposition 4.1.

Recall that the invariant e of a $\mathbb{P}_1$-bundle over a curve is defined by $-e=\min\{C_1^2,\ C_1 \text{ a section}\}$. If the base is an elliptic curve, then $e\geqq -1$, and there is a unique bundle with $e=-1$ [Ha, Theorem V.2.15].

Proposition 4.2. *There exists a curve C on B representing H_B which is mapped to itself by j_B. Moreover there is a commutative diagram*

$$\begin{array}{ccc} C & \xrightarrow{2:1} & \bar{C} = C/j_B \\ \cap\downarrow & & \cap\downarrow \\ B & \xrightarrow{2:1} & \bar{B} = B/j_B \ . \end{array}$$

$\bar{B}$ is the unique $\mathbb{P}_1$-bundle over E with invariant $e = -1$ and $\bar{C}$ is a section of $\bar{B}$.

Remark 4.3. In particular the double cover $C \to \bar{C}$ defines an elliptic involution on C.

Proof. We first claim that

$$E \cdot H_B = F \cdot H_B = 2 \ .$$

Indeed this follows from

$$H_B((\tau_1 + \tau_2, \tau_1 + \tau_2), (1, 1)) = H_B((\tau_1 - \tau_2, \tau_1 - \tau_2), (1, -1)) = 2 \ .$$

In order to find the required curve C we first assume that (B, H_B) is the Jacobian of a smooth curve. By choosing a suitable translate of the theta-divisor we can assume that H_B is represented by a curve C which goes through the origin of B and which is tangent to F [R, Prop. 4.2.]. In particular C intersects E transversally at 0. Since $C \cdot E = 2$ it must intersect E in some other point, say P. By construction $j_B(C)$ and C have the same tangent at 0 and both go through P. Hence the assumption $j_B(C) \neq C$ would imply $C^2 = j_B(C) \cdot C \geqq 3$, a contradiction. If (B, H_B) is the Jacobian of a reducible genus 2 curve it is sufficient to choose C such that it has its singularity at the origin of B. Since j_B is an elliptic involution on B it must interchange the two components of C.

In order to prove the rest of the statement we look at the elliptic curves

$$F_x := F + x \ , \qquad x \in E \ .$$

Then $j_B | F_x = -\mathrm{id}$ for all $x \in E$. The fixed points of $j_B | F_x$ are the four points of $F_x \cap E$. In particular F_x / j_B is a projective line and $\bar{B} = B/j_B$ is a $\mathbb{P}_1$-bundle over the curve

$$E/2\text{-torsion points} \simeq E \ .$$

Two points on C are identified under j_B if and only if they are the points of intersection $C \cap F_x$, $x \in E$.

Hence $\bar{C} = C/j_B$ becomes a section of $\bar{B}$.

It remains to determine the invariant e of the $\mathbb{P}_1$-bundle $\bar{B}$. Since $C^2 = 2$ it follows that $\bar{C}^2 = 1$. Hence e is odd. If $e \geqq 1$ this would imply the existence of a section C_1 of $\bar{B}$ with $C_1^2 < 0$. But this is impossible since on B no effective divisor exists with negative self-intersection.

Finally let (A, H_A) be a bielliptic abelian surface with a polarization of type $(1, p)$ which lies over a principally polarized bielliptic abelian surface (B, H_B). By Proposition 4.1 we can assume that $B = E \times F/\mathbb{Z}_2 \times \mathbb{Z}_2$ and $A = E' \times F/\mathbb{Z}_2 \times \mathbb{Z}_2$ where $E' \to E$ is a p-fold covering. Let C be as in Proposition 4.2 and let D be its preimage in A. If C is smooth then D is a smooth curve of genus $p+1$. Otherwise D consists of two elliptic curves intersecting in p points.

Putting

$$\bar{A}=A/j_A\ , \qquad \bar{B}=B/j_B$$

we have a commutative diagram

$$\begin{tikzcd}
D \arrow[rrr] \arrow[ddd] & & & \bar{D} \arrow[ddd] \\
& A \arrow[r, "\pi_A"'] \arrow[d, "\pi"'] \arrow[ul, phantom, sloped, "\subset"] & \bar{A} \arrow[d, "\pi"] \arrow[ur, phantom, sloped, "\subset"] & \\
& B \arrow[r, "\pi_B"'] \arrow[dl, phantom, sloped, "\subset"] & \bar{B} \arrow[dr, phantom, sloped, "\subset"] & \\
C \arrow[rrr] & & & \bar{C}
\end{tikzcd}$$

where all horizontal maps are $2:1$ and all vertical maps are $p:1$. Note that $\bar{D}\simeq E'$ and $\bar{C}\simeq E$. In particular $D\to\bar{D}$ defines an elliptic involution on D. This is precisely the situation described in Proposition 0.1 resp. Proposition 0.2 (ii).

Proposition 4.4 (i) *$\bar{A}$ is the unique $\mathbb{P}_1$-bundle over E' with invariant $e=-1$.*

(ii) *There is a section $\tilde{D}$ of $\bar{A}$ with $\tilde{D}^2=1$ such that*

$$\mathcal{O}_A(H_A)=\pi_A^*\mathcal{O}_{\bar{A}}(\bar{D})=\pi_A^*\mathcal{O}_{\bar{A}}\left(\tilde{D}+\frac{p-1}{2}f\right)$$

where f is a fibre of $\bar{A}$.

(iii) *The linear system associated to H_A is base point free. The associated map $\phi_A: A\to\mathbb{P}_{p-1}$ factors through a map $\phi_{\bar{A}}:\bar{A}\to\mathbb{P}_{p-1}$ which embeds $\bar{A}$ as an elliptic scroll of degree p in $\mathbb{P}_{p-1}$.*

Proof. (i) It follows from Proposition 4.2 that $\bar{B}$ is the unique $\mathbb{P}_1$-bundle over E with invariant $e=-1$. We can write $\bar{B}=\mathbb{P}(\mathscr{E})$ for a suitable rank 2 bundle $\mathscr{E}$ over E. By the above construction $\bar{A}=\mathbb{P}(\pi^*\mathscr{E})$ where $\pi: E'\to E$ is the p-fold covering from above. By [HL, p. 213] $\bar{A}$ is the unique $\mathbb{P}_1$-bundle over E' with invariant $e=-1$.

(ii) Since $e=-1$ we can find a section $\tilde{D}$ of $\bar{A}$ with $\tilde{D}^2=1$ and $\bar{D}\sim\tilde{D}+af$. Since

$$p=\bar{D}^2=(\tilde{D}+af)^2=1+2a$$

it follows that $a=\frac{1}{2}(p-1)$.

(iii) The map $\pi_A: A\to\bar{A}$ defines an inclusion

$$\pi_A^*:\Gamma\left(\mathcal{O}_{\bar{A}}\left(\tilde{D}+\frac{p-1}{2}f\right)\right)\to\Gamma(\mathcal{O}_A(H_A))\ .$$

Both spaces have dimension p, i.e. π_A^* must be an isomorphism. The assertion then follows from the fact that $|\tilde{D}+kf|$ is very ample for $k\geqq 2$.

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