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On Murasugi's and Traczyk's Criteria for Periodic Links

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1. Introduction

A link L in S^3 is called n-periodic if there is a Z_n action on S^3 with a circle as a fixed point set, which maps L onto itself, and such that L is disjoint from the fixed point set. Furthermore if L is an oriented link, we assume that each generator of Z_n preserves the orientation of L or changes it to the opposite one.

The skein polynomial (also called FLYPMOTH, generalized Jones, HOMFLY, Jones-Conway, twisted Alexander and two-variable Jones) of oriented links in S^3 can be defined uniquely by the conditions $P_{T_1}(a,z)=1$ and $aP_{L_+}(a,z)+a^{-1}P_{L_-}(a,z)=zP_{L_0}(a,z)$ where T_1 is the trivial knot and L_+, L_-, and L_0 are diagrams of oriented links which are identical except near one crossing where they look like in Fig. 1.1.

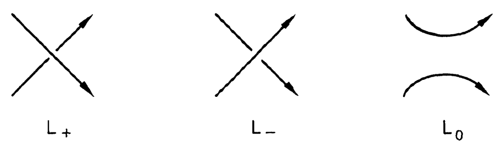

Fig. 1.1

Lemma 1.1. *Let $\mathscr{R}$ be the subring of the ring $Z[a^{\mp 1}, z^{\mp 1}]$ generated by $a^{\mp 1}$, $\frac{a+a^{-1}}{z}$ and z, then for any oriented link L, $P_L(a,z) \in \mathscr{R}$.*

Observe that z is not invertible in $\mathscr{R}$.

Proof. It is true for the trivial link of n components, T_n. Namely $P_{T_n}(a,z) = \left(\frac{a+a^{-1}}{z}\right)^{n-1} \in \mathscr{R}$. Furthermore if $P_{L_-}(a,z)$ [respectively $P_{L_+}(a,z)$] and $P_{L_0}(a,z)$ are elements of $\mathscr{R}$ then $P_{L_+}(a,z)$ [respectively $P_{L_-}(a,z)$] is an element of $\mathscr{R}$. Therefore Lemma 1.1 holds by the standard induction.

Now we can formulate our criterion for n-periodic links. It has especially simple form for a prime period (see Sect. 2 for the general statement).

Theorem 1.2. *Let L be an r-periodic oriented link and r a prime number, then the skein polynomial $P_L(a, z)$ satisfies:*

$$P_L(a, z) \equiv P_L(a^{-1}, z) \bmod (r, z^r),$$

where (r, z^r) is the ideal in $\mathscr{R}$ generated by r and z^r.

The Jones polynomial of oriented links, $V_L(t)$ can be obtained from the skein polynomial $P_L(a, z)$ by substituting $a = it^{-1}$, $z = i\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)$. Therefore

Corollary 1.3 [Tr1]. *Let L be an r-periodic oriented link and r a prime number. Then the Jones polynomial $V_L(t)$ satisfies:*

$$V_L(t) \equiv V_L(t^{-1}) \bmod (r, t^r - 1),$$

where $(r, t^r - 1)$ is the ideal in $Z[t^{\mp 1/2}]$ generated by r and $t^r - 1$.

The Kauffman polynomial of regular isotopy of non-oriented diagrams of links, $\Lambda_D(a, z)$, can be uniquely defined by the following properties:

(i) $\Lambda_O(a, z) = 1$

(ii) $\Lambda_{\rho}(a, z) = a\Lambda_{)}(a, z)$, $\Lambda_{\rho'}(a, z) = a^{-1}\Lambda_{)}(a, z)$,

(iii) $\Lambda_{\times}(a, z) + \Lambda_{\times'}(a, z) = z(\Lambda_{\asymp}(a, z) + \Lambda_{)(}(a, z))$

where the symbols $\times$, $\times'$, $\asymp$, and $)($ stand for diagrams which look like that in a neighborhood of the crossing and are identical elsewhere.

The Kauffman polynomial of (ambient) isotopy of oriented links, $F_L(a, z)$ is defined by

$$F_L(a, z) = a^{-w(D)}\Lambda_D(a, z),$$

where D is any diagram of L and $w(D)$ is the planar writhe (or twist) of D defined by taking the algebraic sum of the crossings, counting $\times$ and $\times'$ as $+1$ and -1 respectively.

For each L, $F_L(a, z) \in \mathscr{R}$, as in the case of the skein polynomial.

Theorem 1.4. *Let L be an r-periodic, oriented link and r a prime number. Then*

$$F_L(a, z) \equiv F_L(a^{-1}, z) \bmod (r, z^r),$$

where (r, z^r) is the ideal in $\mathscr{R}$ generated by r and z^r.

To use practically Theorems 1.2 and 1.4 we need the following fact.

Lemma 1.5. *Let $w(a, z) \in \mathscr{R}$ and $w(a, z) = \sum_i v_i(a) z^i$ where $v_i(a) \in Z[a^{\mp 1}]$. Then $w(a, z) \in (r, z^r)$ if and only if $v_i(a)$ is an element of the ideal $(r, (a + a^{-1})^{r-i})$ in $Z[a^{\mp 1}]$ for each $i \leqq r$.*

Example 1.6. Consider the knot 11_{388} (in the Perko [Pe] notation); Fig. 1.2. The skein polynomial of this knot, $P_{11_{388}}(a, z) = (3 + 5a^2 + 4a^4 + a^6) + (-4 - 10a^2 - 5a^4)z^2 + (1 + 6a^2 + a^4)z^4 - a^2 z^6$ (see [LM] or [P1]). Consider the difference $w(a, z) = P_{11_{388}}(a, z) - P_{11_{388}}(a^{-1}, z)$. $v_0(a)$ for $w(a, z)$ (in the notation of Lemma 1.5) is equal to $5(a^2 - a^{-2}) + 4(a^4 - a^{-4}) + a^6 - a^{-6}$. Therefore for a prime number $r \geqq 7$, $v_0(a) \notin (r, (a + a^{-1})^r) = (r, a^{2r} + 1)$. Therefore by Theorem 1.2 and Lemma 1.5 the knot 11_{388} is not r-periodic for $r \geqq 7$.

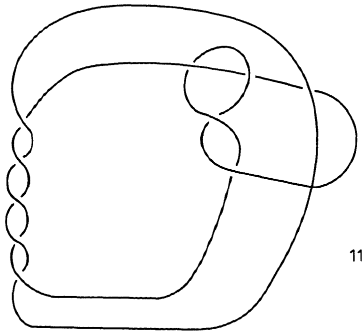

Fig. 1.2

Notice that the knot 11_{388} has the symmetric Jones polynomial $V_{11_{388}}(t) = V_{11_{388}}(t^{-1}) = t^{-2} - t^{-1} + 1 - t + t^2$ and therefore the Traczyk criterion (Corollary 1.3) cannot be applied.

Example 1.7. Consider the knot 10_{48} (in the Rolfsen notation [Ro]); Fig. 1.3. The skein polynomial of 10_{48} is symmetric ($P_{10_{48}}(a,z) = P_{10_{48}}(a^{-1},z)$; [Th, P1]) so Theorem 1.2 cannot be used to analyse periods of 10_{48}. We can use however the Kauffman polynomial to show that 10_{48} is not r-periodic for $r \geqq 7$. One can compute (see [Th] or [P1]) that $w(a,z) = F_{10_{48}}(a,z) - F_{10_{48}}(a^{-1},z) = z(a^5 + 3a^3 + 2a - 2a^{-1} - 3a^{-3} - a^{-5}) + z^2(\dots)$. $v_1(a)$ for $w(a,z)$ (in the notation of Lemma 1.5) is equal to $a^5 + 3a^3 + 2a - 2a^{-1} - 3a^{-3} - a^{-5}$ so for $r \geqq 7$, $v_1(a) \notin (r, (a + a^{-1})^{r-1})$. Therefore by Theorem 1.4 and Lemma 1.5 the knot 10_{48} is not r-periodic for $r \geqq 7$.

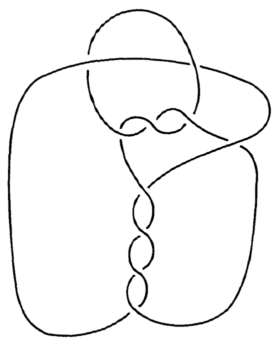

Fig. 1.3

2. Criterion for n-Periodic Links Using the Skein Polynomial

Theorem 2.1. *Let L be an n-periodic oriented link, then the skein polynomial $P_L(a,z)$ satisfies:*

$$P_L(a,z) \equiv P_L(a^{-1},z) \bmod J_n,$$

where J_n is the ideal in $\mathcal{R}$ generated by elements of type $kz^{n/k}$ where k is any divisor of n (e.g. nz and z^n are in J_n).

Corollary 2.2. *Let L be an n-periodic oriented link, then*

(i) *(Theorem 1.2) If n is a prime number then*

$$P_L(a,z) \equiv P_L(a^{-1},z) \bmod (n, z^n)$$

if $n = r^q$ is a power of a prime number, then

(ii) $P_L(a,z) \equiv P_L(a^{-1},z) \bmod (r^q, z^r)$

(iii) $P_L(a,z) \equiv P_L(a^{-1},z) \bmod (r, z^{r^q})$.

Proof of Corollary 2.2. It follows from Theorem 2.1 because the ideals (n, z^n), (r^q, z^r), and (r, z^{r^q}) are bigger then the ideal J_n.

Proof of Theorem 2.1. By the positive solution of the Smith conjecture [Sm, Thur], the fixed point set of our Z_n-action on S^3 is an unknotted circle and the action is conjugated to an orthogonal one. Therefore if we write S^3 as $R^3 \cup \infty$ we can assume that a fixed point set is a vertical axis with ∞ and our Z_n-action is generated by rotation ϕ given by the formula: $\phi(z,t) = (e^{2\pi i/n}z, t)$ where $R^3 = \{z, t : z$ complex and t real numbers$\}$. Each n-periodic link can be represented by ϕ-invariant diagram (also denoted by L); that is $\phi(L) = L$ or $-L$ where $-L$ denote the link (diagram) obtained from L by reversing its orientation; compare Fig. 2.1.

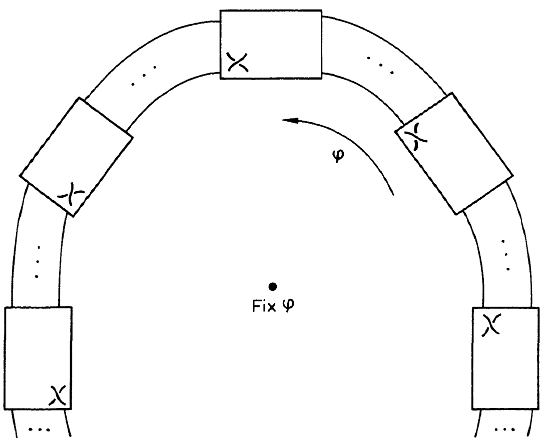

Fig. 2.1

. . .

Let $L_{\mathrm{sym}(\times)}$, $L_{\mathrm{sym}(\times)}$, and $L_{\mathrm{sym}(\asymp)}$ denote three ϕ-invariant diagrams which are identical except near the Z_n orbit of a single crossing where all n crossings of the orbit are positive in $L_{\mathrm{sym}(\times)}$, negative in $L_{\mathrm{sym}(\times)}$ and smoothed in $L_{\mathrm{sym}(\asymp)}$. We have the following fact.

Lemma 2.3.

$$a^n P_{L_{\mathrm{sym}(\times)}}(a,z) + (-1)^{n+1} a^{-n} P_{L_{\mathrm{sym}(\times)}}(a,z) \equiv z^n P_{L_{\mathrm{sym}(\asymp)}}(a,z) \bmod \hat{J}_n,$$

where $\hat{J}_n$ is the ideal in $\mathcal{R}$ generated by elements of type $kz^{n/k}$ where k is any divisor of n but 1.

Proof (based on the idea of Murasugi [Mu2]). Let p be a crossing of $L_{\mathrm{sym}(\times)}$ such that $L_{\mathrm{sym}(\times)}$ and $L_{\mathrm{sym}(\asymp)}$ differ only in crossings $p, \phi(p), \phi^2(p), \ldots, \phi^{n-1}(p)$. Let us build the (part of) binary resolving tree for $L_{\mathrm{sym}(\times)}$, which uses crossings $p, \phi(p), \ldots, \phi^{n-1}(p)$; compare Fig. 2.2. Now we analyse what value is introduced to $P_{L_{\mathrm{sym}(\times)}}(a,z)$ by leaves of our binary tree. The leaf $L_{\mathrm{sym}(\times)}$ gives $(-1)^n a^{-2n} P_{L_{\mathrm{sym}(\times)}}(a,z)$ to $P_{L_{\mathrm{sym}(\times)}}(a,z)$ and the leaf $L_{\mathrm{sym}(\asymp)}$ gives $a^{-n}z^n P_{L_{\mathrm{sym}(\asymp)}}(a,z)$. Observe that Z_n acts on leaves of the binary tree and the only fixed points of the action are $L_{\mathrm{sym}(\times)}$ and $L_{\mathrm{sym}(\asymp)}$. Let L^l be a leaf of the binary tree which is not a fixed point set and let k be the order of its orbit. Of course k divides n and $k>1$. Our leaf has been obtained from $L_{\mathrm{sym}(\times)}$ by applying smoothing at least $\frac{n}{k}$ times so it introduced to $P_{L_{\mathrm{sym}(\times)}}(a,z)$ the value $z^{n/k}P_{L^l}(a,z)(\cdot)$. Furthermore each element of the orbit of L^l is isotopic to L^l so the orbit of L^l introduced to $P_{L_{\mathrm{sym}(\times)}}(a,z)$ the value $kz^{n/k}P_{L^l}(a,z)(\cdot)$ which is an element of the ideal $\hat{J}_n$. Therefore

$$P_{L_{\mathrm{sym}(\times)}}(a,z) \equiv (-1)^n a^{-2n} P_{L_{\mathrm{sym}(\times)}}(a,z) + a^{-r}z^r P_{L_{\mathrm{sym}(\asymp)}}(a,z) \bmod \hat{J}_n .$$

This completes the proof of Lemma 2.3. □

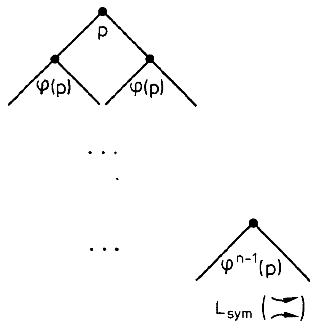

Fig. 2.2

Now we can complete the proof of Theorem 2.1. From Lemma 2.3 it follows immediately that

$$P_{L_{\mathrm{sym}(\times)}}(a,z) \equiv (-a^2)^{-n} P_{L_{\mathrm{sym}(\times)}}(a,z) \bmod J_n$$

(because $z^n \in J_n$). On the other hand one can go from L to its mirror image $\bar{L}$ using changes $L_{\mathrm{sym}(\times)}, \leftrightarrow L_{\mathrm{sym}(\times)}$ therefore $P_L(a,z) \equiv (-a^2)^{-nj} P_{\bar{L}}(a,z) \bmod J_n$ where nj is equal to the writhe number of the diagram L. Now, from the well known equality $P_{\bar{L}}(a,z) = P_L(a^{-1},z)$, it follows that $P_L(a,z) \equiv (-a^2)^{-nj} P_L(a^{-1},z) \bmod J_n$. To complete the proof of Theorem 2.1 we need the following fact.

Lemma 2.4. *$a^{2n}+(-1)^{n+1}$ is in the ideal J_n.*

Proof. J_n is generated by elements $kz^{n/k}$ so the elements $kz^{n/k}\left(\frac{a+a^{-1}}{z}\right)^{n/k} = k(a+a^{-1})^{n/k}$ are also in J_n so $k(a^2+1)^{n/k} \in J_n$. To prove Lemma 2.4 it is sufficient to show that $a^{2n}+(-1)^{n+1}$ is in the ideal generated by elements $k(a^2+1)^{n/k}$ in $Z[a]$. This reduces, after substituting $y=a^2+1$, to showing that $(y-1)^n+(-1)^{n+1}$ is in the ideal generated by the elements $ky^{n/k}$ in $Z[y]$ but it follows from the well known fact that for any natural number i, $\binom{n}{i}$ is a multiplicity of $\frac{n}{gcf(n,i)}$ so $\binom{n}{i} y^i$ is a multiplicity of $\frac{n}{gcf(n,i)} y^{gcf(n,i)}$ which is in the ideal. It completes our proof of Lemma 2.4 and Theorem 2.1. □

We can generalize Theorem 2.1 (or rather show its limits) by considering the following operations on link diagrams:

Definition 2.5 [P2]. (a) A t_k move is an elementary operation on an oriented link diagram L resulting in the diagram $t_k(L)$ as shown on Fig. 2.3. Two oriented links L and L' are said to be t_k equivalent if one can go from L to L' using $t_k^{\mp 1}$ moves (and isotopy).

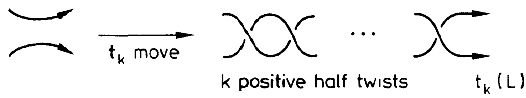

Fig. 2.3

(b) A $\bar{t}_k$ move is an elementary operation on an oriented link diagram L resulting in the diagram $\bar{t}_k(L)$, which is naturally oriented for k even, as shown on Fig. 2.4. Two oriented links L and L' are said to be $\bar{t}_k$ equivalent (k even) if one can go from L to L' using $\bar{t}_k^{\mp 1}$ moves (and isotopy).

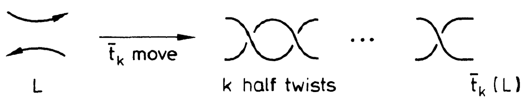

Fig. 2.4

(c) Two unoriented links L and L' are called k-equivalent (or t_k, $\bar{t}_k$ equivalent) if one can go from L to L' using $t_k^{\mp 1}$ or $\bar{t}_k^{\mp 1}$ moves and ignoring orientation. For k even k-equivalence is also well defined for oriented links.

Theorem 2.6. *For every oriented link L*

(a) $P_{t_{2n}(L)}(a,z) \equiv P_L(a,z) \bmod J_n$

(b) $P_{\bar{t}_{2n}(L)}(a,z) \equiv P_L(a,z) \bmod J_n$.

Proof. It follows essentially from Theorems 1.1 and 1.7 of [P2] but we can give a short proof independent from [P2]. We need, first, the lemma corresponding to Lemma 2.3.

Lemma 2.7. (a) $a^n P_{t_n(L)}(a,z) + (-1)^{n+1} a^{-n} P_{t_n^{-1}(L)}(a,z) \equiv z^n P_L(a,z) \bmod \hat{J}_n$

(b) $a^{-n} P_{\bar{t}_{2n}(L)}(a,z) + (-1)^{n+1} a^n P_L(a,z) \equiv z^n \left(\frac{a+a^{-1}}{z}\right)^{n-2} P_{L_\infty}(a,z) \bmod \hat{J}_n$

where $\bar{t}_{2n}(L)$, L, and L_∞ are shown on Fig. 2.5.

Fig. 2.5

Proof. (a) Consider n crossings, say $p_1, \ldots, p_n$, of the diagram $t_n(L)$ in which $t_n(L)$ differ from $t_n^{-1}(L)$ (see Fig. 2.6). Let us build the (part of) binary resolving tree for $t_n(L)$ using the crossings $p_1, p_2, \ldots, p_n$ (compare there proof of Lemma 2.3). L is the leaf of the tree obtained from $t_n(L)$ by performing n smoothings, so it introduces to

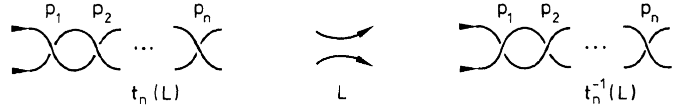

Fig. 2.6

$P_{t_n(L)}(a,z)$ the value $a^{-n}z^nP_L(a,z)$. $t_n^{-1}(L)$ is the leaf obtained from $t_n(L)$ by performing n $\mp$ changes, so it introduces to $P_{t_n(L)}(a,z)$ the value $(-1)^n a^{-2n}P_{t_n^{-1}(L)}(a,z)$. Now consider leaves obtained from $t_n(L)$ by k smoothings and $(n-k)$ $\mp$ changes. The crucial observation is that these leaves are all isotopic (say to $L^{l(k)}$) so there is $\binom{n}{k}$ such leaves and they introduce (together) to $P_{t_n(L)}(a,z)$ the value $\binom{n}{k} z^k a^{-k}(-a^2)^{n-k}P_{L^{l(k)}}(a,z)$. For $0<k<n$, $\binom{n}{k} z^k$ is in the ideal $\hat{J}_n$ (compare Lemma 2.4) so $P_{t_n(L)}(a,z)\equiv a^{-n}z^nP_L(a,z)+(-1)^n a^{-2n}P_{t_n^{-1}(L)}(a,z) \mod \hat{J}_n$ and Lemma 2.7 (a) follows. A proof of Lemma 2.7 (b) is similar to that of (a) and we omit it. □

Theorem 2.6 follows now from Lemmas 2.7 and 2.4. □

Corollary 2.8. *Let L be an oriented link which is t_{2n}, $\bar{t}_{2n}$ equivalent to an n-periodic link, then*

$$P_L(a,z)\equiv P_L(a^{-1},z) \bmod J_n. \quad \square$$

Now we can prove the practical criterion for $w(a,z)\in\mathcal{R}$ to be in the ideal J_n; it generalize Lemma 1.5.

Lemma 2.9. *Let $w(a,z)\in\mathcal{R}$ and $w(a,z)=\sum_i v_i(a)z^i$ where $v_i(a)\in Z[a^{\mp 1}]$. Then $w(a,z)\in J_n$ if an only if for any i, $v_i(a)$ is an element of the ideal $J_n^i(a)$ in $Z[a^{\mp 1}]$ where $J_n^i(a)$ is generated by elements $k(a+a^{-1})^{\max(0,n/k-i)}$ where k is any divisor of n.*

Proof. Let $\mathcal{R}'$ denote the ring $\mathcal{R}$ treated as $Z[a^{\mp 1}]$-module and consider the ideal J_n' in $\mathcal{R}'$ generated by $z^iJ_n^i(a)$ [i.e. by elements $z^ik(a+a^{-1})^{\max(0,n/k-i)}$]. This is chosen so that $v_i(a)\in J_n^i(a)$ for any i iff $w(a,z)\in J_n'$. Now Lemma 2.9 says that $J_n'=J_n$. First of all $J_n'\subset J_n$ because $z^ik(a+a^{-1})^{\max(0,n/k-i)}=kz^{n/k}\left(\frac{a+a^{-1}}{z}\right)^{\max(0,n/k-i)}$ $\times z^{\max(0,i-n/k)}\in J_n$. On the other hand J_n' is an ideal in $\mathcal{R}$, not only in $\mathcal{R}'$, because $J_n^i(a)\subset J_n^{i+1}(a)$ for any i. Therefore $J_n'\supset J_n$ because $kz^{n/k}\in z^{n/k}J_n^{n/k}(a)\subset J_n'$, and therefore $J_n'=J_n$. □

The following corollary is the slight generalization of the Traczyk result [Tr 1].

Corollary 2.10. (a) *Let L be an oriented link t_{2n}, $\bar{t}_{2n}$ equivalent to an n-periodic link. Then the Jones polynomial $V_L(t)$ satisfies:*

(i) $V_L(t)\equiv V_L(t^{-1}) \bmod J_n(t)$ *where $J_n(t)$ is the ideal in $Z\left[t^{\frac{\mp 1}{2}}\right]$ generated by elements of type $k(t-1)^{n/k}$ where k is any divisor of n. In particular*

(ii) *(see Corollary 1.3) If n is a prime number then* $V_L(t)\equiv V_L(t^{-1}) \bmod (n, t^n-1)$. *If $n=r^q$ is a power of a prime number, then*

(iii) $V_L(t)\equiv V_L(t^{-1}) \bmod (r, t^{r^q}-1)$ *and*

(iv) $V_L(t)\equiv V_L(t^{-1}) \bmod (r^q, (t-1)^r)$

(b) *consider the polynomial invariant of (global) isotopy of unoriented links defined by*

$$\hat{V}_L(t)=(t^{1/2})^{-3lk(L')}V_{L'}(t),$$

where L' is an oriented link which reduces to L after ignoring its orientation, and $lk(L')$ is the global linking number of L'. Then for any nonoriented link L, t_{2n}, $\bar{t}_{2n}$ equivalent to an n periodic link:

(i) *If n is odd then $\hat{V}_L(t)\equiv t^{nlk_2 L}\hat{V}_L(t^{-1}) \bmod J_n(t)$ where $lk_2(L)=0$ or 1 is a global linking number of $L \bmod 2$. It does not depend on the orientation of L and for any orientation L' of L $lk_2(L)\equiv lk(L') \bmod 2$.*

In particular if $n=r^q$ is a power of odd prime then

(ii) $\hat{V}_L(t)\equiv \hat{V}_L(t^{-1}) \bmod (r, t^{r^q}-1)$ *and*

(iii) $\hat{V}_L(t)\equiv \hat{V}_L(t^{-1}) \bmod (r^q, (t-1)^r)$.

Proof. $V_L(t)=P_L(a,z)$ for $a=it^{-1}$, $z=i\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)$ therefore (a) is an easy corollary of Theorem 2.1 and Corollaries 2.2 and 2.8. To prove (b) of Corollary 2.10 observe that we can always orient L so that ϕ preserves the orientation (orient first L/Z_n and then lift the orientation to L). For such oriented L, denoted L', we have by part (a):

$$V_{L'}(t)\equiv V_{L'}(t^{-1}) \bmod J_n(t)$$

and therefore

$$(t^{1/2})^{3lkL'}\hat{V}_L(t)\equiv (t^{1/2})^{-3lkL'}\hat{V}_L(t^{-1}) \bmod J_n(t)$$

and so

$$\hat{V}_L(t)=t^{-3lkL'}\hat{V}_L(t^{-1}) \bmod J_n(t).$$

Now if n is an odd number then lkL' is a multiplicity of n. Furthermore $lk(L')+nlk_2(L)$ is a multiplicity of $2n$ and therefore $-3lkL'\equiv nlk_2(L) \bmod 2n$, and so the part b(i) of Corollary 2.10 follows [$t^{2n}-1$ is in $J_n(t)$; compare Lemma 2.4]. b(ii) and (iii) follow similarly as in part (a) or in Corollary 2.2.

3. Criterion for n-Periodic Links Using the Kauffman Polynomial

Theorem 3.1. *Let L be an oriented link which is t_{2n}, $\bar{t}_{2n}$ equivalent to an n-periodic link, then the Kauffman polynomial of L satisfies:*

$$F_L(a,z)\equiv F_L(a^{-1},z) \bmod J_n.$$

Our proof of Theorem 3.1 is very similar to that of Theorem 2.1 and Corollary 2.8 so we will only sketch it here.

We start by considering nonoriented diagrams of n periodic link and its invariant of regular isotopy $\Lambda(a,z)$. We use the notation analogous to that of the proof of Theorem 2.1. In particular we consider nonoriented n-periodic diagrams

$$L_{\mathrm{sym}(\times)}, L_{\mathrm{sym}(\times)}, L_{\mathrm{sym}(\asymp)} \quad \text{and} \quad L_{\mathrm{sym}()()}.$$

Lemma 3.2. (a) $\Lambda_{L_{\mathrm{sym}(\times)}}+(-1)^{n+1}\Lambda_{L_{\mathrm{sym}(\times)}}\equiv z^n(\Lambda_{\mathrm{sym}(\asymp)}+\Lambda_{\mathrm{sym}()()}) \bmod \hat{J}_n$

(b) $\Lambda_{L_{\mathrm{sym}(\times)}}\equiv(-1)^n\Lambda_{\mathrm{sym}(\times)} \bmod J_n$.

Proof. Let p be a crossing of $L_{\mathrm{sym}(\times)}$ such that $L_{\mathrm{sym}(\times)}$ and $L_{\mathrm{sym}(\times)}$ differ only in crossings $p, \phi(p), \ldots, \phi^{n-1}(p)$. Let us consider the (part of) trinary resolving tree for $L_{\mathrm{sym}(\times)}$ which uses crossings $p, \phi(p), \ldots, \phi^{n-1}(p)$. Z_n acts on leaves of the trinary tree and the only fixed points of the action are $L_{\mathrm{sym}(\times)}$, $L_{\mathrm{sym}(\asymp)}$, and $L_{\mathrm{sym}()()}$. They introduce to $\Lambda_{L_{\mathrm{sym}(\times)}}$ the values $(-1)^n \Lambda_{L_{\mathrm{sym}(\times)}}$, $z^n \Lambda_{L_{\mathrm{sym}(\asymp)}}$, and $z^n \Lambda_{L_{\mathrm{sym}()()}}$ respectively. All other orbits of the Z_n action on the leaves introduce to $\Lambda_{L_{\mathrm{sym}(\times)}}$ a value which is in the ideal $\hat{J}_n$ (see the proof of Lemma 2.3) so Lemma 3.2(a) follows. Part (b) follows immediately from (a). $\square$

Now we will show that the equality from Theorem 3.1 holds for any n-periodic oriented link. It follows from Lemma 3.2(b) that for n-periodic oriented link diagram

$$a^{w(L_{\mathrm{sym}(\times)})} F_{L_{\mathrm{sym}(\times)}} = (-1)^n a^{w(L_{\mathrm{sym}(\times)})} F_{L_{\mathrm{sym}(\times)}} \bmod J_n .$$

Because $w(L_{\mathrm{sym}(\times)}) - w(L_{\mathrm{sym}(\times)}) = \mp 2n$ therefore by Lemma 2.4

$$F_{L_{\mathrm{sym}(\times)}} \equiv F_{L_{\mathrm{sym}(\times)}} \bmod J_n .$$

We can go from L to its mirror image $\bar{L}$ using changes $L_{\mathrm{sym}(\times)} \leftrightarrow L_{\mathrm{sym}(\times)}$ and because $F_{\bar{L}}(a,z) = F_L(a^{-1}, z)$ we have

$$F_L(a,z) \equiv F_L(a^{-1}, z) \bmod J_n .$$

To complete the proof of Theorem 3.1 we need the following Lemma.

Lemma 3.3. (a) *Let L_n and L_{-n} be a nonoriented link diagrams which are the same except the part shown on Fig. 3.1. Then*

$$\Lambda_{L_n} \equiv (-1)^n \Lambda_{L_{-n}} \bmod J_n .$$

(b) *If L and L' are oriented $t_{2n}, \bar{t}_{2n}$ equivalent links then* $F_L(a,z) \equiv F_{L'}(a,z) \bmod J_n$.

Fig. 3.1 L_n n positive half twists n negative half twists L_{-n}

Proof. (a) Consider the (part of) trinary resolving tree for L_n using crossings $p_1, p_2, \ldots, p_n$. Let $\psi(p_i)$ be defined $\psi(p_1) = p_2, \psi(p_2) = p_3, \ldots, \psi(p_n) = p_1$. Now consider the Z_n action on leaves of the tree with the generator g defined as follows: Let L^l be any leaf of the tree, then $g(L^l)$ is the leaf related to L^l as follows. If an operation ($\times$ to $\times$ or $\asymp$ or $)($) was performed on p_i in L^l, the same operation is performed on $\psi(p_i)$ in $g(L^l)$. The elements in an orbit of the Z_n action are all isotopic so we can complete our proof of Lemma 3.3(a) in the same manner as that of Lemma 3.2. To proof (b) of Lemma 3.3, observe that $w(L_n) - w(L_{-n}) = \mp 2n$ for any orientation of L_n (and corresponding orientation of L_{-n}) so by Lemma 2.4, $F_{L_n}(a,z) \equiv F_{L_{-n}}(a,z) \bmod J_n$. Hence Lemma 3.3(b) and Theorem 3.1 follows. $\square$

Corollary 3.4. *Consider an invariant of (global) isotopy of nonoriented links, $\hat{F}_L(a,z)$, defined by the formula $\hat{F}_L(a,z) = a^{2lk(L')} F_{L'}(a,z)$ where L' is any orientation on nonoriented link L. Then for any nonoriented link L which is $t_{2n}, \bar{t}_{2n}$ equivalent to a Z_n periodic link*

$$\hat{F}_L(a,z) \equiv \hat{F}_L(a^{-1}, z) \bmod J_n .$$

Proof. Let L' denote L oriented in such a way that Z_n preserves the orientation [compare the proof of Corollary 2.10(b)]. Then by Theorem 3.1 $F_{L'}(a,z) \equiv F_{L'}(a^{-1},z) \bmod J_n$. Therefore $a^{-2lk(L')}\hat{F}_L(a,z) \equiv a^{2lk(L')}\hat{F}_L(a^{-1},z) \bmod J_n$, so $\hat{F}_L(a,z) \equiv a^{4lk(L')}\hat{F}_L(a^{-1},z) \bmod J_n$. Finally by Lemma 2.4 $a^{2n} \equiv (-1)^n \bmod J_n$ so $a^{4lk(L')} \equiv 1 \bmod J_n$ and Corollary 3.4 follows. □

We can consider the simplified version, $S_L(a)$, of the Kauffman polynomial by substituting $z = a + a^{-1}$ in $F_L(a,z)$. $S_L(a)$ is interesting on its own and resemble somehow the Alexander polynomial [if L is a split link then $S_L(a) = 0$; if L is a knot then $S_L(a) = 1 + (a + a^{-1})(\cdot)$]. Our criterion for n periodic links also simplifies when one applies $S_L(a)$. Namely, Theorem 3.1 reduces to

Corollary 3.5. *Let L be an oriented link which is t_{2n}, $\bar{t}_{2n}$ equivalent to an n-periodic link, then*

$$S_L(a) \equiv S_L(a^{-1}) \bmod J_n(a),$$

where $J_n(a)$ is the ideal in $Z[a^{\mp 1}]$ generated by elements of type $k(a+a^{-1})^{n/k}$ where k is any divisor of n. In particular for $n = r^q$ (a power of a prime number)

$$S_L(a) \equiv S_L(a^{-1}) \bmod (r, a^{2r^q} + 1)$$

and

$$S_L(a) \equiv S_L(a^{-1}) \bmod (r^q, (a+a^{-1})^r). \quad \square$$

4. Examples and Further Speculations

Example 4.1. Consider the right handed trefoil knot, 3_1 (Fig. 4.1). The skein polynomial $P_{3_1}(a,z) = (-2a^{-2} - a^{-4}) + a^{-2}z^2$. Then $w(a,z) = P_{3_1}(a,z) - P_{3_1}(a^{-1},z) = (a+a^{-1})^3(a-a^{-1}) - z^2(a+a^{-1})(a-a^{-1})$. Therefore, $w(a,z) \in J_n$ iff $n = 2, 3, 4, 6$ or 12. Hence by Theorem 2.1 the right handed trefoil knot is not n periodic for $n \neq 2, 3, 4, 6$ or 12. If one consider the Kauffman polynomial one cannot do any better so our criteria cannot exclude periods 4, 6, and 12 (compare Lemma 4.5).

Example 4.2. Consider the knot 10_{137} [Ro], Fig 4.1. It is 10_{124} in [Th]. The Kauffman polynomial

$$\begin{aligned} F_{10_{137}}(a,z) = & -a^{-2} - 1 - 2a^2 - 2a^4 - a^6 + z(-a^{-1} - 3a^1 - 5a^3 - 3a^5) \\ & + z^2(a^{-2} + 4 + 7a^2 + 8a^4 + 4a^6) \\ & + z^3(2a^{-1} + 9a + 15a^3 + 8a^5) \\ & + z^4(\cdot); \text{[Th]}. \end{aligned}$$

Consider the difference $w(a,z) = F_{10_{137}}(a,z) - F_{10_{137}}(a^{-1},z)$. One can check that $w(a,z) \in J_n$ iff $n = 2, 3, 4, 6$ or 12 so 10_{137} is not n-periodic for $n \neq 2, 3, 4, 6$, and 12. Let us check it in more detail for $n = 5$. Consider $v_0(a)$ for $w(a,z)$ (in the notation of Lemma 1.5). Modulo the ideal $(5, a^5 + a^{-5})$ one gets:

$$\begin{aligned} v_0(a) & \equiv -a^{-2} - 1 - 2a^2 - 2a^4 - a^6 + a^2 + 1 + 2a^{-2} + 2a^{-4} + a^{-6} \\ & \equiv a^{-2} - a^2 + 2a^{-4} - 2a^4 + a^{-4} - a^4 \equiv a^{-2} - a^2 + 3a^{-4} - 3a^4 \\ & \not\equiv 0 \bmod (5, a^5 + a^{-5}) \quad \text{so} \quad w(a,z) \notin J_5 . \end{aligned}$$

The case $n=5$ is interesting because in [BZ] the knot 10_{137} has, by mistake, associated with it period 5. I would like to thank Murasugi for pointing out the possibility of mistakes in [BZ].

Example 4.3. Consider the right handed Hopf link H_1 (Fig. 4.1). The Kauffman polynomial

$$F_{H_1}(a,z)=(-a^{-3}-a^{-1})z^{-1}+a^{-2}+(a^{-3}+a^{-1})z.$$

Then $w(a,z)=F_{H_1}(a,z)-F_{H_1}(a^{-1},z)$ is in J_n iff $n=2$. Therefore the only period of H_1 is equal to 2.

Example 4.4. Consider the oriented link 6_1^3 as on Fig. 4.1. The skein polynomial $P_{6_1^3}(a,z)=(-a^{-4}-2a^{-2}-1)z^{-2}+(a^{-4}+3a^{-2}+2)+(-a^{-4}-3a^{-2}-1)z^2+a^{-2}z^4$. Then $w(a,z)=P_{6_1^3}(a,z)-P_{6_1^3}(a^{-1},z)=(a+a^{-1})\ (a-a^{-1})\left(\dfrac{a+a^{-1}}{z}\right)^2+(a+a^{-1})^2(\cdot)+z^2(\cdot)$. Therefore $w(a,z)\in J_n$ iff $n=2$. Hence 6_1^3 is not n periodic for $n>2$.

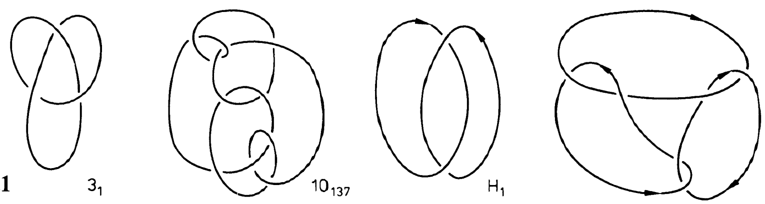

Fig. 4.1

Examples 1.6, 1.7, 4.1, and 4.2 suggest that our criteria cannot exclude periods 2 and 3 for knots. For links of more than one components one can exclude period 3 as shown in Examples 4.3 and 4.4 (notice that the global linking number is equal to ∓ 1 in these examples). In fact the following holds

Lemma 4.5. *For any link L*

(a) $P_L(a,z)-P_L(a^{-1},z)\in J_2$

(b) $F_L(a,z)-F_L(a^{-1},z)\in J_2$.

For any knot K

(c) $P_K(a,z)-P_K(a^{-1},z)\in J_3$

(d) $F_K(a,z)-F_K(a^{-1},z)\in J_3$

(e) $P_K(a,z)-P_K(a^{-1},z)\in J_4$

(f) $F_K(a,z)-F_K(a^{-1},z)\in J_4$.

Therefore our criteria do not work for period 2 *and for knots and periods* 3 *and* 4.

Proof. Let us write $P_L(a,z)$ or $F_L(a,z)$ as

$$\sum_{i\leq 0} u_i(a)\left(\frac{a+a^{-1}}{z}\right)^i+v_1(a)z+\sum_{i\geq 2} v_i(a)z^i.$$

Then $u_i(a)$ is in $Z[a^{\mp 2}]$ so $u_i(a)-u_i(a^{-1})$ is in the ideal $((a+a^{-1})(a-a^{-1}))$ of $Z[a^{\mp 1}]$. $v_1(a)-v_1(a^{-1})$ is in the ideal $(a-a^{-1})$ of $Z[a^{\mp 1}]$ and $\sum_{i\geq 2} v_i(a)z^i$ is in the ideal (z^2) of $\mathcal{R}$. Combining this one gets that $P_L(a,z)-P_L(a^{-1},z)$ and $F_L(a,z)-F_L(a^{-1},z)\in J_2=(2z,z^2)$.

To prove (c) of Lemma 4.5 we use the folklore fact (compare [LM], [L] or [P1]) that for any knot K, $P_K(a,z)-1$ is a multiple of $(a+a^{-1})^2-z^2$. Because $P_K(a,z)=\sum_{i\geq 0} v_{2i}(a)z^{2i}$ where $v_{2i}(a)\in Z[a^{\mp 2}]$ so $v_0(a)-v_0(a^{-1})$ is a multiple of $(a+a^{-1})^3$ and $v_2(a)-v_2(a^{-1})$ is a multiple of $(a+a^{-1})$ so $P_K(a,z)-P_K(a^{-1},z)\in J_3=(3z,z^3)$; compare Lemma 2.9. Now consider $F_K(a,z)=\sum_{i\geq 0} w_i(a)z^i$. $w_0(a)-w_0(a^{-1})$ is a multiple of $(a+a^{-1})^3$ (the first coefficients of $F_K(a,z)$ and $P_K(a,z)$ are the same; compare [L] or [P1]). $w_2(a)-w_2(a^{-1})$ is a multiple of $(a+a^{-1})^2$ because $w_2(a)\in Z[a^{\mp 2}]$. Finally $w_1(a)-w_1(a^{-1})$ is a multiple of $(a+a^{-1})^2$. To prove this it is enough to show that $w_1(a)$ is a multiple of $(a+a^{-1})$. By [LM2], $F_K(a,z)-1$ is a multiple of $(a+a^{-1}-z)$. Therefore $F_K(a,z)-1/a+a^{-1}-z=\sum_{i\geq 0} s_i(a)z^i$; furthermore the first coefficients of $F_K(a,z)-1/a+a^{-1}-z$ and $P_K(a,z)-1/a+a^{-1}-z$ are the same, so $s_0(a)$ is multiple of $(a+a^{-1})$ [we use the fact that $P_K(a,z)-1$ is a multiple of $(a+a^{-1})^2-z^2$]. Finally because $w_1(a)=s_0(a)+(a+a^{-1})s_1(a)$, hence $w_1(a)$ is a multiple of $(a+a^{-1})$. By Lemma 2.9, $F_K(a,z)-F_K(a^{-1},z)\in J_3$ so Lemma 4.5(d) is proven. 4.5(e) and (f) can be proven similarly. One should only additionally observe that if a polynomial $w(a)\in Z[a^{\mp 2}]$ then $w(a)-w(a^{-1})$ is a multiple of

$$(a+a^{-1})(a-a^{-1})=(a+a^{-1})(a+a^{-1}-2a^{-1}).$$

It completes our proof of Lemma 4.5.

Our examples show that even if our link is p periodic our criteria can detect the lack of p^2 periodicity. On the other hand if our criteria do not exclude n and m-periodicity (n and m co-prime numbers) then they cannot exclude nm-periodicity. It is the case because if $w\in J_n$ and $w\in J_m$ then $w\in J_{mn}$. (Lemma 2.9 allows us easily to prove that $J_n\cap J_m=J_{nm}$.) We conclude, combining the above with Lemma 4.5 that our criteria cannot exclude a periodicity of knots for $n=2$, 3, 4, 6, and 12 (compare Examples 4.1 and 4.2).

If a link L is n-periodic then $L_*=L/Z_n$ is a link in the 3-sphere S^3/Z_n. The idea of Murasugi [Mu 2] is to compare polynomial invariant of L and L_* and to show that for some ideal I_n a polynomial invariant W satisfies:

4.6 $$W_L\equiv W^n_{L_*} \bmod I_n.$$

For the Jones polynomial $V_L(t)$, Murasugi has shown [Mu 2] that if L is an r-periodic oriented link and r a prime number then $V_L(t)\equiv V^r_{L_*}(t) \bmod I_r(t)$ where $I_r(t)$ is the ideal in $Z[t^{\mp 1/2}]$ generated by r and $\left(-\frac{t+1}{\sqrt{t}}\right)^{r-1}-1$. The similar result can be obtained for skein and Kaufmann polynomials. In the case of skein polynomial one gets

$$P_L(a,z)\equiv P^r_{L_*}(a,z) \bmod I_r,$$

where I_r is the ideal in $Z[a^{\mp 1},z^{\mp 1}]$ generated by r, $\left(\frac{a+a^{-1}}{z}\right)^{r-1}-1$ and polynomials $P_{T_{k,r}}(a,z)$ where $T_{k,r}$ is the torus link of type k,r. To make 4.6 useful one should analyse how big is the ideal I_r. The idea of a proof of 4.6 is based on the observation that, to the binary resolving tree of L_* corresponds r-periodic binary resolving tree for L (one uses triplet $L_{\mathrm{sym}(\times)}$, $L_{\mathrm{sym}(\times)}$, $L_{\mathrm{sym}(\asymp)}$). The second tree suffices to compute $P_L(a,z) \bmod r$ assuming the values at leaves are known (compare the proof of Theorem 2.1). Finally notice that the above method can be used to prove another Murasugi result [Mu 1] that for an r-periodic knot the Alexander polynomial satisfies:

$$\Delta_L(t) \equiv \Delta^r_{L_*}(t)(1+t+\ldots+t^{\lambda-1})^{r-1} \bmod r$$

for some integer λ.

The method of our paper can be also applied for Z_n invariant links when the fixed point set of the action is a part of the link. Then we have to use Hoste-Kidwell polynomial [HK] or its simplified version introduced in [HP]. We can also analyse Z_n invariants links when Z_n acts freely on S^3. We will describe it in the sequel paper (compare [P 4]).

Finally we hope that the analysis of n-periodic (or symmetric) links in any 3-manifolds can lead to some unified theory of skein modules of coverings (compare [P 3]).

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