Math. Ann. 283, 539–549 (1989)

Mathematische Annalen

The Rate of Convergence of a Harmonic Map at a Singular Point

Robert Gulliver[1] and Brian White[2]

[1] School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
[2] Department of Mathematics, Stanford University, Stanford, CA 94305, USA

0. Introduction

It has been apparent since the 1968 example of De Giorgi that weak solutions of elliptic systems in $m \geqq 3$ independent variables may well have points of discontinuity [dG]. In the geometrically interesting case of harmonic mappings, the example $f(x)=x/|x|$ of a discontinuous (weakly) harmonic mapping, from Euclidean $\mathbb{R}^m$ to the standard sphere of dimension $m-1$, was given by Hildebrandt and Widman [HW]. Thus, the regularity theory for harmonic mappings between Riemannian manifolds requires a clear understanding of the behaviour of the mapping near its singular set. If $f: M^m \to N^n$ is a minimizing harmonic map with a point of discontinuity $O \in M$, then its *homogeneous tangent map* is defined as follows. Let $x=(x^1, \ldots, x^m)$ be Riemannian normal coordinates at 0; for each $0<\lambda<1$, define the blowup $f_\lambda(x) := f(\lambda x)$, as introduced in [GM]. By means of a monotonicity lemma, Schoen and Uhlenbeck showed that f_λ has uniformly bounded Dirichlet integral with respect to the Euclidean metric in the domain [SU, p. 314]. It follows that each blowup sequence $f_{\lambda(i)}$, as $\lambda(i) \to 0$, has a subsequence converging locally weakly to $f_0 : \mathbb{R}^m \to N$. They also show that f_0 is harmonic and homogeneous, and that the subsequence converges in the H^1 norm [SU, p. 329]. We shall adopt the global approach of Schoen and Uhlenbeck: we choose an isometric embedding of N^n into some Euclidean $\mathbb{R}^d$, and define $H^1(M,N)$ to be the subset of $H^1(M,\mathbb{R}^d)$, the space of functions having square-integrable first partial derivatives, having values in N almost everywhere.

The analysis of Schoen and Uhlenbeck left open the important question of the uniqueness of the homogeneous tangent mapping f_0. This was resolved for any smooth f_0 by Simon in [S1] (see also [S2]):

Theorem (Simon). *Let $f \in H^1(M,N)$ be a harmonic mapping which minimizes energy on some neighborhood of $O \in M$, where N is a real-analytic manifold. Let $f_0 \in C^2(\mathbb{R}^m \backslash \{0\}, N)$ be the weak limit of some blowup sequence $f_{\lambda(i)}$ as $\lambda(i) \to 0$. Then f_0 is the unique homogeneous tangent map to f at* 0, *and the restrictions to the sphere of radius* 1 *satisfy, as $\lambda \to 0$,*

$$\|f_\lambda - f_0\|_{C^2(S^{m-1})} + \|D_\varrho f_\lambda\|_{C^1(S^{m-1})} \to 0 .$$

Here D_ϱ denotes partial differentiation in spherical coordinates of $\mathbb{R}^m$ with respect to $\varrho=|x|$. Similarly, we shall write D_α for partial differentiation with respect to x^α, $1\leqq\alpha\leqq m$.

Note that Simon's theorem gives an estimate implying only rather slow convergence of f to its homogeneous tangent mapping f_0 (compare inequality (0.6) below). In the present paper, we show that the order of convergence, in general, *depends on the dimensions* of the domain M^m and the target manifold N^n. In the lowest dimensions for which singularities may occur, namely $m=3$ and $n=2$, the convergence of f to f_0 is controlled by a *positive power of* $|x|$. On the other hand, whenever $m\geqq 3$ and $n\geqq 3$, we construct examples for which this convergence is *slower* than any positive power of $|x|$.

Let us describe harmonic maps in detail. For $f\in H^1(M,N)$, the energy functional is

$$E(f)=1/2\int_M \gamma^{\alpha\beta}\langle D_\alpha f, D_\beta f\rangle d\,\mathrm{vol}_M, \tag{0.1}$$

where the Riemannian metric of M is given by $ds_M^2=\gamma_{\alpha\beta}(x)dx^\alpha dx^\beta$, $(\gamma_{\alpha\beta})$ is the inverse of the $m\times m$ matrix $(\gamma^{\alpha\beta})$, and summation over $1\leqq\alpha,\beta\leqq m$ is assumed. The volume form is $d\,\mathrm{vol}_M=\sqrt{\gamma(x)}\,dx^1\dots dx^m$, where $\gamma=\det(\gamma_{\alpha\beta})$. Since we have chosen an isometric embedding of N^n into Euclidean $\mathbb{R}^d$, the inner product $\langle\ ,\ \rangle$ may be understood as the standard inner product of $\mathbb{R}^d$. Via integration by parts, one sees that $f\in H^1(M,N)$ is stationary for E if and only if the weak Laplace-Beltrami operator

$$\Delta_M f:=-\gamma^{-1/2}D_\alpha(\sqrt{\gamma}\,\gamma^{\alpha\beta}D_\beta f) \tag{0.2}$$

is normal to N almost everywhere. Equivalently, the vector function f satisfies

$$\Delta_M f+\gamma^{\alpha\beta}(x)B(D_\alpha f, D_\beta f)=0 \tag{0.3}$$

weakly, where B is the second fundamental form of N in $\mathbb{R}^d$. For any vector fields U, V tangent to N, we may define $B(U,V):=(D_U V)^\perp$, where D_U is covariant differentiation in $\mathbb{R}^d$, and at the relevant point of N, a vector $W\in\mathbb{R}^d$ is given the orthogonal decomposition $W=W^\perp+W^T$ into vectors $W^\perp$ normal to N and W^T tangent to N. Note that in Eq. (0.3), the coefficients of B depend on $f(x)\in N$.

The homogeneous tangent mapping $f_0:\mathbb{R}^m\to N$ is also harmonic, but with respect to the Euclidean metric on $\mathbb{R}^m$. If we write $\Delta f:=-D_\alpha D_\alpha f$ for the standard Laplacian, then the equation satisfied weakly by f_0 is

$$Lf_0:=\Delta f_0+B(D_\alpha f_0, D_\alpha f_0)=0.$$

In addition to Simon's result stated above, there is an earlier method developed by Allard and Almgren in [AA] in the context of minimal varieties, which requires an additional hypothesis but yields a stronger conclusion. The analogous proof in the context of harmonic mappings has been carried out by Simon in [S2]. Given a harmonic mapping $f_0:S^{m-1}\to N$, that is, one whose homogeneous extension to $\mathbb{R}^m$ satisfies $Lf_0=0$, a vector field $\phi:S^{m-1}\to TN$ along f_0 is called a *harmonic-Jacobi field* if Lf_t vanishes to first order in t for any family of mappings $f_t:S^{m-1}\to N$ with $\partial f_t/\partial t=\phi$ at $t=0$. Equivalently, ϕ is a solution of the linearized equation

$$\Delta\phi+2B(D_\alpha\phi^T, D_\alpha f)+D_\phi B(D_\alpha f, D_\alpha f)=0, \tag{0.4}$$

where $D_\phi B$ is the covariant derivative with respect to ϕ of the second fundamental form as a tensor with values in $\mathbb{R}^d$, and using the natural connection D^T of TN for its arguments. For example if $f_t: S^{m-1} \to N$, $-\varepsilon < t < \varepsilon$, is a one-parameter family of harmonic mappings, then it follows that $\phi = \partial f_t / \partial t$ is a harmonic-Jacobi field; in this case, we may say that ϕ is *integrable.*

Theorem (Almgren-Allard [AA]; cf. [S2]). *Let $f \in H^1(M, N)$ be a harmonic mapping which minimizes energy on some neighborhood of $O \in M$.*

Let $f_0 \in C^2(\mathbb{R}^m \setminus \{0\}, N)$ be the weak limit of some blowup sequence $f_{\lambda(i)}(x) = f(\lambda(i)x)$ where $\lambda(i) \to 0$. Assume that f_0 satisfies the following integrability hypothesis:

There is a k-parameter family $f: (U \subset \mathbb{R}^k) \times S^{m-1} \to N$ of harmonic maps such that $f(0, \cdot) = f_0$ and such that each harmonic-Jacobi field φ along f_0 is equal to $\frac{d}{dt} f(tv, \cdot)(t=0)$ for some $v \in \mathbb{R}^k$. (0.5)

Then f_0 is the unique homogeneous tangent map to f at 0, and

$$\|f_\lambda - f_0\|_{C^2(S^{m-1})} + \|D_\varrho f_\lambda\|_{C^1(S^{m-1})} \leqq C\lambda^\alpha \tag{0.6}$$

for some C and $\alpha > 0$ depending on N and on f_0.

Remark 0.1. As stated in [S2, pp. 272–273], this theorem requires N to be analytic. But in the proof there, analyticity is used only to conclude (0.5) from a weaker integrability hypothesis. (cf. [S2, pp. 271–272]). Thus with (0.5), analyticity is not needed. □

It might appear likely to a casual observer that the two theorems are but special cases of a stronger result, as yet undiscovered, that concludes the λ^α convergence but does not require the strong hypothesis (0.5). However, as we shall show

(Section 1) *the λ^α convergence does* not *hold in dimensions m, $n \geqq 3$ for stationary harmonic mappings in the absence of the integrability hypothesis;*

and, on the other hand:

(Section 2) *the integrability hypothesis* (0.5), *and therefore λ^α convergence,* always *hold when the domain M has dimension* 3 *and the target manifold N has dimension* 2.

It is interesting to note that this universal integrability holds precisely in the first dimensions in which regularity fails. In fact if $n = 1$, with any m, then we are dealing with weak solutions of a single uniformly elliptic equation, which are as smooth as the coefficients allow (for reference see [G, p. 53]). If the domain dimension $m = 1$, then harmonic mappings become geodesics with constant speed parametrization, and weak solutions are again smooth [M, p. 28]. In the case of a two-dimensional domain, we may refer to Morrey's result on general variational problems [M, Theorem 1.10.4(iii) and pp. 34–37], from which it may be seen that a weakly harmonic mapping is as regular as the target manifold N.

It is also interesting to note that the method of [S1] requires analyticity of N to conclude uniqueness of the limit map f_0, whereas the method used here gives uniqueness and fast convergence without assuming analyticity.

The first author would like to thank the Consiglio Nazionale delle Ricerche for its hospitality at the University of Trento. The second author would like to acknowledge the support of the Institute for Mathematics and its Applications at the University of Minnesota, and of the Alfred P. Sloan Foundation.

1. An Example of Logarithmic Convergence

We begin by presenting an example of a stationary harmonic mapping $f: M^3 \to N^3$, having a single point of discontinuity, where the domain M and the target manifold N each have dimension three. Once this example is constructed, it may be extended to form examples $f_1: M_1^m \to N_1^n$ of harmonic mappings for arbitrary dimensions $m \geqq 3$ and $n \geqq 3$, having the same rate of convergence to their homogeneous tangent mappings. In fact, we may choose $M_1 := M \times (S^1)^{m-3}$ and $N_1 := N \times R^{n-3}$ as Riemannian product manifolds and then define $f_1(x, \theta) = (f(x), 0) \in N_1$, where $(x, \theta) \in M \times (S^1)^{m-3}$, $\theta = (\theta_4, \ldots, \theta_m)$ and where $(u, t) \in N \times \mathbb{R}^{n-3}$, $t = (t_4, \ldots, t_n)$. Then the logarithmic convergence of f to its homogeneous tangent mapping f_0 will imply a similar property for f_1 (although the homogeneous tangent mapping for f_1 will have a singular set of the form $\mathbb{R}^{m-3} \subset \mathbb{R}^m$; see Remark 1.1 below).

We choose $M^3 = B_1^3 \subset \mathbb{R}^3$, the unit ball with the standard, Euclidean metric, and refer to standard coordinates $x = (x^1, x^2, x^3) = \varrho\omega$, $\omega \in S^2$, $\varrho \geqq 0$. The target manifold N^3 shall be a hypersurface of revolution in $\mathbb{R}^4$, generated by the curve $r = \Gamma_0(z)$ in the (r, z)-plane:

$$N = \{(v, z) \in \mathbb{R}^3 \times \mathbb{R} : |v| = \Gamma_0(z)\}.$$

We shall also write $v = r\omega \in \mathbb{R}^3$ where $\omega \in S^2, r \geqq 0$. For simplicity, we may replace the coordinate z by the arc-length parameter $u = u(z)$ for the generating curves: $(du/dz)^2 = 1 + (d\Gamma_0/dz)^2$, and define $\Gamma(u)$ such that $\Gamma(u(z)) := \Gamma_0(z)$. Then the coordinates $(\omega, u) \in S^2 \times \mathbb{R}$ may be used to describe the Riemannian metric of N induced from the Euclidean metric of $\mathbb{R}^4$:

$$ds_N^2 = \Gamma(u)^2 ds_\Sigma^2(\omega) + du^2,$$

where $\omega \in S^2$, $u \in \mathbb{R}$ and ds_Σ^2 is the canonical metric of constant Gauss curvature 1 on the sphere $\Sigma = S^2$. We consider the $O(3)$-equivariant mapping $f(x) = f(\varrho\omega) = (\omega, u(\varrho)) \in N$ in terms of the coordinates $(\omega, u) \in S^2 \times \mathbb{R}$ for N, determined by a real-valued function $u = u(\varrho)$ of one real variable. Then the energy of f may be computed in terms of $u(\varrho)$:

$$\begin{aligned} E(f) &= \int_M [(du/d\varrho)^2 + 2(\Gamma(u)/\varrho)^2]\varrho^2 d\varrho \, d\mathrm{vol}_\Sigma(\omega) \\ &= 4\pi \int_0^1 [\varrho^2(du/d\varrho)^2 + 2\Gamma(u)^2] d\varrho, \end{aligned} \tag{1.1}$$

as follows from (0.1). The Euler-Lagrange equations may be computed directly from this formula for $E(f)$, to show that a continuous mapping $f: M \to N$ of the form $f(\varrho\omega)=(\omega, u(\varrho))$ is stationary for E if and only if $u(\varrho)$ is a weak solution of

$$\frac{d}{d\varrho}\left(\varrho^2 \frac{du}{d\varrho}\right)=2\Gamma(u(\varrho))\Gamma'(u(\varrho)). \tag{1.2}$$

The reader will observe that the ordinary differential Eq. (1.2) has a singular point at $\varrho=0$.

Let us consider in particular the family of functions

$$u(\varrho)=\frac{1}{\sqrt{C-2\log\varrho}}, \quad 0 \leqq \varrho \leqq 1, \tag{1.3}$$

for various constants $C \geqq 0$. As $\varrho \to 0$, $u(\varrho)$ converges to zero more slowly than any positive power of ϱ. A direct computation yields $d(\varrho^2 du/d\varrho)/d\varrho = u^3 + 3u^5$ for each value of C. This computation and (1.2) lead us to consider the specific function

$$\Gamma(u)=\sqrt{1+u^4/4+u^6/2}. \tag{1.4}$$

This choice for Γ leads to $\Gamma(u)=\Gamma_0(z)$, via a hyperelliptic integral, defined for all $-\infty < z < \infty$. In other words, (1.4) corresponds to a *complete hypersurface of revolution* $N^3 \subset \mathbb{R}^4$.

The homogeneous tangent mapping $f_0: \mathbb{R}^3 \to N$ is given by $f_0(\varrho\omega)=(\omega, 0)$, which is an isometric parameterization of the totally geodesic sphere $\Sigma_0=\{(\omega,u)\in N : \omega \in S^2, u=0\}$ at the narrowest point of N. Recalling the Allard-Almgren theorem stated above, it is of interest to consider whether harmonic Jacobi fields $\phi: S^2 \to TN$ along the restricted mapping $f_0: S^2 \to N$ are integrable. In fact, *any* harmonic mapping $f: S^2 \to N$ must have its image in Σ_0. Namely, the parallel sphere $\Sigma_K := \{(\omega, u)\in N : \omega \in S^2, u=K\}$ has principal curvature vectors, as a submanifold of N, with negative or positive component in the direction of $\partial/\partial z$, when $\Gamma_0'(z)$ is positive or negative, respectively. With Γ as in (1.4), we have $u\Gamma'(u)>0$ for $u \neq 0$. It follows that $|u|$ may not have a positive local maximum along a harmonic mapping, and in particular, $u \equiv 0$ for any harmonic mapping $g: S^2 \to N$, by compactness of S^2. On the other hand, $\Gamma''(0)=0$ implies that $f_t(\omega) := (\omega, t)\in N$ is harmonic *to first order* at $t=0$, which means that $\partial/\partial z = \partial f_t(\omega)/\partial t$, at $t=0$, is a harmonic-Jacobi field along f_0. In particular, the integrability hypothesis is violated.

It is apparent that the mapping $f: B_1 \to N$ we have constructed is not continuous at 0; for this reason, we have yet to show that it is a weak solution of the Euler-Lagrange Eqs. (0.3). To show that $f \in H^1(M,N)$, it is enough to show that the energy integral (1.1) is finite. But $\varrho du/d\varrho = u^3$ for the family of functions (1.3), which implies that the integrand of (1.1) is uniformly bounded on $0 \leqq \varrho \leqq 1$. Next, recall that f is a weak solution of (0.3) provided that for all $h \in C^\infty(M, \mathbb{R}^4)$ with compact support, there holds

$$\int_M \gamma^{\alpha\beta}(\langle D_\alpha f, D_\beta h\rangle + \langle B(D_\alpha f, D_\beta f), h\rangle) d\mathrm{vol}_M = 0. \tag{1.5}$$

Write $h=h_1+h_0$, where $h_1(x)=0$ for $|x|\leqq \varepsilon$. Then it follows from (1.2) that (1.5) holds with h replaced by h_1, since f is smooth on $\mathrm{supp}\,t(h_1)$. By choosing h_0 to be h

times a standard cutoff function, we may achieve that $h_0(x)=0$ for $|x|\geqq 3\varepsilon$ and that $|Dh_0|\leqq |Dh|+|h|/\varepsilon$. Since h is uniformly bounded, we have $\|h\|_{L^2(B_{3\varepsilon})}\leqq C\varepsilon^{3/2}$, and in particular, $h_0\to 0$ in $H^1(M,\mathbb{R}^4)$ as $\varepsilon\to 0$, implying that $\int_M \langle D_\alpha f, D_\beta h_0\rangle dx\to 0$. Meanwhile, since $f\in H^1(M,N)$, we have $B(D_\alpha f, D_\beta f)\in L^1(M,\mathbb{R}^4)$ and hence

$$\left|\int_M \langle B(D_\alpha f, D_\beta f), h_0\rangle dx\right| \leqq \|h\|_{L^\infty}\int_{B_{3\varepsilon}} |B(D_\alpha f, D_\beta f)|dx\to 0 .$$

This shows that f is a weak solution of the Euler-Lagrange equations. We have proved the following

Theorem 1. *Given any $m, n\geqq 3$, there is a real-analytic Riemannian manifold N^n, and a harmonic mapping $f: B_1^m\to N^n$ with a discontinuity at $O\in M$, such that $f(\varrho\omega)\to f_0(\omega)$ and $\varrho D_\varrho f(\varrho\omega)\to 0$, as $\varrho\to 0$, uniformly in $C^2(S^2)$, both more slowly than any positive power of ϱ.*

Remark 1.1. As observed above, for $m>3$ the example leads to a homogeneous tangent map with singularities on an $\mathbb{R}^{m-3}$. The analysis of [AA] and of [S1] is problematic in this case. However, we may modify the example to construct an *isolated* singularity, by defining $M=B_1^m\subset\mathbb{R}^m$ and $N=\{(v,z)\in\mathbb{R}^m\times\mathbb{R}: |v|=\Gamma_0(z)\}$ in close analogy to the example above. Let $f: M\to N$ be the $O(m)$-invariant mapping $f(\varrho\omega)=(\omega, u(\varrho))$ where $u(\varrho)$ belongs to the one-parameter family (1.3). Choose the hypersurface of revolution N so that

$$\Gamma(u)=\sqrt{1+\frac{m-2}{2(m-1)}u^4+\frac{1}{m-1}u^6}\,, \tag{1.6}$$

in terms of the arc-length parameter u. Then $f: M^m\to N^m$ is a stationary harmonic mapping as in Theorem 1, which moreover has an *isolated* singularity. This carries over to any dimensions $n\geqq m\geqq 3$.

Remark 1.2. It might be noted that both theorems on convergence to the homogeneous tangent map, as stated in the introduction, require f to be locally mimimizing, while our examples are only stationary (see however Remark 2.1 below). In fact, it is a rather nontrivial exercise to prove that any specific discontinuous mapping minimizes energy. Recently, Schoen and Brézis-Coron-Lieb have announced independent proofs that the mapping $f(x)=x/|x|$ from the Euclidean ball to S^2 has minimum energy. In analogy with these results, we expect that the examples with isolated singularities just constructed have minimum energy with respect to their Dirichlet boundary data.

Remark 1.3. Suppose that the function $\Gamma(u)$ is an *arbitrary* real-analytic function which assumes its positive minimum value at the unique critical point $u=0$, which is degenerate: $\Gamma''(0)=0$. Let $k+2\geqq 4$ be the order of the first nonzero derivative of Γ at 0 (k must be even). Then there are solutions of the ordinary differential Eq. (1.2) with $u(\varrho)\to 0$. Specifically, these solutions have the asymptotic behaviour $u(\varrho)(C-ka\log\varrho)^{1/k}\to 1$ as $\varrho\to 0$, for some real constants a and C. This behavior may be proved by first finding an invariant manifold of the form $\varrho du/d\varrho=\Phi(u)$, where Φ is a function of the form $\Phi(u)=au^{k+1}+O(u^{k+2})$, and where $2\Gamma(u)\Gamma'(u)$ has the same leading term.

2. The Integrability Hypothesis for $m=3$ and $n=2$

As observed in the introduction, a harmonic mapping: $M^m \to N^n$ must be smooth if the target dimension $n=1$, if the domain dimension $m=1$ or if $m=2$. On the other hand, if m and n are both $\geqq 3$, then as we have just seen, there are discontinuous harmonic mappings with only logarithmic convergence to their homogeneous tangent mappings. This leaves exactly one pair of dimensions to be investigated: $m=3$ and $n=2$. In these dimensions, the homogeneous tangent mapping is a harmonic mapping from S^2 into N^2, which may be expected to show a rigidity not apparent in higher dimensions. In fact, we have the following

Theorem 2. *Let M^3 and N^2 be Riemannian manifolds of dimension 3 and 2, respectively. Let $f: M^3 \to N^2$ be a locally minimizing harmonic map near $O \in M$. Then f converges to a unique homogeneous tangent mapping f_0 at a rate controlled by a positive power of the distance ϱ from 0, as in inequality (0.6). Moreover, if f has a discontinuity at 0, then N has the topological type of the two-dimensional sphere or projective plane.*

In order to apply the Allard-Almgren theorem stated in the introduction, we need the integrability hypothesis for an arbitrary harmonic mapping $f_0: S^2 \to N^2$. First, we shall observe that f_0 necessarily enjoys a much stronger property than harmonicity, as is widely known. We write g in place of f_0 for the three lemmas.

Lemma 1. *Let Σ^2, N^2 be two-dimensional Riemannian manifolds, $N^2 \subset \mathbb{R}^d$, and $g \in H^1(\Sigma, N)$ a harmonic mapping. If Σ has the topological type of S^2 or of $\mathbb{R}P^2$, then g is a conformal mapping. If moreover N is not of the topological type of S^2 or $\mathbb{R}P^2$, then g is constant.*

For completeness, we give a proof for Lemma 1. Let $z \in \mathbb{C}$ be a local conformal parameter for Σ: that is, $z=x+iy$ where $ds_\Sigma^2 = \lambda(z)^2(dx^2+dy^2)$, and $\lambda(z)>0$. The existence of z follows from the uniformization theorem. Then as a mapping into $\mathbb{R}^d$, g has the differential

$$dg = g_x dx + g_y dy = g_z dz + g_{\bar{z}} d\bar{z},$$

where the subscripts denote partial derivatives, and the complex partial derivatives are defined by $g_z = (g_x - i g_y)/2$ and $g_{\bar{z}} = (g_x + i g_y)/2$, as usual. Let the Euclidean inner product $\langle \, , \rangle$ of $\mathbb{R}^d$ be extended as a *complex-bilinear* form on $\mathbb{C}^d$, and similarly, let the second fundamental form B of N, as a submanifold of $\mathbb{R}^d$, be extended as a symmetric bilinear tensor on the complexified tangent bundle to N, with values in its complexified normal bundle. We may compute $\lambda^2 \Delta_\Sigma g = -4 g_{z\bar{z}}$ and $\lambda^2 \gamma^{\alpha\beta} B(D_\alpha g, D_\beta g) = 4B(g_z, g_{\bar{z}})$. The Eq. (0.3) for a harmonic mapping becomes

$$g_{z\bar{z}} = B(g_z, g_{\bar{z}}). \tag{2.1}$$

As noted in the introduction, since Σ has dimension 2, a weak solution $g \in H^1(\Sigma, N)$ must be of class C^2.

Recall that a *conformal* mapping $g: \Sigma \to N$ is one which preserves the Riemannian metric up to a variable factor $\sigma: \Sigma \to [0, \infty)$, that is, such that $g^* ds_N^2 = \sigma(z) ds_\Sigma^2$. For a conformal parameter $z=x+iy$, this is equivalent to $|g_x|^2 = |g_y|^2 (= \sigma\lambda^2)$ and $\langle g_x, g_y \rangle = 0$. These two equations may be written in complex notation as $\langle g_z, g_z \rangle = 0$.

We shall first show that $\langle g_z, g_z\rangle$ is locally *holomorphic*, or equivalently, that $\langle g_z, g_z\rangle_{\bar z}=0$. But $\langle g_z, g_z\rangle_{\bar z}=2\langle g_z, g_{z\bar z}\rangle=2\langle g_z, B(g_z, g_{\bar z})\rangle$ by Eq. (2.1). On the other hand, the real and imaginary parts of g_z are tangent vectors to N, while $B(g_z, g_{\bar z})$ is a (real) normal vector, so this last quantity vanishes identically. This shows that $\langle g_z, g_z\rangle$ is holomorphic on the domain of definition of the parameter z.

Now if Σ has the topological type of $\mathbb{R}P^2$, then we may compose g with the covering projection from the Riemannian universal covering space of Σ. Thereby, we may assume that Σ has the topological type of S^2. It follows from the uniformization theorem that Σ is conformally equivalent to the standard $S^2=\mathbb{C}P^1$, and in particular, we may cover Σ by two conformal coordinate charts $z:\mathbb{C}\to\Sigma$ and $\zeta:\mathbb{C}\to\Sigma$ satisfying $z\zeta\equiv 1$. By the chain rule, we obtain $z^2\langle g_z, g_z\rangle=\zeta^2\langle g_\zeta, g_\zeta\rangle$. Now $\langle g_\zeta, g_\zeta\rangle$ is a holomorphic function at $\zeta=0$, which implies that $|\langle g_z, g_z\rangle|\leqq C|z|^{-4}$ for $|z|$ sufficiently large, and hence $\langle g_z, g_z\rangle\equiv 0$ by the maximum principle. This shows that g is a conformal mapping (some readers may prefer to invoke the Riemann-Roch theorem here).

Finally, if N^2 is not topologically a sphere or projective plane, then its universal cover $\tilde N$ is conformally the disk or the plane, and g lifts to a conformal mapping $\tilde g: S^2\to\tilde N$. But a nonconstant conformal mapping must be an open mapping, which would imply that $\tilde g(S^2)$ is both compact and open as a subset of $\tilde N$, a contradiction. Therefore $\tilde g$ and g must be constant mappings. This finishes the proof of Lemma 1. □

In analogy with harmonic-Jacobi fields along a harmonic mapping, we may define a vector field $\varphi:\Sigma\to TN$ along a conformal mapping $g:\Sigma^2\to N^2$ to be a *conformal-Jacobi field* (or infinitesimal conformal mapping) if it satisfies

$$\langle \varphi_z, g_z\rangle=0. \tag{2.2}$$

This is immediately seen to be equivalent to the vanishing of the first variation of $\langle g_z, g_z\rangle$ when g is varied in the direction of φ.

Lemma 2. *Let Σ, N be two-dimensional Riemannian manifolds, $g:\Sigma\to N$ a harmonic mapping, and $\varphi:\Sigma\to TN$ a harmonic-Jacobi field along g. If Σ has the topological type of S^2 or of $\mathbb{R}P^2$, then g is a conformal mapping, and φ is a conformal-Jacobi field.*

We shall show that $\langle\varphi_z, g_z\rangle$ is a holomorphic function on the domain of any conformal parameter z. Lemma 2 will then follow as in the proof of Lemma 1.

We may rewrite (0.4) in the form

$$\varphi_{z\bar z}=B(\varphi_z^T, g_{\bar z})+B(g_z, \varphi_{\bar z}^T)+D_\varphi B(g_z, g_{\bar z}). \tag{2.3}$$

Observe that φ_z is the same as $D_{g_z}\varphi$, since φ is a vector field along g. The component of φ_z normal to N is therefore $\varphi_z^\perp=(D_{g_z}\varphi)^\perp=:B(g_z,\varphi)=B(\varphi, g_z)=(D_\varphi g_z)^\perp$. Further, since the values of B are normal vectors to N, we have $\langle g_z, B(g_z, g_{\bar z})\rangle=0$. Differentiating this last expression in the direction of φ yields

$$\langle g_z, D_\varphi B(g_z, g_{\bar z})\rangle=-\langle D_\varphi g_z, B(g_z, g_{\bar z})\rangle=-\langle\varphi_z, B(g_z, g_{\bar z})\rangle.$$

Using twice more the fact that B has values normal to N, we obtain from (2.3) that

$$\langle g_z, \varphi_{z\bar{z}}\rangle = \langle g_z, D_\varphi B(g_z, g_{\bar{z}})\rangle = -\langle \varphi_z, B(g_z, g_{\bar{z}})\rangle .$$

Recalling (2.1), we conclude that $\langle g_z, \varphi_z\rangle_{\bar{z}} = 0$, so that $\langle g_z, \varphi_z\rangle$ is holomorphic. This completes the proof of Lemma 2. □

Lemma 3. *Let $g: \Sigma \to N$ be a nonconstant (branched) conformal mapping of degree d, where Σ is topologically S^2 and N is topologically S^2 or $\mathbb{R}P^2$. Then there is a $(4d+2)$-parameter family*

$$f: (U \subset \mathbb{R}^{4d+2}) \times \Sigma \to N$$

of conformal mappings with $f(0, \cdot) = g(\cdot)$ such that every conformal-Jacobi field along g is equal to

$$\frac{d}{dt} f(tv, \cdot)(t=0)$$

for some $v \in \mathbb{R}^{4d+2}$.

To begin the proof of Lemma 3, we may assume that N is topologically S^2 (otherwise lift g to the universal cover). Recall that Σ and N are conformally equivalent to $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; we write conformal diffeomorphisms $T_1: \hat{\mathbb{C}} \to \Sigma$ and $T_2: \hat{\mathbb{C}} \to N$. Then g is represented by a (branched) conformal mapping $w_0: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (That is, $T_2 \circ w_0 = g \circ T_1$). The conformality relation $\langle g_z, g_z\rangle = 0$ implies that $w_{0z} = 0$ or $w_{0\bar{z}} = 0$. We may orient Σ so that $w_{0\bar{z}} = 0$ holds. Then as is well known, w_0 must be a rational function: $w_0(z) = P(z)/Q(z)$ for some polynomials P, Q having no common factor other than constants. Since g has mapping degree d,

$$\max\{\deg P, \deg Q\} = d .$$

Suppose that $w_0(\infty) \neq \infty$, or equivalently,

$$\deg(P) \leqq \deg(Q) = d$$

(otherwise modify T_2). Note that w_0 belongs to the $(4d+2)$ parameter family

$$w_{A,B} = \frac{P+A}{Q+B},$$

where A and B are polynomials with degree $(A) \leqq d$ and degree $(B) \leqq d-1$. Each $w_{A,B}$ corresponds to a conformal map $g_{A,B}$, and $g_{0,0} = g$.

Now let φ be a conformal-Jacobi field along g. If $w_0(z) \neq \infty$, then for some real $\alpha(z)$ and $\beta(z)$

$$T_{2*}^{-1}(\varphi(T_1(z)) = \alpha(z) D_u(w_0(z)) + \beta(z) D_v(w_0(z)),$$

where $w = u + iv$ and $\{D_u, D_v\}$ is the coordinate basis of vectorfields on $\mathbb{C}$. Write $\gamma(z) := \alpha(z) + i\beta(z)$. The equation $\langle g_z, \varphi_z\rangle = 0$ for a conformal-Jacobi field becomes $\alpha_z u_{0z} + \beta_z v_{0z} = 0$ where $w_0(z) := u_0(z) + iv_0(z)$. This is immediately equivalent to $\gamma_{\bar{z}} = 0$ since w_0 has only isolated branch points. Thus φ is represented by the function γ, which is holomorphic except at the poles of w_0.

We need some information about these singularities. Near any pole z_1 of w_0 (so $w_0(z_1)=\infty$) we may consider the conformal parameter $\hat{w}:=1/w$. Then g is represented by the meromorphic function $\hat{w}_0$ with $\hat{w}_0(z_1)=0$. The vector field φ is represented in terms of $D_{\hat{u}}$ and $D_{\hat{v}}$ by a locally holomorphic function $\hat{\gamma}(z)$. It follows from the chain rule that $\gamma(z)=-w_0(z)^2\hat{\gamma}(z)$. Therefore, γ has at most a pole at z_1, whose order is at most twice the order of the pole of w_0. It follows that $\gamma(z)$ may be written *globally* as

$$\gamma(z)=R(z)/Q(z)^2$$

for some complex polynomial $R(z)$. Also, since $w_0(\infty)\neq\infty$, $\gamma(\infty)\neq\infty$ and thus $\deg(R)\leqq\deg(Q^2)=2d$. Now we must show that for some polynomials A and B with $\deg A\leqq d$ and $\deg B\leqq d-1$,

$$\varphi=\frac{d}{dt}g_{tA,tB}(t=0)$$

i.e.,

$$\gamma=\frac{d}{dt}(P+tA)/(Q+tB)=\frac{AQ-BP}{Q^2}$$

i.e.,

$$R=AQ-BP.$$

Now since the complex polynomials form a Euclidean domain, and since P and Q are relatively prime, there exist polynomials A_1 and B_1 such that

$$R=A_1Q-B_1P.$$

Now divide B_1 by Q to get polynomials S and B such that

$$B_1=SQ+B$$
$$\deg(B)<\deg(Q).$$

Then, letting $A=A_1-SP$, we have

$$R=AQ-BP.$$

Since $\deg(R)\leqq 2d$ and $\deg(BP)<\deg(Q^2)=2d$, it follows that $\deg(A)\leqq d$. This completes the proof of Lemma 3. □

Theorem 2 is a direct consequence of the three lemmas. In fact, since f is locally energy minimizing, it has at least one homogeneous tangent map $f_0:\mathbb{R}^3\to N$ [SU, p. 314]. If N is not S^2 or $\mathbb{R}P^2$, then f_0 is constant by Lemma 1, and therefore f is C^α (indeed smooth) in a neighborhood of 0 [SU, p. 315]. In particular, (0.6) holds.

On the other hand, if N is S^2 or $\mathbb{R}P^2$ and f_0 is not constant, then by Lemmas 1–3, it satisfies the integrability hypothesis (0.5) of Simon's version of the Allard-Almgren theorem. This completes the proof. □

Remark 2.1. Theorem 2 (as well as the theorems quoted in the introduction) also apply to mappings that are harmonic but not necessarily energy minimizing. In that case, however, one must assume that at least one blow-up sequence $f_{\lambda(i)}$ converges strongly (say in C^2 on compact subsets of $\mathbb{R}^m \setminus \{0\}$) to the homogeneous limit f_0.

References

[G] Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. Ann. Math. Study 105. Princeton (1983)

[SU] Schoen, R., Uhlenbeck, K.: A regularity theory for harmonic maps. J. Differ. Geom. **17**, 307–335 (1982)

[GM] Giusti, E., Miranda, M.: Sulla regolarità delle solutioni deboli di una classe di sistemi ellittici quasilineari. Arch. Rat. Mech. Anal. **31**, 173–184 (1968)

[M] Morrey, C.B.: Multiple integrals in the calculus of variations. Berlin Heidelberg New York: Springer 1966

[AA] Allard, W., Almgren, F.: On the radial behaviour of minimal surfaces and the uniqueness of their tangent cones. Ann. Math. **113**, 215–265 (1981)

[S1] Simon, L.: Asymptotics for a class of non-linear evolution equations, with applications to geometric problems. Ann. Math. **118**, 525–571 (1983)

[S2] Simon, L.: Isolated singularities of extrema of geometric variational problems. Lecture Notes Math., Vol. 1161, pp. 206–277. Berlin Heidelberg New York: Springer 1985

[dG] De Giorgi, E.: Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. Boll. U.M.I. **4**, 135–137 (1968)

[HW] Hildebrandt, S., Widman, K.-Ø.: Some regularity results for quasilinear elliptic systems of second order. Math. Z. **142**, 67–86 (1975)

Received December 2, 1986; in revised form August 5, 1987