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On Essential Singularities of Meromorphic Mappings

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1. Introduction

The notion of meromorphic mapping which is used in this article is the one of Stein [6, 7]. It is equivalent to the notion "SR-meromorph" of Stoll [10], and the idea to this notion was already given by Remmert [5]. In [10], Stoll also introduced a notion of essential singularity of a meromorphic mapping: Let X^*, Y be normal complex spaces, $A \subset X^*$ a thin subset, $X = X^* \setminus A$ and $f: X \to Y$ a meromorphic mapping. Then a point $P \in A$ is called a "SR-Singularität" of f if there doesn't exist a neighbourhood U of P in X^* and a meromorphic extension $g: U \to Y$ of $f:(U \cap X) \to Y$.

But defining the notion of essential singularity like "SR-Singularität" has at least two disadvantages:

Firstly a "SR-Singularität" only is a very weak form of a singularity, e.g. the holomorphic function

$$z^{-1}: (\mathbb{C} \setminus \{0\}) \to \mathbb{C}$$

has a "SR-Singularität" in the zero point.

Secondly the assumption that A has to be thin in X^* is very restrictive, e.g. for the holomorphic function

$$\sqrt{z}: (\mathbb{C} \setminus \mathbb{R}_0^+) \to \mathbb{C}$$

$\mathbb{R}_0^+$ is not thin in $\mathbb{C}$.

Hence in this article the notion of essential singularity of meromorphic mappings is defined differently:

Firstly it is allowed to replace Y by a "bigger" normal complex space Z (that shall mean that Y is an open subspace of Z) before extending f into the point P. Further examples, some of which are given in the Sects. 3–5 of this paper, let it seem to be sensible to define three "versions" of an essential singularity: The first version is the one discribed above. For the second (resp. the third) version it also is allowed to replace Y by a normal complex space Y' which is a bit "smaller" than Y and then to replace Y' by a "bigger" space Z before extending f into P. Here "smaller" shall mean that there exists a closed nowhere dense subset M of Y and a holomorphic mapping $h: Y \to Y'$ so that $h(Y \setminus M)$ is an open subspace of Y' and $h:(Y \setminus M) \to h(Y \setminus M)$ is biholomorphic (resp. local biholomorphic).

Secondly it is allowed that $A \neq X^*$ is an arbitrary closed subset of X^* and P is any point of ∂X (where ∂X denotes the border of X in respect of X^*). Now the situation around P can be very complicated (e.g. it can happen that for any connected neighbourhood U of P, $U \cap X$ has infinitely many connected components), so it is not clear at once what an extension of f into P shall be. For $g: U \to Z$ being an extension of the mapping $f: (U \cap X) \to Z$ one should demand at least that there is an open subset $O \subset U \cap X$ with $P \in \partial O$ so that the equality $f = g$ holds on O. Using the identity-lemma for meromorphic mappings [6, p. 830] it can easily be shown that this is equivalent to demand that there exists a sequence $(G_\nu)_{\nu \in \mathbb{N}}$ of connected components G_ν of $U \cap X$ with $P \in \partial \left(\bigcup_{\nu \geqq \nu_0} G_\nu \right)$ for all $\nu_0 \in \mathbb{N}$ so that the equality $f = g$ holds on $\bigcup_{\nu \geqq 1} G_\nu$. Since in this paper it is intended to define the notion of essential singularity as strong as possible this already is the right concept of extension, but it should at least be added that other concepts of extension are possible and, more generally, there are also other possibilities to define the notion of essential singularity than the one given in this article.

In Sect. 2 we first define the three notions "essential singularity of the i-th kind, $i = 1, 2, 3$" (Definition 2.2). It is an immediate consequence of this definition that if f in P has an essential singularity of the i-th kind, it there also has an essential singularity of the $(i-1)$-th kind, $i = 2, 3$. Then a proposition is proved that shows that if A is nowhere dense in X^* and for every neighbourhood U of P there exists a subneighbourhood W so that $W \cap X$ is connected (these assumptions hold e.g. if A is thin in X^*), then Definition 2.2 gives the "right" notions.

In Sects. 3–5 there are given a lot of examples of meromorphic mappings with essential singularities which especially show that there exist mappings f and points P, in which f has an essential singularity of the i-th kind, but no essential singularity of the $(i+1)$-th kind, $i = 1, 2$. Beyond that there are given two theorems which prove:

(a) If X^* is connected with $\dim X^* = n$, $A \neq X^*$ is an arbitrary closed subset of X^* and m any number equal to or greater than n and 2, there exists a pure m-dimensional normal complex space Y_m and a meromorphic mapping $f_m: X \to Y_m$ which has essential singularities of the first kind in all points of ∂X, but no single essential singularity of the second kind.

(b) Let X^* and A be as in (a) and X^* be 1-dimensional. Then there exists a meromorphic mapping which has essential singularities of the third kind in every point of ∂X.

(c) Let Y be 1-dimensional. Then every essential singularity of the first kind is one of the second kind.

Notice that (a) also is interesting in connection with extension problems for meromorphic mappings as they were examined by Stein in [7, 8] (cf. also Theorem 6.3 of this paper), because for thin A (a) especially yields that the correspondence given by $\overline{G_{f_m}} \subset X^* \times Y_m$, where G_{f_m} is the graph of $f_m: X \to Y_m$, doesn't yield a meromorphic mapping in any point $P \in A$. This result even holds if Y_m is enlarged to a "bigger" space Z as it was described above.

In Sect. 6 there are collected some propositions which can be helpful when trying to prove that a meromorphic mapping in a given point has **no** essential singularity. Among them there is an important theorem of Stein which deals with the case that A is thin in X^* and $\dim X^* - \dim A > \dim Y$ and a further theorem of Stoll which can be applied if Y is projective-algebraic.

2. Definition of Essential Singularities of Meromorphic Mappings

First we introduce some notations which will be kept up during this paper. Let X^* and Y be normal complex spaces, where X^* is connected and has a countable basis of topology. Let $A \neq X^*$ be a non-empty closed subset of X^*, $X := X^* \backslash A$ and $P \in \partial X$, where ∂X denotes the border of X in respect of X^*. Let further $f: X \to Y$ be a meromorphic mapping, $G_f \subset X \times Y$ its graph, $\check{f}: G_f \to X$ and $\hat{f}: G_f \to Y$ its canonical projections and $S_f \subset X$ the set of its (non-essential) singularities.

Definition 2.1 *(c-sequence).* Let U be a connected neighbourhood of P in X^*. For every $\nu \in \mathbb{N}$ let G_ν be (not necessary different) connected components of $U \cap X$ with

$$P \in \partial\Big(\bigcup_{\nu \geqq \nu_0} G_\nu\Big) \quad \text{for all } \nu_0 \in \mathbb{N}.$$

Then $\mathscr{G} := (G_\nu)_{\nu \in \mathbb{N}}$ is called *c-sequence (in resp. of P and U).* For $\mathscr{G}$ we set $|\mathscr{G}| := \bigcup_{\nu \geqq 1} G_\nu \subset U \cap X$.

Definition 2.2 *(essential singularities of meromorphic mappings). P* is said to be an *essential singularity of the i-th kind (*ess *i*-sing*) of f (in resp. of X^*), $i = 1, 2, 3$,* if for every
connected neighbourhood U of P in X^*,
c-sequence $\mathscr{G}$ in resp. of P and U,
normal complex space Z,
$h \in \mathscr{H}_i(Y, Z)$ (see below),
there doesn't exist a meromorphic extension $g: U \to Z$ of the mapping $h \circ f: |\mathscr{G}| \to Z$.
The sets $\mathscr{H}_i(Y, Z)$ are defined as subsets of the set of *holomorphic mappings* from Y to Z as follows:
$\mathscr{H}_1(Y, Z)$ consists of all $h: Y \to Z$ for which there exists an open subset $Z_0 \subset Z$, so that $h: Y \to Z_0$ is biholomorphic.
$\mathscr{H}_2(Y, Z)$ consists of all $h: Y \to Z$ for which there exists an open subset $Z_0 \subset Z$ and a closed and nowhere dense subset $M \subset Y$, so that $h: (Y \backslash M) \to Z_0$ is biholomorphic.
$\mathscr{H}_3(Y, Z)$ consists of all $h: Y \to Z$ for which there exists a closed and nowhere dense subset M of Y, so that $h: (Y \backslash M) \to Z$ is locally biholomorphic.

It can be easily proved that Definition 2.2 defines a local property of f. Definition 2.2 becomes simpler if A has additional properties:

Proposition 2.3. *Let A be nowhere dense in X^* and assume that for every neighbourhood V of P in X^* there exists a subneighbourhood W for which $W \cap X$ is connected.*
Then P is an ess *i*-sing *of f in resp. of X^* if and only if for every*
connected neighbourhood U of P in X^*,
normal complex space Z,
$h \in \mathscr{H}_i(Y, Z)$ (cf. Definition 2.2),
there doesn't exist a meromorphic extension $g: U \to Z$ of $h \circ f: (U \cap X) \to Z$.

The *proof* is straightforward.

3. Some Relations Between Ess 1-Sing and Ess 2-Sing

Theorem 3.1. (a) *Let X^* be n-dimensional, $m \in \mathbb{N}$ with $m \geqq \max(n, 2)$. Then there exists a pure m-dimensional normal complex space Y_m and a meromorphic mapping $f_m: X \to Y_m$, so that every point of ∂X is an* ess 1-sing, *but no point of ∂X is an* ess 2-sing.

(b) *Let Y be 1-dimensional. Then if P is an* ess 1-sing *of f, it also is an* ess 2-sing *of f.*

Remark. See Remark 2 of Theorem 6.3 for a supplementation to this theorem.

Before we start with the proof of Theorem 3.1, we prove two lemmas:

Lemma 3.2. *Let M_1, $M_2 \subset X^*$ be open subsets, M_2 be connected and M_3 be a connected component of $M_1 \cap M_2$. If $M_1 \cap M_2 \neq \emptyset \neq (X^* \setminus M_1) \cap M_2$, then*

$$M_2 \cap \partial M_1 \cap \partial M_3 \neq \emptyset .$$

For the *Proof of Lemma 3.2,* we refer, if necessary, to [2, p. 39].

Lemma 3.3. *Let S be the singular locus of X^*. Then there exists a sequence $F = (x_\mu)_{\mu \in \mathbb{N}}$ with the following properties:*

(a) *$x_\mu \in X \setminus S$ for all $\mu \in \mathbb{N}$; $x_{\mu_1} \neq x_{\mu_2}$ for $\mu_1 \neq \mu_2$.*

(b) *We set $|F| := \{x_\mu : x_\mu \in F\}$. Then $|F|$ is a discrete subset of X.*

(c) *For all $x \in \partial X$, for all connected neighbourhoods U of x in X^* and for all connected components G of $U \cap X$ the set $G \cap |F|$ contains infinitely many points: $\#(G \cap |F|) = \infty$.*

Proof of Lemma 3.3. Let $\mathscr{B} := \{B_\nu, \nu \in \mathbb{N}\}$ be a countable basis of the topology of X^* consisting of connected sets and $\mathscr{Z}_A$ be the set of all connected components of $B_\nu \cap X$ of those $B_\nu \in \mathscr{B}$ with $B_\nu \cap A \neq \emptyset$. The set $\mathscr{Z}_A$ is countable, so we can enumerate its elements:

$$\mathscr{Z}_A = \{G_\mu, \mu \in \mathbb{N}\} .$$

Since X^* has a countable basis of topology, we can introduce a metric $\delta(\cdot, \cdot)$ on X^*. Now define

$$A_\nu := \left\{ x \in X^* : \delta(x, A) < \frac{1}{\nu} \right\} \quad \text{for every } \nu \in \mathbb{N} .$$

Then $A_\nu \cap G_\nu \neq \emptyset$: Since A_ν is a neighbourhood of A, it is enough to show $\partial G_\nu \cap A \neq \emptyset$. This directly follows if we apply Lemma 3.2.

Now we can construct the sequence F:

Choose x_1 from $(A_1 \cap G_1) \setminus S$. If $x_1, \ldots, x_{\mu-1}$ are already constructed, choose x_μ from $(A_\mu \cap G_\mu) \setminus (S \cup \{x_1, \ldots, x_{\mu-1}\})$.

It follows directly from this construction that the properties (a) and (b) are fulfilled. From Lemma 3.2, applied with $M_1 = X$, $M_2 = U$, $M_3 = G$, it follows that there exists a $x_0 \in U \cap \partial X \cap \partial G$.

Assume now $\#(|F| \cap G) < \infty$. Then $V := U \setminus (G \cap |F|)$ is a neighbourhood of x_0, so there is an open set $B \in \mathscr{B}$ with $x_0 \in B \subset V$. Since $B \cap G \neq \emptyset$ there exists a connected component G_ν of $B \cap X$ in $\mathscr{Z}_A$ with $G_\nu \cap (G \setminus |F|) \neq \emptyset$. Hence $G_\nu \subset G \setminus |F|$, but this contradicts $|F| \cap G_\nu \supset \{x_\nu\}$. $\square$

Proof of Part (a) of the Theorem in the Case $n \geqq 2$. First we construct a sequence F like in Lemma 3.3. Then we construct Y from X by blowing up X simultaneously in every point of $|F|$ by Hopf's σ-process [4]. If $\pi: Y \to X$ is the canonical projection, we have:

$$f := \pi^{-1}: X \to Y; \quad x \to \pi^{-1}(x) \quad \text{is a meromorphic mapping,} \tag{1}$$

$$S_f = |F|, \tag{2}$$

$$\pi: Y \to X \quad \text{is a holomorphic mapping with } \pi \circ f = \mathrm{id}|_X, \tag{3}$$

$$Y \backslash f(X \backslash |F|) \quad \text{is a 1-codimensional analytic set in } Y. \tag{4}$$

We now define $Y_m := Y \times \mathbb{C}^{m-n}$ and $f_m: X \to Y_m$; $x \to (f(x), 0, \ldots, 0)$ and prove the assertion of part (a) of the theorem for $n \geqq 2$:

Take a point $P \in \partial X$. It is easy to see that f_m has no ess 2-sing in P: Define U any connected neighbourhood of P, $\mathscr{G}$ any c-sequence in resp. of P and U, $Z := X^* \times \mathbb{C}^{m-n}$, $h := \pi \times \mathrm{id}^{(m-n)}: Y_m \to Z$; $(y, z_1, \ldots, z_{m-n}) \to (\pi(y), z_1, \ldots, z_{m-n})$, $g: X^* \to X^* \times \mathbb{C}^{m-n}$; $x \to (x, 0, \ldots, 0)$. As a consequence of (3) and (4) we have $h \in \mathscr{H}_2(Y_m, Z)$, and $h \circ f_m = g$ on $|\mathscr{G}|$ is obvious.

To prove that f_m has an ess 1-sing in P, we assume the contrary. Under this assumption there exist U, $\mathscr{G}$, Z, h and g like in Definition 2.2. We especially have $h \circ f_m = g$ on $|\mathscr{G}|$, hence (cf. [6, p. 837])

$$S_g \cap |\mathscr{G}| = |\mathscr{G}| \cap |F|. \tag{5}$$

S_g is an analytic subset of U, for which there exists a unique decomposition in irreducible analytic sets $(S_g)_\iota$, $\iota \in I$, so that $\{(S_g)_\iota, \iota \in I\}$ is a local finite covering of S_g, hence there exists a connected subneighbourhood $V \subset U$ of P with

$$V \cap (S_g)_\iota \neq \emptyset \quad \text{only for a finite number of } \iota \in I. \tag{6}$$

We have $V \cap |\mathscr{G}| \cap |F| = V \cap |\mathscr{G}| \cap S_g$ [cf. (5)]. From this we conclude with Lemma 3.3b and (6):

$$\#(V \cap |\mathscr{G}| \cap |F|) < \infty. \tag{7}$$

Because of the properties of $\mathscr{G}$ there exists a $\nu_0 \in \mathbb{N}$ with $G_{\nu_0} \cap V \neq \emptyset$, hence there is a connected component G of $V \cap X$ with $G \subset G_{\nu_0}$. An application of Lemma 3.3c to P, V and G yields $\#(|F| \cap G) = \infty$, hence $\#(V \cap |\mathscr{G}| \cap |F|) = \infty$, which contradicts (7). □

To prove part (a) of the theorem in the case $n = 1$, we need another lemma:

Lemma 3.4. *Let X^* be 1-dimensional. Then there exists a sequence $F = (x_\mu)_{\mu \in \mathbb{N}}$ in X with the properties* (a), (b), *and* (c) *of Lemma 3.3 and a holomorphic function $f: X \to \mathbb{C}$ with*

$$\{x \in X : f(x) = 0\} = |F|.$$

Proof. Choose the sequence F like in Lemma 3.3. It is sufficient to construct such a function on every connected component of X. X^* is a Riemann surface (because the singular locus is at least 2-codimensional, hence empty), so the connected components of X are Riemann surfaces, too. On every connected component of X, $|F|$ yields a Cousin-II-distribution. Since such connected components are not compact (this is a simple consequence of the properties of F), this distribution has a solution, and this solution has the desired properties. □

Proof of Part (a) of the Theorem in the Case $n=1$: Let $f: X \to \mathbb{C}$ be the function defined in the previous lemma and define $Y_m := \mathbb{C}^m$ and $f_m: X \to Y_m$; $x \to (f(x), 0, \ldots, 0)$.

Take any point $P \in \partial X$. Again, it is easy to see that P is no ess 2-sing of f_m: Let U be any connected neighbourhood of P, $\mathscr{G}$ any c-sequence in resp. of P and U, $Z := \mathbb{C}^m$, $h: Y_m \to Z$; $(z_1, z_2, \ldots, z_m) \to (z_1 z_2, z_2, \ldots, z_m)$ and $g=0: U \to Z$.

To prove that f_m has an ess 1-sing in P, we again assume the contrary. Under this assumption there exist U, $\mathscr{G}$, Z, h, and g like in Definition 2.2. First we show:

$$P \in \partial(|F| \cap |\mathscr{G}|), \qquad |F| \cap |\mathscr{G}| \neq \emptyset. \tag{8}$$

It suffices to show that for every connected subneighbourhood $U' \subset U$ of P the nonequality $|F| \cap U' \cap |\mathscr{G}| \neq \emptyset$ holds. There exists a G_ν with $G_\nu \cap U' \neq \emptyset$ and hence a connected component G of $U' \cap X$ with $G \subset G_\nu$. An application of Lemma 3.3 yields $\#(|F| \cap G) = \infty$ and hence $|F| \cap |\mathscr{G}| \cap U' \neq \emptyset$.

Let $z_0 := h(0)$. Then $g(x) = z_0$ for all $x \in |F| \cap |\mathscr{G}|$, and with $S_g = \emptyset$ (cf. [10, p. 224]) and (8) we can conclude:

$$\{x \in U: g(x) = z_0\} \supset (|F| \cap |\mathscr{G}|) \cup \{P\}. \tag{9}$$

There exists a neighbourhood W of z_0 in Z which is mapped biholomorphically on a closed analytic subspace of a domain in a $\mathbb{C}^r$. Hence g yields r holomorphic functions which, because of (9), are constant on $((|F| \cap |\mathscr{G}|) \cup \{P\}) \cap g^{-1}(W)$. Therefore if $V \subset g^{-1}(W)$ is a connected neighbourhood of P, we have with (8) and the identity-lemma on Riemann surfaces:

$$g(x) \equiv z_0 \quad \text{for all} \quad x \in V. \tag{10}$$

Since we have $h \circ f_m = g$ on $V \cap |\mathscr{G}| (\neq \emptyset)$ and h is injective, we get from (10) and (8):

$$f_m(x) \equiv 0 \quad \text{for all} \quad x \in V \cap |\mathscr{G}|.$$

But this is impossible, since $\{x \in X: f_m(x) = 0\} = |F|$ and $|F|$ only is a discrete subset of the open set $V \cap |\mathscr{G}|$. □

Before we start with the proof of part (b) of the theorem, we prove a topological lemma:

Lemma 3.5. *Let S, T be topological spaces which locally admit a metric, $C \subset S$ a closed and nowhere dense subset. Let $f: S \to T$ be continuous and $f: (S \setminus C) \to T$ be injective. Let $O_1, O_2 \subset S$ be open sets with $O_1 \cap O_2 = \emptyset$ for which $f(O_i)$ are open subsets of T and $f: O_i \to f(O_i)$ are topological maps.*

Then $f(O_1) \cap f(O_2) = \emptyset$.

Proof. Assume $W := f(O_1) \cap f(O_2) \neq \emptyset$. Since $f: O_i \to f(O_i)$ are topological maps and T locally admits a metric $W' := (W \cap [f(O_1 \cap C) \cup f(O_2 \cap C)])$ is closed and nowhere dense in W. Hence there exists $w_0 \in W \setminus W'$ and $w_1 \in O_1 \setminus C$, $w_2 \in O_2 \setminus C$ with $f(w_1) = f(w_2) = w_0$, but this is impossible because $f: (S \setminus C) \to T$ was injective. □

Proof of Part (b) of the Theorem. Let Z be a normal complex space. It suffices to show $\mathscr{H}_2(Y, Z) \subset \mathscr{H}_1(Y, Z)$. Let $h \in \mathscr{H}_2(Y, Z)$ and Y_1 be a connected component of Y. Then $h(Y_1) \subset Z$ is an open subset and $h: Y_1 \to h(Y_1)$ is biholomorphically: For $\dim Y_1 = 0$ this is a direct consequence of $h \in \mathscr{H}_2(Y, Z)$, for $\dim Y_1 = 1$ we will prove that below. The previous lemma now shows that $h: Y \to Z$ is injective, hence $h \in \mathscr{H}_1(Y, Z)$.

We still have to show that if $h \in \mathscr{H}_2(Y, Z)$ with a Riemann surface Y then $h(Y)$ is an open subset of Z and $h: Y \to h(Y)$ is biholomorphic. It is enough to show that $h: Y \to Z$ is locally biholomorphic, since then an application of Lemma 3.5 completes the proof.

Since $h(Y)$ is connected, we may assume that Z is a Riemann surface. If we introduce local charts in Y and Z in an appropriate way (cf. [1, p. 164]), we reduce our assertion to the following one:

Let $\varepsilon \in \mathbb{R}^+$, $U_\varepsilon(0) := \{z \in \mathbb{C} : |z| < \varepsilon\}$, $N \subset U_\varepsilon(0)$ a closed and nowhere dense subset, $f(z) := z^p$ with $p \in \mathbb{N}$ so that $f : (U_\varepsilon(0) \setminus N) \to \mathbb{C}$ is injective. Then $p = 1$.

Assume $p \geqq 2$. Then the two points $z_1 = \frac{\varepsilon}{2}$, $z_2 = \frac{\varepsilon}{2} e^{\frac{2\pi i}{p}}$ are different, so there are neighbourhoods O_1 (resp. O_2) of z_1 (resp. z_2) with $O_i \subset U_\varepsilon(0)$ and $O_1 \cap O_2 = \emptyset$, for which the mappings $f : O_i \to f(O_i)$ are biholomorphic. Then the Lemma 3.5 yields $f(O_1) \cap f(O_2) = \emptyset$, but this is wrong since $f(z_1) = f(z_2)$. $\square$

4. Some Relations Between Ess 2-Sing and Ess 3-Sing

First, we introduce some special notations for this section:

$$\begin{aligned} &G := \{z = r \cdot e^{2\pi i\alpha} : r \in \mathbb{R}^+, \alpha \in \mathbb{R}, 0 < \alpha < 1\}. \\ &H := \{z = r \cdot e^{2\pi i\alpha} : r \in \mathbb{R}^+, \alpha \in \mathbb{R}, 0 < \alpha < \tfrac{1}{2}\}. \\ &\tilde{f} : G \to H; \quad r \cdot e^{2\pi i\alpha} \to \sqrt{r} \cdot e^{\pi i\alpha}. \\ &X^* = Y := \mathbb{C}^n, \quad X := G \times \mathbb{C}^{n-1}, \\ &A := \mathbb{C}^n \setminus X, \quad S := \{z \in A : z_1 = 0\} \end{aligned} \tag{11}$$

and, for

$$\begin{aligned} \varepsilon \in \mathbb{R}, x = \{x_1, \ldots, x_n\} \in \mathbb{C}^n: \quad &U_\varepsilon(x) := \{z \in \mathbb{C}^n : |z - x| < \varepsilon\}, \\ &U_\varepsilon(x_1) := \{z \in \mathbb{C} : |z - x_1| < \varepsilon\}. \end{aligned}$$

We define

$$f : X \to Y; \quad (z_1, z_2, \ldots, z_n) \to (\tilde{f}(z_1), z_2, \ldots, z_n).$$

Proposition 4.1. (a) *P is no* ess 1-sing *of f for all $P \in A \setminus S$.*
(b) *P is an* ess 2-sing *of f for all $P \in S$.*
(c) *P is no* ess 3-sing *of f for all $P \in A$.*

Proof. (a) is obvious, since, if $P = (p_1, \ldots, p_n) \in A \setminus S$ and $\varepsilon \in \mathbb{R}^+$ with $\varepsilon < p_1$ we can extend $\tilde{f}$ holomorphically from $U_\varepsilon(p_1) \cap \{\operatorname{Im} z_1 > 0\}$ to $U_\varepsilon(p_1)$.

(c) is easy: Choose $Z = \mathbb{C}^n$, $h : \mathbb{C}^n \to \mathbb{C}^n$; $(z_1, z_2, \ldots, z_n) \to (z_1^2, z_2, \ldots, z_n)$. To prove (b), let $P \in S$ be arbitrary. Assume that P is no ess 2-sing of f. Then there exist U, Z, h, M, and g like in Proposition 2.3. First we want to prove:

There are points $Q_1 = (q_1, q_2, \ldots, q_n)$, $Q_2 = (-q_1, q_2, \ldots, q_n)$ in $\mathbb{C}^n$ with $q_1 \in \mathbb{R}^+$ and $\delta \in \mathbb{R}^+$ with $\delta < q_1$ in such a way, that for every two points $R_1 = (r_1, r_2, \ldots, r_n)$, $R_2 = (-r_1, r_2, \ldots, r_n)$ in $\mathbb{C}^n$ with $r_1 \in \mathbb{R}^+$ and $R_1 \in U_\delta(Q_1)$ the equality $h(R_1) = h(R_2)$ holds. (12)

Since S_g is a 2-codimensional analytic subset of U there exist a point $P' = (p'_1, \ldots, p'_n) \in A \cap U$ with $p'_1 > 0$ and an $\eta \in \mathbb{R}^+$ with $\eta < p_1$, so that we have:

$$U_\eta(P') \subset U, \quad U_\eta(P') \cap S_g = \emptyset. \tag{13}$$

Define $Q_1 := (+\sqrt{p'_1}, p'_2, \ldots, p'_n)$, $Q_2 := (-\sqrt{p'_1}, p'_2, \ldots, p'_n)$ and $\delta \in \mathbb{R}^+$ so small, that, if q denotes the mapping $\mathbb{C}^n \to \mathbb{C}^n$; $(z_1, z_2, \ldots, z_n) \to (z_1^2, z_2, \ldots, z_n)$, we have $\delta < \sqrt{p'_1}$ and $q(U_\delta(Q_1)) \subset U_\eta(P')$.

Let R_1, R_2 be like in (12) and $R := q(R_1)$. Let $(z_\nu^{(1)})_{\nu \in \mathbb{N}}, (z_\nu^{(2)})_{\nu \in \mathbb{N}}$ be sequences with $z_\nu^{(1)} \in X \cap \{\operatorname{Im} z_1 > 0\}$, $z_\nu^{(2)} \in X \cap \{\operatorname{Im} z_1 < 0\}$ and $z_\nu^{(1)} \to R \leftarrow z_\nu^{(2)}$ for $\nu \to \infty$. From (11), we conclude $f(z_\nu^{(1)}) \to R_1$, $f(z_\nu^{(2)}) \to R_2$ and hence, because $R \notin S_g$ [cf. (13)],

$$g(R) = \lim_{\nu \to \infty} g(z_\nu^{(i)}) = \lim_{\nu \to \infty} h \circ f(z_\nu^{(i)}) = h(R_i) \quad \text{for} \quad i = 1, 2$$

which proves (12).

Now define $s: \mathbb{C}^n \to \mathbb{C}^n$; $(z_1, z_2, \ldots, z_n) \to (-z_1, z_2, \ldots, z_n)$. The set $\{z \in U_\delta(Q_1) : h \circ s = h\}$ is an analytic subset of $U_\delta(Q_1)$, which contains the set $U_\delta(Q_1) \cap A$ [cf. (12)], hence $h \circ s = h$ on $U_\delta(Q_1)$.

Choose z' from the set $U_\delta(Q_1) \setminus [(U_\delta(Q_1) \cap M) \cup s(U_\delta(Q_2) \cap M)]$. Then $s(z') \in U_\delta(Q_2) \setminus M$, especially $z', s(z') \in \mathbb{C}^n \setminus M$ and $z' \neq s(z')$, but $h(z') = h(s(z'))$, what is impossible, because h is injective on $\mathbb{C}^n \setminus M$. □

5. Some Examples for Ess 3-Sing

We again use the notations introduced in Sect. 2.

Theorem 5.1. *Let X^* be a Riemann surface. Then there is a holomorphic function $f: X \to \mathbb{C}$ so that every point $P \in \partial X$ is an* ess 3-sing *of f.*

Proof. First we apply Lemma 3.4 and get a sequence $F = (x_\mu)_{\mu \in \mathbb{N}}$ with the properties (a), (b), and (c) of Lemma 3.3 and a holomorphic function $f: X \to \mathbb{C}$ with

$$\{x \in X : f(x) = 0\} = |F|. \tag{14}$$

Let us assume that there exists a point $P \in \partial X$ which is no ess 3-sing of f. Then there exist U, $\mathcal{G}$, Z, h, M, and g like in Definition 2.2. Since $\mathbb{C}$ is connected, we may assume that Z is connected, too, and hence a Riemann surface. Now we can prove (8, 9) literally as it was done in Sect. 3. From (8, 9) we can conclude with the identity-lemma for holomorphic mappings between Riemann surfaces (where $z_0 = h(0)$):

$$g(x) \equiv z_0 \quad \text{for all} \quad x \in U. \tag{15}$$

Since $|F|$ is a discrete subset of X and (14, 8) there exists a connected component G_ν of $|\mathcal{G}|$ where f is not constant, hence locally biholomorphic outside a discrete subset of G_ν. Since h is locally biholomorphic outside M, too, there exists an open subset V of $|\mathcal{G}|$ where $h \circ f$ is locally biholomorphic. This contradicts (15). □

Proposition 5.2. *Let H_n be the n-dimensional Hopf-manifold and $\pi: (\mathbb{C}^n \setminus \{0\}) \to H_n$ the canonical projection.*

Then the zero point of $\mathbb{C}^n$ is an ess 3-sing *of π.*

Proof. Define $X^* := \mathbb{C}^n$, $A := \{0\}$, $Y := H_n$ and $f := \pi$. Let $p \in \mathbb{R}^+$ be the smallest number so that for an arbitrary $z \in X$ we have $f(p \cdot z) = f(z)$, and, for this p and any $r \in \mathbb{R}^+$, define $F_r := \{z \in X : r < |z| < p \cdot r\}$ (cf. [2, p. 146]).

Assume that the zero point is no ess 3-sing of f. Then there exist U, Z, h, and g like in Proposition 2.3. There exist open subsets $U_0 \subset Y$ and $W_0 \subset Z$, so that $h: U_0 \to W_0$ is biholomorphic, especially W_0 is n-dimensional. We further may assume that $U_0 \subset f(F_r)$ for suitable chosen $r \in \mathbb{R}^+$. For all $k \in \mathbb{N}_0$ define $V_k := (f|_{F_{rp^{-k}}})^{-1}(U_0)$. Then all mappings $f: V_k \to f(V_k) = U_0$ are biholomorphic.

Now let $w_0 \in W_0$ be arbitrary. Then there exists a point $u_0 \in U_0$ and for all $k \in \mathbb{N}_0$ a point $v_k \in V_k$ with $f(v_k) = u_0$, $h(u_0) = w_0$, hence $(v_k, w_0) \in G_g$. Since $v_k \to 0$ if $k \to \infty$ and G_g is closed in $U \times Z$ we have $(0, w_0) \in G_g$, and, because $w_0 \in W_0$ was arbitrary:

$$\{0\} \times W_0 \subset G_g, \qquad \dim W_0 = n. \tag{16}$$

Since G_g is an irreduzile n-dimensional analytic set, we therefore get the contradiction $G_g = G_g \cap (\{0\} \times Z)$. □

6. When do Ess *i*-Sing not Exist?

Proposition 6.1 (Product-Spaces). *Let $Y_1, \ldots, Y_t$ be normal complex spaces, $Y = Y_1 \times \ldots \times Y_t$ and pr_j, $j = 1, \ldots, t$, the canonical projections from Y to Y_t.*

(a) *If there exists a connected neighbourhood U of P in X^*, a c-sequence $\mathscr{G}$ in resp. of P and U and for every $j \in \{1, \ldots, t\}$ a normal complex space Z_j and a holomorphic mapping $h_j \in \mathscr{H}_i(Y_j, Z_j)$ such that $h_j \circ pr_j \circ f : |\mathscr{G}| \to Z_j$ can be extended to a meromorphic map $g_j : U \to Z_j$, then P is no* ess *i*-sing *of f.*

(b) *Let A be nowhere dense in X^* and assume that for every neighbourhood V of P in X^* there exists a subneighbourhood W such that $W \cap X$ is connected. Then if P is no* ess *i*-sing *of any mapping $pr_j \circ f : X \to Y_j$, $j = 1, \ldots, t$, P also is no* ess *i*-sing *of f.*

Remark. There exist meromorphic mappings, for which some $pr_j \circ f$ may have ess *i*-sing in P, but f hasn't: The mapping f_m constructed in the proof of part (a) of Theorem 3.1 in the case $n = 1$ has no ess 2-sing, but $pr_1 \circ f_m$ has ess 3-sing, as we showed in the proof of Theorem 5.1.

Proof of Proposition 6.1. (a) Define $Z := Z_1 \times \ldots \times Z_t$, $h := h_1 \times \ldots \times h_t : Y \to Z$. It is easily proved that $h \in \mathscr{H}_i(Y, Z)$. Now we have to construct g: Let $G_{g^*} := \{(x, z_1, \ldots, z_t) : x \in U,\ z_i \in g_i(x),\ i = 1, \ldots, t\} \subset U \times Z$. Then there exists a meromorphic map $g : U \to Z$ with $G_g \subset G_{g^*}$, (cf. [6, p. 839]). There further exists a closed an thin subset M^* of U such that $G_{g^*} \cap [(U \setminus M^*) \times Z]$ gives a holomorphic map, hence $G_g \cap [(U \setminus M^*) \times Z] = G_{g^*} \cap [(U \setminus M^*) \times Z]$. From the last equality it follows $g = h \circ f$ on $(|\mathscr{G}| \setminus M^*)$, hence an application of the identity-lemma for meromorphic maps (cf. [6, p. 830]) yields $g = h \circ f$ on $|\mathscr{G}|$. So g is an extension of $h \circ f$ from $|\mathscr{G}|$ to U.

(b) The proof is straightforward if we firstly apply Proposition 2.3, then the special assumption on the structure of A and at last part (a). □

Proposition 6.2 (Closed Complex Subspaces). *Let A be nowhere dense in X, U a connected neighbourhood of P in X^*, Z a normal complex space and $h \in \mathscr{H}_i(Y, Z)$. Let Z be a closed complex subspace of a normal complex space Z_0.*

Then, if $g: U \to Z_0$ *is a meromorphic extension of* $h \circ f: U \cap X \to Z$, *we have* $g(U) \subset Z$ *and* $g: U \to Z$ *is a meromorphic mapping; especially P is no* ess *i*-sing *of f.*

Proof. $\check{g}^{-1}(A \cup S_g)$ is a closed, nowhere dense subset of G_g (cf. [6, p. 823], [3, p. 167]). Since $\check{g}^{-1}(U \cap X) \subset U \times Z$ we therefore have $G_g \subset U \times Z$. Since G_g is irreducible in $U \times Z_0$ and $\check{g}$ is a proper map these properties hold in $U \times Z$, too. □

The next theorem is due to Stein [8, 9]:

Theorem 6.3. *Define* $\operatorname{fmd} f := \operatorname{Min}_{z \in G_j} \dim \hat{f}^{-1}(\hat{f}(z))$.

Let $A \subset X^*$ *be analytic and* $\operatorname{fmd} f > \dim A$.

Then the topological closure $\overline{G_f}$ *of* G_f *in* $X^* \times Y$ *is a meromorphic extension of f from X to* X^*; *especially f has no* ess 1-sing *in any point* $P \in A$.

Remark 1. A simple dimension-theoretic calculation shows that $\operatorname{fmd} f \geqq \dim G_f - \dim Y = \dim X^* - \dim Y$. So Theorem 6.3 is especially true if $\operatorname{codim} A > \dim Y$.

Remark 2. Theorem 6.3 shows that the inequality "$m \geqq \max(n, 2)$" in part (a) of Theorem 3.1 can't be improved: If $m < n$, we take A an isolated point P of X^*. Then Theorem 6.3 tells that $f_m: X \to Y_m$ has no ess 1-sing in P. The fact "$m \geqq 2$" already follows from part (b) of Theorem 3.1.

The next theorem actually only is an application of Stoll's theorem (4.3) in his paper [10]:

Theorem 6.4. *Let A be thin of codimension 2 in* X^* *and Y be a projective-algebraic space.*

Then there exists a meromorphic extension $g: X^* \to Y$ *of* $f: X \to Y$; *especially f has no* ess 1-sing *in any point* $P \in A$.

Proof. We may assume that $Y = \mathbb{P}^r$ for a suitable $r \in \mathbb{N}$, since then the assertion for arbitrary Y follows from Proposition 6.2. An application of Stoll's theorem (4.3) yields:

There exists a $\mu \in \{0, \ldots, r\}$ such that $X \setminus S_f \not\subset (f|_{X \setminus S_j})^{-1}(E_\mu)$ and for all $P \in X \setminus (S_f \cup (f|_{X \setminus S_j})^{-1}(E_\mu))$ the equation

$$f(P) = (f_0(P) : f_1(P) : \ldots : f_{\mu-1}(P) : 1 : f_{\mu+1}(P) : \ldots : f_r(P))$$

with meromorphic functions $f_i: X \to \mathbb{C}$ holds, where

$$E_\mu = \{(w_0 : w_1 : \ldots : w_r) \in \mathbb{P}^r : w_\mu = 0\}.$$

Now Levi's extension-theorem (cf. [3, p. 185]) tells us that we can extend the f_i to meromorphic functions $f_i^*: X^* \to \mathbb{C}$. Let $P(f_i^*)$ be the polar sets of f_i^* and $R = \bigcup_{i=0, \ldots, n; i \neq \mu} P(f_i^*)$. Then

$$f^*: (X^* \setminus R) \to \mathbb{P}^r;\ P \to (f_0^*(P) : f_1^*(P) : \ldots : f_{\mu-1}^*(P) : 1 : f_{\mu+1}^*(P) : \ldots : f_r^*(P))$$

is a holomorphic mapping on which we can apply Theorem (4.3) of Stoll a second time, but this time the other way around. We get that $\overline{G_{f^*}} \subset X^* \times \mathbb{P}^r$ yields a meromorphic mapping $g: X^* \to \mathbb{P}^r$. As is easily seen with the identity-lemma for meromorphic mappings, g is an extension of f. $\square$

The following proposition shows that in Theorem 6.4 the assumption that A is thin of codimension 2 in X^* cannot be weakened. It also gives the connection between essential singularities like they are defined in Definition 2.2 and isolated singularities like they occur in the function theory of **one** complex variable:

Proposition 6.5. *Let $B \subset \mathbb{C}$ be a domain with $0 \in B$ and $f: B \setminus \{0\} \to \mathbb{C}$ be a holomorphic function. Then the zero point is an* ess 1-sing *of f if and only if it is an isolated essential singularity in the sense of function theory of one complex variable* [1]. *In this case it even is an* ess 3-sing *of f.*

Proof. Since the zero point is no ess 1-sing of f if it is a removable singularity or a pole we only have to show:

If the zero point is an isolated essential singularity then it is an ess 3-sing of f.

Assume that it is no ess 3-sing. Then there exist U, Z, h, M, and g like in Proposition 2.3. Take $y^{(1)}, y^{(2)} \in \mathbb{C}$ with $h(y^{(1)}) \neq h(y^{(2)})$. Now with the theorem of Casorati-Weierstraß there exist two sequences $(x_\nu^{(1)})_{\nu \in \mathbb{N}}$, $(x_\nu^{(2)})_{\nu \in \mathbb{N}}$ in $U \setminus \{0\}$ with

$$x_\nu^{(1)} \to 0 \leftarrow x_\nu^{(2)}, \qquad f(x_\nu^{(1)}) \to y^{(1)}, \qquad f(x_\nu^{(2)}) \to y^{(2)} \quad \text{for} \quad \nu \to \infty .$$

Now we have

$$g(0) = \lim_{\nu \to \infty} g(x_\nu^{(1)}) = \lim_{\nu \to \infty} h(f(x_\nu^{(1)}) = h\left(\lim_{\nu \to \infty} f(x_\nu^{(1)})\right) = h(y^{(1)})$$

$$\neq h(y^{(2)}) = h\left(\lim_{\nu \to \infty} f(x_\nu^{(2)})\right) = \lim_{\nu \to \infty} h(f(x_\nu^{(2)})) = \lim_{\nu \to \infty} g(x_\nu^{(2)}) = g(0) . \square$$

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