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A Characterization of Sun-Reflexivity*

B. de Pagter

Department of Mathematics, Delft University of Technology, Julianalaan 132, 2628 BL Delft, The Netherlands

1. Introduction

The duality theory for strongly continuous semigroups of bounded linear operators (i.e., C_0-semigroups) in a Banach space was initiated by Phillips in [8]. One of the difficulties in dealing with adjoint semigroups is that the adjoint semigroup $\{T^*(t)\}_{t \geqq 0}$ of a C_0-semigroup $\{T(t)\}_{t \geqq 0}$ in a Banach space X, need not be strongly continuous in X^*. However, restricting $\{T^*(t)\}_{t \geqq 0}$ to the closed subspace $X^{\odot}$ of X^* on which $\{T^*(t)\}_{t \geqq 0}$ is strongly continuous, we obtain a C_0-semigroup $\{T^{\odot}(t)\}_{t \geqq 0}$ in $X^{\odot}$. Now we can repeat this construction with the semigroup $\{T^{\odot}(t)\}_{t \geqq 0}$, and we get a C_0-semigroup $\{T^{\odot\odot}(t)\}_{t \geqq 0}$ in the Banach space $X^{\odot\odot}$ (we refer the reader to Sect. 2 for a more detailed exposition of this construction). Then it may happen that the space $X^{\odot\odot}$ coincides with X and $\{T^{\odot\odot}(t)\}_{t \geqq 0} = \{T(t)\}_{t \geqq 0}$. If this occurs we say that X is $\odot$-reflexive ("sun-reflexive") with respect to $\{T(t)\}_{t \geqq 0}$.

In recent years this duality theory for C_0-semigroups has found various applications, in particular to differential equations (e.g. second order elliptic boundary value problems, evolution equations), and $\odot$-reflexivity plays an important role in these applications (e.g. [1, 2]). It was already shown by Phillips (see also [7, Theorem 14.6.1]) that a Banach space X is $\odot$-reflexive with respect to a C_0-semigroup $\{T(t)\}_{t \geqq 0}$ if and only if the resolvent operator $R(\lambda, A)$ is $\sigma(X, X^{\odot})$-compact for all λ in the resolvent set of A, where A denotes the infinitesimal generator of $\{T(t)\}_{t \geqq 0}$. This implies in particular that X is $\odot$-reflexive whenever $R(\lambda, A)$ is a weakly compact operator.

The purpose of the present paper is to show that $\odot$-reflexivity is in fact equivalent to weak compactness of $R(\lambda, A)$. Moreover, it will be shown that in certain Banach spaces (e.g. in L^1-spaces) $\odot$-reflexivity in equivalent to compactness of $R(\lambda, A)$.

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2. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space and suppose that $\{T(t)\}_{t \geqq 0}$ is a C_0-semigroup of bounded linear operators in X. The infinitesimal generator of $\{T(t)\}_{t \geqq 0}$ is denoted by A, with domain $\operatorname{dom}(A)$. We refer the reader for the basic properties of C_0-semigroups and their generators to one of the books [3, 5, 7]. For a complex number λ in the resolvent set $\varrho(A)$ of A we denote by $R(\lambda, A)$ the bounded linear operator $(\lambda I - A)^{-1}$. For convenience of the reader and to establish notation we briefly recall the duality theory for such semigroups. For details and proofs see e.g. [7, Chap. 14].

The adjoint semigroup $\{T^*(t)\}_{t \geqq 0}$ in X^* is clearly weak*-continuous, but in general it is not a C_0-semigroup in X^*. The adjoint A^* of A is a closed and weak*-densely defined operator in X^*, which is the weak*-generator of $\{T^*(t)\}_{t \geqq 0}$, i.e.,

$$\operatorname{dom}(A^*) = \left\{ x^* \in X^* : w^* - \lim_{t \downarrow 0} \frac{T^*(t)x^* - x^*}{t} \text{ exists in } X^* \right\}$$

and $\langle x, A^*x \rangle = \lim_{t \downarrow 0} t^{-1} \langle x, T^*(t)x^* - x^* \rangle$ for all $x \in X$ and $x^* \in \operatorname{dom}(A^*)$. Recall that $\varrho(A^*) = \varrho(A)$ and $R(\lambda, A^*) = R(\lambda, A)^*$ for all $\lambda \in \varrho(A)$.

Now let $X^{\odot}$ be the largest subspace of X^* on which $\{T^*(t)\}_{t \geqq 0}$ is strongly continuous, i.e.,

$$X^{\odot} = \{x^* \in X^* : \|T^*(t)x^* - x^*\| \to 0 \quad \text{as} \quad t \downarrow 0\}.$$

Clearly $X^{\odot}$ is a closed subspace of X^* and $T^*(t)(X^{\odot}) \subseteq X^{\odot}$ for all $t \geqq 0$. Define $T^{\odot}(t) = T^*(t)|_{X^{\odot}} : X^{\odot} \to X^{\odot}$ for all $t \geqq 0$. Then $\{T^{\odot}(t)\}_{t \geqq 0}$ is a C_0-semigroup. We collect some important facts concerning this "sun-dual" in the following proposition.

Proposition 2.1 (see [7, Chap. 14]). (i) $X^{\odot} = \overline{\operatorname{dom}(A^*)}$, *the norm closure of* $\operatorname{dom}(A^*)$ *in* X^*; *equivalently*, $X^{\odot} = \overline{R(\lambda, A)^*(X^*)}$ *for all* $\lambda \in \varrho(A)$.

(ii) *If we put* $\|x\|_1 = \sup\{|\langle x, x^* \rangle| : x^* \in X^{\odot}, \|x^*\| \leqq 1\}$ *for all* $x \in X$, *then* $\|\cdot\|_1$ *is a norm in* X *which is equivalent to the original norm in* X; *in fact*, $\|x\|_1 \leqq \|x\| \leqq M_0 \|x\|_1$ *for all* $x \in X$, *where* $M_0 = \liminf_{\lambda \to \infty} \|\lambda R(\lambda, A)\|$.

(iii) *The infinitesimal generator* $A^{\odot}$ *of* $\{T^{\odot}(t)\}_{t \geqq 0}$ *is given by* $\operatorname{dom}(A^{\odot}) = \{x^* \in \operatorname{dom}(A^*) : A^*x^* \in X^{\odot}\}$ *and* $A^{\odot}x^* = A^*x^*$ *for all* $x^* \in \operatorname{dom}(A^{\odot})$ *(i.e.,* $A^{\odot}$ *is the part of* A^* *in* $X^{\odot}$*).*

(iv) $\varrho(A^{\odot}) = \varrho(A^*) = \varrho(A)$ *and* $R(\lambda, A^{\odot}) = R(\lambda, A)^*|_{X^{\odot}}$ *for all* $\lambda \in \varrho(A)$.

Note that it follows from Proposition 2.1 (i) that $\{T^*(t)\}_{t \geqq 0}$ is a C_0-semigroup in X^* if and only if $\operatorname{dom}(A^*)$ is norm dense in X^*, which is in particular the case if X is reflexive or, of course, if $\{T(t)\}_{t \geqq 0}$ is a uniformly continuous semigroup.

Now we can repeat the above procedure with X and $\{T(t)\}_{t \geqq 0}$ replaced by $X^{\odot}$ and $\{T^{\odot}(t)\}_{t \geqq 0}$ respectively. Thus $\{T^{\odot *}(t)\}_{t \geqq 0}$ is a weak*-continuous semigroup in $X^{\odot *}$ with weak*-generator $A^{\odot *}$. Furthermore, the domain of strong continuity of $\{T^{\odot *}(t)\}_{t \geqq 0}$ in $X^{\odot *}$ is $X^{\odot\odot} = \overline{\operatorname{dom}(A^{\odot *})}$ and $\{T^{\odot\odot}(t)\}_{t \geqq 0}$ is a C_0-semigroup in $X^{\odot\odot}$ with generator $A^{\odot\odot}$. Note in particular that $\varrho(A^{\odot\odot}) = \varrho(A^{\odot *}) = \varrho(A^{\odot}) = \varrho(A^*) = \varrho(A)$ and $R(\lambda, A^{\odot\odot}) = R(\lambda, A^{\odot})^*|_{X^{\odot\odot}}$ for all $\lambda \in \varrho(A)$.

Let $j: X \to X^{**}$ be the canonical embedding of X into its bidual, and let $r_\odot: X^{**} \to X^{\odot *}$ be the restriction mapping, i.e., $r_\odot(x^{**}) = x^{**}|_{X^\odot}$ for all $x^{**} \in X^{**}$. Now $j_\odot = r_\odot \circ j$ is a mapping from X into $X^{\odot *}$ such that $\langle j_\odot(x), x^\odot \rangle = \langle x, x^\odot \rangle$ for all $x \in X$ and $x^\odot \in X^\odot$. Since

$$\|j_\odot(x)\| = \sup\{|\langle x, x^\odot \rangle| : x^\odot \in X^\odot, \|x^\odot\| \leqq 1\} = \|x\|_1$$

for all $x \in X$, it follows from Proposition 2.1 (ii) above that $j_\odot$ is a linear norm isomorphism from X onto $j_\odot(X)$. In particular, $j_\odot(X)$ is a closed subspace of $X^{\odot *}$. In general, however, $j_\odot$ is not an isometric isomorphism. Furthermore, it follows from

$$\langle T^{\odot *} j_\odot x, x^\odot \rangle = \langle x, T^\odot(t) x^\odot \rangle = \langle T(t) x, x^\odot \rangle$$

for all $x^\odot \in X^\odot$, that $T^{\odot *} j_\odot x = j_\odot T(t) x$ for all $x \in X$ and all $t \geqq 0$. Now it is clear that $j_\odot(X) \subseteqq X^{\odot\odot}$.

If we identify, for a moment, X with its image $j_\odot(X)$, then the C_0-semigroup $\{T^{\odot\odot}(t)\}_{t \geqq 0}$ is an extension of $\{T(t)\}_{t \geqq 0}$, $\operatorname{dom}(A^{\odot\odot}) \cap X = \operatorname{dom} A$, $A^{\odot\odot} x = Ax$ for all $x \in \operatorname{dom} A$ and $R(\lambda, A^{\odot\odot})_{|X} = R(\lambda, A)$ for all $\lambda \in \varrho(A)$.

Now we recall the following definition.

Definition 2.2. *The space X is called $\odot$-reflexive ("sun-reflexive") with respect to $\{T(t)\}_{t \geqq 0}$ if $j_\odot(X) = X^{\odot\odot}$.*

Since $X^{\odot\odot} = \overline{\operatorname{dom} A^{\odot *}}$, it is clear that X is $\odot$-reflexive with respect to $\{T(t)\}_{t \geqq 0}$ if and only if $R(\lambda, A^\odot)^*(X^{\odot *}) \subseteqq j_\odot(X)$ for all $\lambda \in \varrho(A)$.

Next we will present two simple examples to illustrate the above concepts.

Examples 2.3. (i) Let $X = C_0(\mathbb{R})$, the space of all complex continuous functions f on $\mathbb{R}$ such that $\lim_{|x| \to \infty} f(x) = 0$, with the sup-norm. For $t \geqq 0$ and $f \in C_0(\mathbb{R})$ define $T(t) f(x) = f(x + t)$ for all $x \in \mathbb{R}$. Clearly $\{T(t)\}_{t \geqq 0}$ is a C_0-semigroup in $C_0(\mathbb{R})$ and the generator A is given by

$$\operatorname{dom}(A) = \{f \in C_0(\mathbb{R}) : f \text{ is differentiable and } f' \in C_0(\mathbb{R})\},$$

$Af = f'$ for all $f \in \operatorname{dom}(A)$. The dual space of $C_0(\mathbb{R})$ can be identified with the space $M_b(\mathbb{R})$ of all bounded (complex) Borel measures on $\mathbb{R}$. The adjoint of A is given by

$$\operatorname{dom}(A^*) = \{\mu \in M_b(\mathbb{R}) : D\mu \in M_b(\mathbb{R})\}, A^* \mu = -D\mu \text{ for all } \mu \in \operatorname{dom}(A^*),$$

(where $D\mu$ denotes the distributional derivative of μ). A Borel measure μ with $D\mu \in M_b(\mathbb{R})$ is absolutely continuous with respect to Lebesgue measure m. As usual, we identify the subspace of $M_b(\mathbb{R})$ consisting of all measures which are absolutely continuous with respect to m with the space $L^1(\mathbb{R}, m)$, via the Radon-Nikodym derivative. Then $\operatorname{dom}(A^*)$ consists of all functions in $L^1(\mathbb{R}, m)$ which are of bounded variation. It is now easy to see that $X^\odot = \overline{\operatorname{dom}(A^*)} = L^1(\mathbb{R}, m)$ and that $T^\odot(t) g(x) = g(x - t)$ for all $t \geqq 0$. Furthermore,

$$\operatorname{dom}(A^\odot) = \{g \in L^1(\mathbb{R}, m) : g \text{ is absolutely continuous and } g' \in L^1(\mathbb{R}, m)\}$$

and $A^\odot g = -g'$ for all $g \in \operatorname{dom}(A^\odot)$.

As usual, the dual of $L^1(\mathbb{R}, m)$ is identified with $L^\infty(\mathbb{R}, m)$. For $f \in L^\infty(\mathbb{R}, m)$ we then have $T^{\odot *}(t) f(x) = f(x + t)$ for all $t \geqq 0$, and the weak*-generator of

$\{T^{\odot *}(t)\}_{t \geqq 0}$ is given by

$$\operatorname{dom}(A^{\odot *}) = \{f \in L^\infty(\mathbb{R}, m) : f \text{ is absolutely continuous and } f' \in L^\infty(\mathbb{R}, m)\}$$

and $A^{\odot *} f = f'$ for all $f \in \operatorname{dom}(A^{\odot *})$. Moreover, $X^{\odot\odot} = BUC(\mathbb{R})$, the space of all bounded uniformly continuous functions on $\mathbb{R}$. The generator of $\{T^{\odot\odot}(t)\}_{t \geqq 0}$ is given by

$$\operatorname{dom}(A^{\odot\odot}) = \{f \in BUC(\mathbb{R}) : f \text{ is differentiable and } f' \in BUC(\mathbb{R})\},$$

$A^{\odot\odot} f = f'$ for all $f \in \operatorname{dom}(A^{\odot\odot})$. We see that the space $C_0(\mathbb{R})$ is not $\odot$-reflexive with respect to $\{T(t)\}_{t \geqq 0}$. Note that the spectrum of A is $\varrho(A) = \{ia : a \in \mathbb{R}\}$ and that

$$R(\lambda, A) f(x) = \int_x^\infty f(\xi) e^{\lambda(x-\xi)} d\xi$$

for all $f \in C_0(\mathbb{R})$ and $\operatorname{Re}\lambda > 0$. It is easy to verify that $R(\lambda, A)$ is not weakly compact (see also the comments at the end of this paper).

(ii) Let X be the Banach space $C(S^1)$ of all continuous functions on the unit circle S^1. For $t \geqq 0$ and $f \in C(S^1)$ we define $T(t) f(\theta) = f(\theta + t)$ (as usual we write $f(\theta)$ for $f(e^{i\theta})$). The dual of $C(S^1)$ can be identified with the space $M(S^1)$ of all Borel measures on S^1. As in (i) we find that $X^\odot = L^1(S^1, m)$ (where m denotes normalized Lebesgue measure on S^1) and $T^\odot(t) g(\theta) = g(\theta - t)$ for all $t \geqq 0$ and all $g \in L^1(S^1, m)$. The adjoint semigroup $\{T^{\odot *}(t)\}_{t \geqq 0}$ in $L^\infty(S^1, m)$ satisfies $T^{\odot *}(t) f(\theta) = f(\theta + t)$ for all $f \in L^\infty(S^1, m)$. It is now clear that $X^{\odot\odot} = C(S^1)$, and hence $C(S^1)$ is $\odot$-reflexive with respect to the C_0-semigroup $\{T(t)\}_{t \geqq 0}$. We note already that in this situation the resolvent operator $R(\lambda, A)$ of the generator A is in fact compact for all $\lambda \in \varrho(A)$.

3. A Characterization of $\odot$-Reflexivity

First we recall some relevant facts concerning weakly compact operators. Let $\mathscr{L}(X, Y)$ denote the Banach space of all bounded linear operators from Banach space X into Banach space Y. An operator $T \in \mathscr{L}(X, Y)$ is called weakly compact if the image $T(B_X)$ of the closed unit ball B_X is relatively weakly compact in Y, i.e., if $\overline{T(B_X)}$ is weakly compact (note that the norm and weak closure of the convex set $T(B_X)$ coincide). If $T \in \mathscr{L}(X, Y)$, then T is weakly compact if and only if $T^{**}(X^{**})$ is contained in Y (identifying Y with its canonical image in Y^{**}). Furthermore, the set of all weakly compact operators from a Banach space X into itself is a norm closed two-sided ideal in $\mathscr{L}(X)$. The proofs of these well-known results can be found in e.g. [5, Sect. VI.4].

Now let $\{T(t)\}_{t \geqq 0}$ be a C_0-semigroup in the Banach space X with generator A. It is useful to note that for any $\lambda \in \varrho(A)$ we have the following commutative diagram

$$\begin{tikzcd}
 & X^{**} \arrow[r, "R(\lambda, A)^{**}"] \arrow[dd, "r_\odot"] & X^{**} \arrow[dd, "r_\odot"] \\
X \arrow[ur, "j"] \arrow[dr, "j_\odot"'] & & \\
 & X^{\odot *} \arrow[r, "R(\lambda, A^\odot)^*"'] & X^{\odot *}
\end{tikzcd}$$

(where the mappings j, $j_\odot$, and $r_\odot$ are as introduced in the previous section). Furthermore observe that it is immediate from the resolvent equation that $R(\lambda, A)$ is (weakly) compact for all $\lambda \in \varrho(A)$ if and only if $R(\lambda, A)$ is (weakly) compact for some $\lambda \in \varrho(A)$. The proof of the following proposition is now simple.

Proposition 3.1 (cf. [7, Corollary to Theorem 14.6.1]). *If $R(\lambda, A)$ is weakly compact for $\lambda \in \varrho(A)$, then X is $\odot$-reflexive.*

Proof. Take $\lambda \in \varrho(A)$. If $R(\lambda, A)$ is weakly compact, then $R(\lambda, A)^{**}(X^{**}) \subseteq j(X)$. Since the restriction mapping $r_\odot$ is surjective, it follows from the commutativity of the above diagram that $R(\lambda, A^\odot)^*(X^{\odot *}) \subseteq j_\odot(X)$. As observed in Sect. 2, this implies that X is $\odot$-reflexive with respect to $\{T(t)\}_{t \geqq 0}$. □

As mentioned in the introduction, it is shown in [7, Theorem 14.6.1], that $\odot$-reflexivity is equivalent to $\sigma(X, X^\odot)$-compactness of $R(\lambda, A)$. Since weak compactness of $R(\lambda, A)$ clearly implies that $R(\lambda, A)$ is $\sigma(X, X^\odot)$-compact, the above proposition is an immediate consequence of this result. The direct proof above is included for the reader's convenience. Our next objective is to show that $\odot$-reflexivity of X is in fact *equivalent* to weak compactness of $R(\lambda, A)$. The proof is divided into three lemmas.

Lemma 3.2. *Let Y be a closed subspace of a Banach space Z. We denote by $i: Y \to Z$ the inclusion mapping (so $i^*: Z^* \to Y^*$ is the restriction mapping). Let W be a closed subspace of Z^* and put $W_1 = i^*(W)$. Now suppose that $T: Z \to Z$ is a bounded linear operator which satisfies the following three conditions:* (i) $T(Z) \subseteq Y$; (ii) $T^*(W) \subseteq W$; (iii) $T_0^*(Y^*) \subseteq W_1$, *where $T_0 = T_{|Y}: Y \to Y$.*

Then $(T^*)^2(Z^*) \subseteq W$.

Proof. Defining $T_1: Z \to Y$ by $T_1 z = Tz$ for all $z \in Z$ we clearly get the commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "T"] \arrow[dr, "T_1"] & Z \arrow[r, "T"] & Z \\
Y \arrow[u, "i"] \arrow[r, "T_0"'] & Y \arrow[u, "i"] \arrow[r, "T_0"'] & Y \arrow[u, "i"]
\end{tikzcd}$$

and so by taking adjoints we find the commutative diagram

$$\begin{tikzcd}
Z^* \arrow[r, "T^*"] \arrow[d, "i^*"] & Z^* \arrow[r, "T^*"] \arrow[d, "i^*"] & Z^* \arrow[d, "i^*"] \\
Y^* \arrow[r, "T_0^*"'] & Y^* \arrow[r, "T_0^*"'] \arrow[ur, "T_1^*"] & Y^*
\end{tikzcd}$$

Hence, $(T^*)^2(Z^*) = T_1^* \circ T_0^* \circ i^*(Z^*) = T_1^* \circ T_0^*(Y^*) \subseteq T_1^*(W_1) = T_1^* \circ i^*(W)$
$= T^*(W) \subseteq W$. □

Lemma 3.3. *If the Banach space X is $\odot$-reflexive with respect to the C_0-semigroup $\{T(t)\}_{t \geqq 0}$, then $R(\lambda, A)^2$ is weakly compact for all $\lambda \in \varrho(A)$.*

Proof. Take in the above lemma $Z=X^*$, $Y=X^{\odot}$, $W=j(X)$ and $T=R(\lambda, A)^*$. Note that $W_1=j_{\odot}(X)$ and $T_0=R(\lambda, A^{\odot})$. It follows now from the $\odot$-reflexivity of X that $R(\lambda, A^{\odot})^*(X^{\odot *})\subseteq j_{\odot}(X)$, i.e., that $T_0^*(Y^*)\subseteq W_1$. Hence, we may conclude that $(T^*)^2(Z^*)\subseteq W$, and so $[R(\lambda, A)^2]^{**}(X^{**})\subseteq j(X)$, which shows that $R(\lambda, A)^2$ is weakly compact. □

Lemma 3.4. *If $\{T(t)\}_{t\geqq 0}$ is a C_0-semigroup in the Banach space X with generator A, then $\|[\lambda R(\lambda, A)]^2 R(\mu, A)-R(\mu, A)\|\to 0$ as $\lambda\to\infty$ for all $\mu\in\varrho(A)$.*

Proof. Fix $\mu\in\varrho(A)$ and let $M\geqq 1$ be such that $\|\lambda R(\lambda, A)\|\leqq M$ for all $\lambda\geqq\lambda_0$, for some $\lambda_0\in\mathbb{R}_+$. For $\lambda\neq\mu$ we have

$$\begin{aligned}\lambda R(\lambda, A)R(\mu, A)-R(\mu, A)&=\frac{\lambda}{\lambda-\mu}\{R(\mu, A)-R(\lambda, A)\}-R(\mu, A)\\&=\frac{1}{\lambda-\mu}\{\mu R(\mu, A)-\lambda R(\lambda, A)\},\end{aligned}$$

and hence

$$\|\lambda R(\lambda, A)R(\mu, A)-R(\mu, A)\|\leqq\frac{1}{\lambda-|\mu|}(\|\mu R(\mu, A)\|+M)$$

for all $\lambda>\max(\lambda_0, |\mu|)$. This shows that $\|\lambda R(\lambda, A)R(\mu, A)-R(\mu, A)\|\to 0$ as $\lambda\to\infty$. Now

$$\begin{aligned}&\|[\lambda R(\lambda, A)]^2 R(\mu, A)-R(\mu, A)\|\\&\quad\leqq\|\lambda R(\lambda, A)[\lambda R(\lambda, A)R(\mu, A)-R(\mu, A)]\|+\|\lambda R(\lambda, A)R(\mu, A)-R(\mu, A)\|\\&\quad\leqq(M+1)\|\lambda R(\lambda, A)R(\mu, A)-R(\mu, A)\|\end{aligned}$$

for all $\lambda\geqq\lambda_0$, by which the lemma is proved. □

We now formulate the main result of the paper.

Theorem 3.5. *Given a C_0-semigroup $\{T(t)\}_{t\geqq 0}$ in the Banach space X with generator A, the following two statements are equivalent.*

(i) *X is $\odot$-reflexive with respect to $\{T(t)\}_{t\geqq 0}$.*

(ii) *$R(\lambda, A)$ is weakly compact for $\lambda\in\varrho(A)$.*

Proof. Assume that X is $\odot$-reflexive. It follows from Lemma 3.3 that $R(\lambda, A)^2$ is weakly compact for all $\lambda\in\varrho(A)$. Since the set of weakly compact operators is a closed two-sided ideal in $\mathscr{L}(X)$, Lemma 3.4 now implies that $R(\lambda, A)$ is weakly compact for all $\lambda\in\varrho(A)$. The converse implication is Proposition 3.1. □

In certain Banach spaces the result of the above theorem can be strengthened. For this purpose, recall that a Banach space X has the *Dunford-Pettis property* if every weakly compact operator from X into any Banach space Y maps weakly compact subsets of X onto compact subsets of Y (see e.g. Sect. II.9 in the book [9]). Clearly, if X has the Dunford-Pettis property and $T\in\mathscr{L}(X)$ is weakly compact, then T^2 is a compact operator.

Corollary 3.6. *Suppose that X is a Banach space with the Dunford-Pettis property, and let $\{T(t)\}_{t \geqq 0}$ be a C_0-semigroup in X with generator A. The following statements are equivalent.*

(i) *X is $\odot$-reflexive with respect to $\{T(t)\}_{t \geqq 0}$.*

(ii) *$R(\lambda, A)$ is compact for $\lambda \in \varrho(A)$.*

Proof. Only (i) $\Rightarrow$ (ii) needs proof. Suppose that X is $\odot$-reflexive. It follows from Theorem 3.5 that $R(\lambda, A)$ is weakly compact for all $\lambda \in \varrho(A)$. Since X has the Dunford-Pettis property, this implies that $R(\lambda, A)^2$ is compact for all $\lambda \in \varrho(A)$. Now it is a simple consequence of Lemma 3.4 that $R(\lambda, A)$ is compact for all $\lambda \in \varrho(A)$. □

We mention two important examples of Banach spaces to which the result of the corollary applies.

1) For any σ-finite measure space (Ω, Σ, μ), the space $X = L^1(\Omega, \mu)$ has the Dunford-Pettis property (this is the classical result of Dunford and Pettis [4]).

2) For any locally compact Hausdorff space Ω, the Banach space $C_0(\Omega)$ of all continuous functions on Ω vanishing at infinity (with sup-norm) has the Dunford-Pettis property. In particular, the Banach space $C(\Omega)$ of all continuous functions on a compact space Ω, has the Dunford-Pettis property (these results go back to Grothendieck [6]).

We end this paper by mentioning a criterion for weak compactness of $R(\lambda, A)$. Let A be a closed and densely defined linear operator in the Banach space X with $\varrho(A) \neq \phi$. For $x \in \operatorname{dom}(A)$ define $\|x\|_A = \|x\| + \|Ax\|$. Then $(\operatorname{dom}(A), \|\cdot\|_A)$ is a Banach space with unit ball $B_A = \{x \in \operatorname{dom}(A) : \|x\| + \|Ax\| \leqq 1\}$. Given $\lambda \in \varrho(A)$ it is easy to verify that

$$(\|R(\lambda, A)\| + \|AR(\lambda, A)\|)^{-1} R(\lambda, A)(B_X) \subseteqq B_A \subseteqq \max(|\lambda|, 1) R(\lambda, A)(B_X),$$

so $R(\lambda, A)(B_X)$ is relatively weakly compact if and only if B_A is a relatively weakly compact subset of X. Furthermore, by the Eberlein-Smulian theorem (see e.g. [5, Theorem V.6.1]), a subset S of X is relatively weakly compact if and only if S is relatively sequentially weakly compact. Combining these observations we get the equivalence of the following three statements:

(1) $R(\lambda, A)$ is a weakly compact operator for $\lambda \in \varrho(A)$;

(2) the embedding of $(\operatorname{dom}(A), \|\cdot\|_A)$ into X is weakly compact;

(3) any sequence $\{x_n\}_{n=1}^{\infty}$ in $\operatorname{dom}(A)$ with $\sup_n \|x_n\| < \infty$ and $\sup_n \|Ax_n\| < \infty$ has a subsequence which is weakly convergent to an element in X.

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