Math. Ann. 283, 551–572 (1989)

Mathematische Annalen

The Symmetric-Square L-Function Attached to a Cuspidal Automorphic Representation of GL_3

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In [13] Rankin introduced a new method into the theory of automorphic forms which he used to determine the analytic properties of $\sum_{n\geqq 1} \tau(n)^2 n^{-s}$ where τ is the Ramanujan function. We can reformulate his results as follows. Let π_Δ be the automorphic representation of $PGL_2(\mathbf{Q}_\mathbf{A})$ associated with Δ. Let $L(s,\pi,r)$ be the L-function associated with an automorphic representation π of $G_\mathbf{A}$ where G is a reductive algebraic group over the base field k, and is a finite dimensional representation of the L-group ${}^LG$. Then Rankin proved that

$$s\mapsto L(s,\pi_\Delta,\mathrm{Sym}^2)\zeta(s)$$

where Sym^2 is the symmetric square representation $SL_2(\mathbf{C})$ and ζ is the zeta function of $\mathbf{Q}$ (including the archimedean factor), has an analytic continuation as a meromorphic function into $\mathbf{C}$. It is invariant under the replacement of s by $1-s$. The poles of this function are located at $s=0,1$ and are simple.

The same method was rediscovered shortly after Rankin's work by Selberg [14] and is usually called the Rankin-Selberg method.

At the Antwerp conference in 1972 Shimura [16] described a variant of the Rankin-Selberg method which yielded the Dirichlet series $\sum_{n\geqq 1} \tau(n^2)n^{-s}$, i.e. $L(s,\pi_\Delta,\mathrm{Sym}^2)$ which he showed to be holomorphic. This was a significant sharpening of Rankin's result and had important consequences – see, for example, [4].

The method of Rankin-Selberg can be generalized very substantially; see [9, 12] for more details. In particular, if π is an automorphic representation of $GL_r(k_\mathbf{A})$ with central quasicharacter χ one has an Euler product $L(s,\pi\times\pi)$ which has an analytic continuation as a meromorphic function into the entire plane. If π is unitary then $s=1$ is a simple pole of $L(s,\pi\times\pi)$ if and only if $\pi\cong\check{\pi}$, the contragredient of π; still assuming π to be unitary the function $L(s,\pi\times\pi)$ is holomorphic except possibly at $s=0,1$, and there is a functional equation relating $L(s,\pi\times\pi)$ and $L(1-s,\ \check{\pi}\times\check{\pi})$.

Since $L(s,\pi\times\pi)=L(s,\pi,\mathrm{Ten}^2)$ where Ten^2 denotes the tensor square of the standard representation of $GL_r(\mathbf{C})$, and since

$$\mathrm{Ten}^2=\mathrm{Sym}^2\oplus\Lambda^2$$

(Sym^2 = symmetric square, Λ^2 = alternating square) gives the decomposition of Ten^2 into irreducible representations we have

$$L(s,\pi\times\pi)=L(s,\pi,\mathrm{Sym}^2)\cdot L(s,\pi,\Lambda^2).$$

In particular, when $r=3$ we have moreover

$$L(s,\pi,\Lambda^2)=L(s,\check{\pi}\otimes\chi)$$

where $L(s,\pi)$ is the L-function of [8].

Since the analytic continuation and functional equation of $L(s,\pi\otimes\pi\chi)$ are also known we deduce the analytic continuation as a meromorphic function, and the functional equation of $L(s,\pi,\mathrm{Sym}^2)$. What does not follow from this is that $L(s,\pi,\mathrm{Sym}^2)$ should have only finitely many poles. This has, however been proved by a different method by Shahidi [15]. Indeed if π is cuspidal one might expect $L(s,\pi,\mathrm{Sym}^2)$ to be entire, but this is not the case for the following reason. Let π_1 be a cuspidal automorphic representation of $GL_2(k_{\mathbf{A}})$ with central quasicharacter χ_1; let π be the Gelbart-Jacquet [4] lift of π_1 to $GL_3(k_{\mathbf{A}})$. Then π is cuspidal and

$$L(s,\pi,\mathrm{Sym}^2)=L(s,\pi_1,\mathrm{Sym}^4)\cdot L(s,\chi_1^2)$$

which has a pole at $s=1$ if $\chi_1^2=1$. In fact $L(s,\pi\times\pi)$ has a pole at $s=1$ precisely when $\pi\cong\check{\pi}$; see [9, p. 368]. The condition $\chi_1^2=1$ ensures that the condition $\pi\cong\check{\pi}$ is fulfilled. Conversely, by a theorem of Flicker's [3, II, Theorem 2.9] this condition implies that π is a lift of an automorphic representation π_1 with $\chi_1^2=1$. Thus the only case in which $L(s,\pi,\mathrm{Sym}^2)$ can have a pole is that which we have just discussed.

It is the objective of this paper to give a integral representation of $L(s,\pi,\mathrm{Sym}^2)$ analogous to the one given by Shimura in the case of GL_2 in [15, 17] (see also [4] for an adelic representation-theoretic account of Shimura's method). Shimura's method is based on the consideration of the Rankin-Selberg convolution of an automorphic form of the representation π with a theta-function. Here we shall do the same, although the theta-function can no longer be constructed using the Weil representation but by the technique of Eisenstein series used in [10]. The integral representation is given in Proposition 3.2 and Corollary 4.2. A peculiarity of our method is that we exploit the fact that the representation to which the theta-function belongs is degenerate i.e. it has no Whittaker model.

From this integral representation we would expect to be able to deduce that $L(s,\pi,\mathrm{Sym}^2)$ has only finitely many poles. Unfortunately we do not reach this goal in general. Let $L_v(s,\pi,\mathrm{Sym}^2)$ denote the v^{th} factor of $L(s,\pi,\mathrm{Sym}^2)$. We shall be able to limit the poles of a function

$$\prod_{v\in S} M_v(s)\cdot\prod_{v\notin S} L_v(s,\pi,\mathrm{Sym}^2)$$

where M_v runs through a certain vector space. Unfortunately we have not been able to show that for a archimedean v this space has no common zeros in the halfplane $\mathrm{Re}(s) \geqq \frac{1}{2}$. The "local integrals" M_v have a more complicated structure than is usual since they involve three representations, all of which are infinite dimensional.

It also follows from our method that when the characteristic of k is neither 0 or 2 that $L(s, \pi, \mathrm{Sym}^2)$ has a pole at $s=1$ if and only if

$$\int_{G_k \backslash G_{\mathbf{A}} / Z_{\mathbf{A}}} \varphi(g) \theta_1(g) \theta_2(g) dg \neq 0$$

where φ is some automorphic form of π and θ_1, θ_2 are certain "theta functions," albeit not ones arising from a Weil representation. In view of the discussion above this integral is not identically zero when π is a lift of an automorphic representation of $GL_2(k_{\mathbf{A}})$ for which χ_1^2 is the fourth power of a Größencharakter. We presume that the same holds when the characteristic of k is 0 but we have not been able to overcome the technical difficulties (see Proposition 5.3 and the remarks following it).

For our construction we shall have to make use of the theory of Eisenstein series. In Sect 2 we shall recall those facts which we shall need. In Sect. 3 we shall construct the Rankin-Selberg integral which is the central subject of this paper. In Sect. 4 we shall evaluate the local integral at a generic place. In Sect. 5 we prove the appropriate local functional equation. Finally in Sect. 6 we shall summarize our results.

We should note that Shahidi has given in [15] a quite different approach to the investigation of L-functions of the type considered here. He derives these results from the theory of Eisenstein series, and he proves results which are much more general than ours, and in this special case, more precise [15, Corollary 6.7]. Nevertheless the method described here is interesting in itself and may well have applications which the method of [15] cannot have.

In this paper k will denote a fixed $\mathbf{A}$-field of characteristic $\neq 2$. We shall denote for an algebraic group G defined over k the group of k-points of G by G_k, *of* k_v-points by G_v where v is a place of k, and of $k_{\mathbf{A}}$-points by $G_{\mathbf{A}}$ where $k_{\mathbf{A}}$ denotes the adele ring of k. Let $\Sigma(k)$ denote the set of places of k; let $\Sigma_\infty(k)$ (resp. $\Sigma_f(k)$) be the subset of archimedean (resp. non-archimedean) places.

We shall work with GL_2, GL_3 and subgroups of these. Let P (resp. Q) be the standard $(2,1)$ (resp. $(1,2)$) parabolic subgroup of GL_3. Let N denote the upper triangular unipotent subgroup of GL_3 or GL_2 (it will be clear from the context which is meant). Let $M(P)$, $M(Q)$ be the standard Levi factor of P, Q; let $N(P) = M(P) \cap N$, $N(Q) = M(Q) \cap N$. Let $U(P)$, U(Q) be the unipotent radical of P, Q. Let H denote the diagonal subgroups of GL_2 or GL_3; again it will be clear from the context which is intended. Let M be the normalizer of H, $W = M/H$. Let Z be the centre of GL_2 or GL_3. Let $B = H \cdot N$.

Over a local field F let μ (resp. $\mu(P)$, $\mu(Q)$) denote the square root of the modulus of the adjoint action of $H(F)$ (resp. $M(P)(F)$, $M(Q)(F)$) on the Lie algebra over F of N (resp. $U(P)$, $U(Q)$). This also yields positive quasi-cahracters $\mu_{\mathbf{A}}$ of $H_{\mathbf{A}}$, $\mu_{P,\mathbf{A}}$ of $M(P)_{\mathbf{A}}$ and $P_{\mathbf{A}}$, $\mu_{Q,\mathbf{A}}$ of $M(Q)_{\mathbf{A}}$ and $Q_{\mathbf{A}}$.

Let $\mu_2 = \{\pm 1\}$. Let us denote by $\tilde{G}$ the 2-fold metaplectic over of G when this is defined; if $G_1 \subset G$ is such that there exists a natural splitting of this covering then we denote by G_1^* the corresponding isomorphic copy of G_1 in $\tilde{G}$. We shall take those facts which we need concerning metaplectic groups from [10].

We recall here briefly the description of the metaplectic covers of GL_r over local fields and rings of adeles. These covers are characterised by their restrictions to the diagonal subgroup H. Let F be a local field and let $(\cdot,\cdot)_F$ be the 2-Hilbert symbol on F. Let $\mathrm{diag}(a_1,\ldots,a_r)$ be the diagonal matrix with a_i as the ii^{th} entry. The function $c: F^\times \times F^\times \to \{\pm 1\}$ given by

$$c(\mathrm{diag}(a_1,\ldots,a_r),\mathrm{diag}(a'_1,\ldots,a'_r)) = \prod_{i<j} (a_i, a'_j)_F$$

defines a 2-cocycle on H which extends to a 2-cocycle of $GL_r(F)$. The centre of the covering group is the lift of Z_F where

$$Z_F = \{zI \mid z \in F^\times\} \quad (r \text{ odd})$$

and

$$Z_F = \{zI \mid z \in F^{\times 2}\} \quad (r \text{ even}).$$

In the case of GL_3 we can identify $M(P)$ (resp. $M(Q)$) with $GL_2(F)$. Z_F by embedding GL_2 in the upper (resp. lower) 2×2 diagonal block. As the lift of Z_F is the centre of $GL_3(F)$ the lifts of the two factors commute. This carries over to the adelic case. The covering groups are therefore given by 2-cocycles and are so endowed with a section which we shall denote by s.

Finally we shall write π (or a similar letter) for a class of representations. If π is automorphic we shall write π_v for the local component at v. We shall often choose a representation space V of π; on this we shall write the action of G on V as left-multiplication; i.e. $G \times V \to V$; $(g,v) \mapsto gv$. We shall take an automorphic representation to be irreducible unless the contrary is stated.

2. Eisenstein Series

We shall be concerned here with Eisenstein series associated with representations of the two-fold covers of $P_{\mathbf{A}}$ and $Q_{\mathbf{A}}$ induced to that of $GL_3(k_{\mathbf{A}})$. Let $\tilde{V}$ be an automorphic representation of $\widetilde{GL}_2(k_{\mathbf{A}})$ and let $L: \tilde{V} \to \mathbf{C}$ be a $GL_2(k)^*$-invariant linear form. Let

$$\Lambda(f) = \int_{N_k^* \backslash N_{\mathbf{A}}^*} L(nf)\,dn, \qquad f \in V$$

be the corresponding $N_{\mathbf{A}}^*$-invariant linear form. This is identically zero if $(\tilde{V}, L)$ is cuspidal, otherwise not. Let χ be a quasicharacter of $\tilde{Z}_{\mathbf{A}}$ trivial on Z_k^* and such that $\chi \mid \mu_2$ is non-trivial. We shall also assume that μ_2 acts non-trivially on $\tilde{V}$; we say then that $\tilde{V}$ and χ are "genuine." Considering $\widetilde{GL}_2(k_{\mathbf{A}})\tilde{Z}_{\mathbf{A}}$ as $\tilde{M}(P)_{\mathbf{A}}$ or $\tilde{M}(Q)_{\mathbf{A}}$ as above we construct two new representations, which we denote by $\tilde{V}_P$ and $\tilde{V}_Q$, of $\tilde{M}(P)_{\mathbf{A}}$ and $\tilde{M}(Q)_{\mathbf{A}}$. The vector space is the original $\tilde{V}$, the action of $\widetilde{GL}_2(k_{\mathbf{A}})$ is the original one, the action of $\tilde{Z}_{\mathbf{A}}$ is by χ. The linear forms L, Λ yield linear forms on $\tilde{V}$ yield linear forms L_P, Λ_P (resp. L_Q, Λ_Q) on $\tilde{V}_P$ (resp. $\tilde{V}_Q$). Note that L_P (resp. L_Q) is $M(P)_k^*$- (resp. $M(Q)_k^*$)-invariant.

Let Ω_P (resp. Ω_Q) be the group of quasicharacters of $M(P)_{\mathbf{A}}$ (resp. $M(Q)_{\mathbf{A}}$) which are trivial on $M(P)_k Z_{\mathbf{A}}$) (resp. $M(Q)_k Z_{\mathbf{A}}$). Recall that Ω_P and Ω_Q have the structure of complex manifolds. For $\omega \in \Omega_P$ (resp. $\omega \in \Omega_Q$) the real number $\sigma(\omega)$ by $|\omega(x)| = \mu_P(x)^{\sigma(\omega)}$ (resp. $|\omega(x)| = \mu_Q(x)^{\sigma(\omega)}$). This is well-defined. For $\omega \in \Omega_P$ (resp. Ω_Q) we define $\tilde{\mathbf{V}}_P(\omega)=\tilde{V}_P\otimes(\omega\mu_P)$, $\tilde{\mathbf{V}}_Q(\omega)=\tilde{\mathbf{V}}_Q\otimes(\omega\mu_Q)$. We regard $\tilde{\mathbf{V}}_P(\omega)$ and $\tilde{\mathbf{V}}_Q(\omega)$ as the fibres of holomorphic vector bundles $\tilde{\mathbf{V}}_P$ and $\tilde{\mathbf{V}}_Q$ over Ω_P and Ω_Q respectively. We regard $\tilde{V}_P(\omega)$ and $\tilde{V}_Q(\omega)$ as representation spaces of $\tilde{P}_{\mathbf{A}}$ and $\tilde{Q}_{\mathbf{A}}$ on which $U(P)^*_{\mathbf{A}}$ and $U(Q)^*_{\mathbf{A}}$ act trivially. We define $\mathbf{F}_P(\omega)$ to be the space of functions $f:\widetilde{GL}_3(k_{\mathbf{A}})\to\tilde{V}_P(\omega)$ satisfying

$$f(\gamma g)=\gamma f(g),\quad \gamma\in\tilde{P}_A,\quad g\in\widetilde{GL}_3(k_A),$$

and which satisfies the usual smoothness conditions (cf. [10, II.1]). This is, by right multiplication a representation space of $\widetilde{GL}_3(k_{\mathbf{A}})$. We can construct $F_Q(\omega)$, $\omega\in\Omega_Q$ analogously. Note that for $f\in F_P(\omega)$ the map

$$f\mapsto L_P(f(g))$$

is a linear form left-invariant under $M(P)^*_k U(P)^*_A$. The spaces $\mathbf{F}_P(\omega)$ and $\mathbf{F}_Q(\omega)$ are the fibres of holomorphic vector bundles $\mathbf{F}_P$ and $\mathbf{F}_Q$ over Ω_P and Ω_Q. We represent sections $f\in\mathbf{F}_P(U)$, U an open subset of Ω_P, by $g\mapsto f(g,\omega)$, $g\in GL_3(k_{\mathbf{A}})$, $\omega\in U$ and analogously with Q instead of P.

There exists $c(\tilde{V})\in\mathbf{R}$ so that over the open sets $\{\omega\in\Omega_P|\sigma(\omega)>c(\tilde{V})\}$ and $\{\omega\in\Omega_Q|\sigma(\omega)>c(\tilde{V})\}$ we have maps

$$E_P:\mathbf{F}_P\to\mathcal{O}_P\quad\text{and}\quad E_Q:\mathbf{F}_Q\to\mathcal{O}_Q$$

where $\mathcal{O}_P$ (resp. $\mathcal{O}_Q$) is the structure sheaf of Ω_P (resp. Ω_Q). The maps E_P and E_Q (Eisenstein series) are defined by

$$E_P(f,\omega)=\sum_{\gamma\in P^*_k\backslash GL_3(k)^*}L_P(f(\gamma,\omega))$$

and

$$E_Q(f,\omega)=\sum_{\gamma\in Q^*_k\backslash GL_3(k)^*}L_Q(f(\gamma,\omega)).$$

Let $\mathcal{M}_P$ (resp. $\mathcal{M}_Q$) be the sheaf of meromorphic functions on Ω_P (resp. Ω_Q). Then E_P (resp. E_Q) can be continued to maps (over Ω_P (resp. Ω_Q))

$$E_P:\mathbf{F}_P\to\mathcal{M}_P,\qquad E_Q:\mathbf{F}_Q\to\mathcal{M}_Q.$$

This is one of the central results of the theory of Eisenstein series (see [11, p. 276]). One can also give a functional equation for the E_P, E_Q and describe the singularities. This we shall now discuss.

We fix an additive character e_0 of $k_{\mathbf{A}}$ which is non-trivial but trivial on k. We shall assume that measures on $k_{\mathbf{A}}$ (or k_v) are self-dual with respect to e_0 (resp. $e_{0,v}$). In particular the measure of $k_{\mathbf{A}}/k$ is 1.

We shall assume first that $\tilde{V}$ is cuspidal – we shall later assume that it is exceptional, that is, that $\tilde{V}$ is a cuspidal Weil representation (see [5, 3.3]; [6, 4.4]). Let

$$w_{123}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad w_{132}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

so that

$$w_{123}M(P)w_{123}^{-1}=M(Q), \qquad w_{132}M(Q)w_{132}^{-1}=M(P).$$

We define $\Omega_P\to\Omega_Q$; $\omega\mapsto\tilde{\omega}$ where

$$\tilde{\omega}(x)=\omega(w_{132}xw_{132}^{-1})$$

and analogously $\Omega_Q\to\Omega_P$; $\omega\mapsto\tilde{\omega}$. Note that $\tilde{\tilde{\omega}}=\omega$. Also $\sigma(\omega)+\sigma(\tilde{\omega})=0$. There are intertwining operators defined if $\sigma(\omega)>c(\tilde{V})$

$$I_{PQ}:\tilde{\mathbf{V}}_Q(\omega)\to\tilde{\mathbf{V}}_P(\omega);\ f\mapsto\left(g\mapsto\int_{U(Q)_{\mathbb{A}}^*} f(w_{132}ng,\omega)dn\right)$$

$$I_{QP}:\tilde{\mathbf{V}}_P(\omega)\to\tilde{\mathbf{V}}_Q(\omega);\ f\mapsto\left(g\mapsto\int_{U(P)_{\mathbb{A}}^*} f(w_{123}ng,\omega)dn\right)$$

and one has the following standard evaluations of the constant terms:

$$\int_{U(P)_k^*\backslash U(P)_{\mathbb{A}}^*} E_P(nf,\omega)dn=L_P(f(I,\omega)),$$

$$\int_{U(Q)_k^*\backslash U(Q)_{\mathbb{A}}^*} E_P(nf,\omega)dn=L_Q((I_{QP}f)(I,\omega)),$$

$$\int_{U(Q)_k^*\backslash U(Q)_{\mathbb{A}}^*} E_Q(nf,\omega)dn=L_Q(f(I,\omega))$$

and

$$\int_{U(P)_k^*\backslash U(P)_{\mathbb{A}}^*} E_Q(nf,\omega)dn=L_P((I_{PQ}f)(I,\omega)).$$

The study of Eisenstein series reduces to a large extent to the study of the I_{PQ} and I_{QP}. If $V\cong\hat{\otimes}V_v$ then the operators I_{PQ} and I_{QP} are determined by their local analogues. In particular this allows us to regularize the I_{PQ} and I_{QP}, as we shall now describe.

Although one can do this without restricting $\tilde{V}$ further we shall assume that $\tilde{V}$ is exceptional as well as being cuspidal. This means that there exists a Größencharakter α of $k_{\mathbb{A}}^*$ so that:

i) if $\alpha_v(-1)=1$ then $\tilde{V}_v\cong V_0(\alpha_v^*)$ with

$$\alpha^*(s(h^2))=\mu(h)\alpha_v(\det(h)) \qquad (h\in H_v)$$

where $V_0(\alpha^*)$ has the meaning of [10, I.2]. This means that $V_0(\alpha^*)$ is the irreducible quotient of a principal series representation (of $\bar{\varrho}(\alpha_v^{1/2}|\,|_v^{1/4},\ \alpha_v^{-1/2}|\,|_v^{1/4})$ in the notation of [5, Sect. 2]). The covariants of $V_0(\alpha^*)$ with respect to the lift of the upper triangular unipotent group is an irreducible $\tilde{H}_v$-module on which $s\{h^2\,|\,h\in H_v\}$ acts by $\alpha^*\mu^{-1}$. This suffices to identify $V_0(\alpha^*)$ with the r_α of [5, Sect. 2].

ii) there exist places v such that $\alpha_v(-1)=-1$; at such places $\tilde{V}_v$ is cuspidal if v is non-archimedean and square-integrale if v is archimedean. Here V_v is again r_α (see [5, Proposition 3.3.3]).

The construction of the global V is given in [5, Sect. 8].

$$W_{\Pi,v}(\omega_v)=\{f:\widetilde{GL}_3(k_v)\to\tilde{V}_v\,|\,f(\gamma g)=\omega_v(\gamma)\mu_{\Pi,v}(\gamma)\cdot(\gamma f(g)),$$
$$\gamma\in\tilde{\Pi}_v,\ g\in\widetilde{GL}_3(k_v),\ \text{and } f \text{ locally constant}\},$$

where $\Pi = P$ or Q. Then we have maps

$$I_{QP}: W_{P,v}(\omega_v) \to W_{Q,v}(\omega_v)$$

defined as the regularization of

$$(I_{QP}f)(g) = \int_{U(Q)_v^*} f(w_{132}ng)dn$$

and

$$I_{PQ}: W_{Q,v}(\omega_v) \to W_{P,v}(\omega_v)$$

defined as the regularization of

$$(I_{PQ}f)(g) = \int_{U(P)_v^*} f(w_{123}ng)dn.$$

These are the local factors of the global intertwining operators defined above. If $v_v^0 \in W_{\Pi,v}(\omega_v)$ is such that

$$v_v^0(gk) = v_v^0(g) \quad \text{for} \quad k \in K_v^*$$

where this is meaningful, and normalized by

$$v_v^0(I) = v_v^0$$

where v_v^0 is the standard K_v^*-invariant vector of $\tilde{V}_v$ (see [10, I.2]). This defines the family of vectors with respect to which the tensor product of the $W_{\Pi,v}(\omega_v)$ can be taken. One has then at an unramified place that ([10, Proposition I.2.4])

$$I_{QP}v_v^0 = \frac{L(\omega_v^6\alpha_v^3\chi_v^{-2}|\,|_v^{-1/2})}{L(\omega_v^6\alpha_v^3\chi_v^{-2}|\,|_v^{3/2})} v_v^0$$

and

$$I_{PQ}v_v^0 = \frac{L(\omega_v^6\alpha_v^{-3}\chi_v^2|\,|_v^{-1/2})}{L(\omega_v^6\alpha_v^{-3}\chi_v^2|\,|_v^{3/2})} v_v^0.$$

where

$$\chi_v^2(x) = \chi_v(\mathrm{s}(xI))$$

and

$$\omega_v(x) = \omega_v\begin{pmatrix}1 & 0 & 0\\ 0 & x & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \left(\text{resp. } \omega_v(x) = \omega_v\begin{pmatrix}1 & 0 & 0\\ 0 & x & 0\\ 0 & 0 & 1\end{pmatrix}^{-1}\right).$$

It follows from this and [10, Proposition I.2.3] by the usual techniques of the theory of Eisenstein series

$$L(\omega^6\alpha^3\chi^{-2}\|\,\|_{\mathbf{A}}^{3/2})E_P(f,\omega)$$

and

$$L(\omega^6\alpha^{-3}\chi^2\|\,\|_{\mathbf{A}}^{3/2})E_Q(f,\omega)$$

are holomorphic in ω. Moreover one has the functional equation

$$\begin{aligned}&L(\omega^6\alpha^{-3}\chi^2\|\,\|_{\mathbf{A}}^{3/2})E_Q(f,\omega)\\ &\quad = \varepsilon(\omega^6\alpha^{-3}\chi^2\|\,\|_{\mathbf{A}}^{-1/2})L(\omega^{-6}\alpha^3\chi^{-2}\|\,\|_{\mathbf{A}}^{3/2})E_P\left(\frac{L(\omega^6\alpha^{-3}\chi^2\|\,\|_{\mathbf{A}}^{3/2})}{L(\omega^6\alpha^{-3}\chi^2\|\,\|_{\mathbf{A}}^{-1/2})} I_{PQ}f, \tilde{\omega}\right)\end{aligned}$$

and an analogous one with P and Q interchanged. Here L and ε are the usual Tate functions; see [11, pp. 110, 111]. Note that if $f = \otimes f_v$, $f_v = v_v^0$ almost everywhere, then

$$\frac{L(\omega^6\alpha^{-3}\chi^2\| \|_{\mathbf{A}}^{3/2})}{L(\omega^6\alpha^{-3}\chi^2\| \|_{\mathbf{A}}^{1/2})} I_{PQ}f = \bigotimes_v \left\{ \frac{L(\omega_v^6\alpha_v^{-3}\chi_v^2| |_v^{3/2})}{L(\omega_v^6\alpha_v^{-3}\chi_v^2| |_v^{-1/2})} I_{PQ}f_v \right\}$$

and almost all factors here are equal to v_v^0.

This suffices for the discussion of the case where $\tilde{V}$ is cuspidal. The case where $\tilde{V}$ is an exceptional non-cuspidal representation is similar although a little more complicated. In this case we see that

$$L(\omega^6\alpha^3\chi^{-2}\| \|_{\mathbf{A}}^{3/2})E_P(f,\omega)$$

and

$$L(\omega^6\alpha^{-3}\chi^2\| \|_{\mathbf{A}}^{3/2})E_Q(f,\omega)$$

have at most simple poles where

$$\omega^6\alpha^3\chi^{-2} = \| \|_{\mathbf{A}}^{\pm 3/2} \quad \text{resp. } \omega^6\alpha^{-3}\chi^2 = \| \|_{\mathbf{A}}^{\pm 3/2}.$$

One has the same functional equation as before. These assertions follow from the general results on Eisenstein series [10, II.1]; [11, p. 278].

3. The Rankin-Selberg Integral

In this section we shall prove the central global result needed for our investigation. It is a formula of Rankin-Selberg type which we now formulate.

Let W be an irreducible cuspidal representation of $GL_3(k_{\mathbf{A}})$ and let $L_W: W \to \mathbf{C}$ be a non-trivial $GL_3(k)$-invariant linear form. Let θ be an irreducible automorphic representation of $\widetilde{GL}_3(k_{\mathbf{A}})$ with genuine central quasicharacter $\tilde{\chi}$; this is to be an exceptional representation of the type of [10, Theorem II.2.1]. Let $L_\theta: \theta \to \mathbf{C}$ be a non-trivial $GL_3(k)^*$-invariant linear form. We recall that it follows from [10, Theorem I.3.5] that if e is a non-degenerate character of $N_{\mathbf{A}}^*$ trivial on N_k^* then

$$\int_{N_k^* \backslash N_{\mathbf{A}}^*} L_\theta(nv)\bar{e}(n)dn = 0, \qquad v \in \theta.$$

Let χ be the central quasicharacter of W. Let V be an irreducible exceptional automorphic representation of $\widetilde{GL}_2(k_{\mathbf{A}})$ (possibly cuspidal) and extend it to $\tilde{P}_{\mathbf{A}}$ and $\tilde{Q}_{\mathbf{A}}$ with central quasicharacter $(\chi\tilde{\chi})^{-1}$. Let $L: V \to \mathbf{C}$ be a non trivial $GL_2(k)^*$-invariant linear form. We form the corresponding series $E_P(f,\omega)$ and $E_Q(f,\omega)$ as in Sect. 2. We shall now prove:

Proposition 3.1. *With the notations above one has for ω with $\sigma(\omega)$ large enough, $w \in W$, $t \in \theta$, $f \in \mathbf{F}_P(\omega)$, $f' \in \mathbf{F}_Q(\omega)$ the integrals*

$$\int_{GL_3(k)\backslash GL_3(k_{\mathbf{A}})/Z_{\mathbf{A}}} L_W(gw)L_\theta(gt)E_P(gf,\omega)dg$$

and

$$\int_{GL_3(k)\backslash GL_3(k_{\mathbf{A}})/Z_{\mathbf{A}}} L_W(gw)L_\theta(gt)E_Q(gf',\omega)dg$$

converge absolutely. Define for a non-degenerate character e of $N_{\mathbf{A}}$ trivial on N_k the following functionals

$$\Lambda_W(\omega)=\int_{N_k\backslash N_{\mathbf{A}}} L_W(nw)\bar{e}(n)dn\,,\qquad w\in W$$

$$\Lambda_\theta^{01}(t)=\int_{U(P)_k\backslash U(P)_{\mathbf{A}}} L_\theta(nt)e(n)dn\,,\qquad t\in\theta$$

$$\Lambda_\theta^{10}(t)=\int_{U(Q)_k\backslash U(Q)_{\mathbf{A}}} L_\theta(nt)e(n)dn\,,\qquad t\in\theta$$

$$\Lambda_P(f(g,\omega))=\int_{(N\cap M(P))_k\backslash(N\cap M(P))_{\mathbf{A}}} L(nf(g,\omega))e(n)dn\,,\qquad (f\in\mathbf{F}_P(\omega))$$

$$\Lambda_Q(f(g,\omega))=\int_{(N\cap M(Q))_k\backslash(N\cap M(Q))_{\mathbf{A}}} L(nf(g,\omega))dn\,,\qquad f\in\mathbf{F}_Q(\omega)\,.$$

Then Λ_θ^{01} (resp. Λ_θ^{10}) is $U(Q)_{\mathbf{A}}$- (resp. $U(P)_{\mathbf{A}}$-) invariant. One has

$$\int_{GL_3(k)\backslash GL_3(k_{\mathbf{A}})/Z_{\mathbf{A}}} L_W(gW)L_\theta(gt)E_P(gf,\omega)dg$$
$$=\int_{N_{\mathbf{A}}Z_{\mathbf{A}}\backslash GL_3(k_{\mathbf{A}})} \Lambda_W(gw)\Lambda_\theta^{01}(gt)\Lambda_P(f(g,\omega))dg$$

and

$$\int_{GL_3(k)\backslash GL_3(k_{\mathbf{A}})/Z_{\mathbf{A}}} L_W(gw)L_\theta(gt)E_Q(gf',\omega)dg$$
$$=\int_{N_{\mathbf{A}}Z_{\mathbf{A}}\backslash GL_3(k_{\mathbf{A}})} \Lambda_W(gw)\Lambda_\theta^{10}(gt)\Lambda_Q(f'(g,\omega))dg\,.$$

Proof. We shall deal only with the integral involving P; the one involving Q can be treated analogously. We shall first consider the integrals formally leacing aside questions of convergence.

The usual Rankin-Selberg transformation shows that

$$\int_{GL_3(k)\backslash GL_3(k_{\mathbf{A}})/Z_{\mathbf{A}}} L_W(gw)L_\theta(gt)E_P(gf,\omega)dg$$

is equal to

$$\int_{P_k\backslash GL_3(k_{\mathbf{A}})/Z} L_W(gw)L_\theta(gt)L(f(g,\omega))dg\,.$$

Now we have the Fourier expansion

$$L_W(gw)=\sum_{p\in N_kZ_k\backslash P_k} \Lambda_W(pgw)$$

since W is cuspidal. Using this the integral becomes

$$\int_{N_kZ_{\mathbf{A}}\backslash GL_3(k_{\mathbf{A}})} \Lambda_W(gw)L_\theta(gt)L(f(g,\omega))dg$$
$$=\int_{N_{\mathbf{A}}Z_{\mathbf{A}}\backslash GL_3(k_{\mathbf{A}})}\int_{N_k\backslash N_{\mathbf{A}}} \Lambda_W(gw)L_\theta(ngt)L(f(ng,\omega))e(n)dndg\,.$$

The inner integral can be written as

$$\int_{(N\cap M(P))_k\backslash(N\cap M(P))_{\mathbf{A}}}\int_{U(P)_k\backslash U(P)_{\mathbf{A}}} \Lambda_W(gw)L_\theta(n_1n_2gt)L(f(n_2g,\omega))e(n_1)e(n_2)dn_1dn_2\,.$$

Since θ is exceptional the $(N\cap M(P))_k$-invariant function

$$n_2 \mapsto \int_{U(P)_k\backslash U(P)_{\mathbf{A}}} L_\theta(n_1 n_2 g t) e(n_1) dn$$

is constant (otherwise there would be a non-trivial non-degenerate Fourier coefficient); thus our integral becomes

$$\Lambda_W(gw) \int_{U(P)_k\backslash U(P)_{\mathbf{A}}} L_\theta(n_1 g t) e(n_1) dn_1 \int_{(N\cap M(P))_k\backslash(N\cap M(P))_{\mathbf{A}}} L(n_2 f(g,\omega)) e(n_2) dn_2$$

which is

$$\Lambda_W(gw)\Lambda_\theta^{01}(gt)\Lambda_P(f(g,\omega)).$$

This yields the equality asserted in the proposition. The convergence of the latter integral follows from [7, Sect. 2] and the intermediate ones follow from this.

From Proposition 3.1 we deduce now:

Proposition 3.2. *The functions*

$$Z_P(\omega; w, t, f) = L(\omega^6\alpha^3\chi^2\tilde{\chi}^2\|\ \|_{\mathbf{A}}^{3/2}) \int_{GL_3(k)\backslash GL_3(k_{\mathbf{A}})/Z_{\mathbf{A}}} L_W(gw) L_\theta(gt) E_P(gf, \omega) dg$$

and

$$Z_Q(\omega; w, t, f') = L(\omega^6\alpha^{-3}\chi^{-2}\tilde{\chi}^{-2}\|\ \|_{\mathbf{A}}^{3/2}) \int_{GL_3(k)\backslash GL_3(k_{\mathbf{A}})/Z_{\mathbf{A}}} L_W(gw) L_\theta(gt) E_Q(gf', \omega) dg$$

where α is derived from V as in Sect. 2 have analytic continuations as meromorphic functions to Ω_P and Ω_Q respectively. If V is cuspidal then Z_P and Z_Q are holomorphic; if V is not cuspidal then Z_P has at most simple poles where

$$\omega^6\alpha^3\chi^2\tilde{\chi}^2 = \|\ \|_{\mathbf{A}}^{\pm 3/2};$$

likewise Z_Q has at most simple poles where

$$\omega^6\alpha^{-3}\chi^{-2}\tilde{\chi}^{-2} = \|\ \|_{\mathbf{A}}^{\pm 3/2}.$$

One has the functional equation:

$$Z_Q(\omega; w, t, f') = \varepsilon(\omega^6\alpha^{-3}\chi^{-2}\tilde{\chi}^{-2}\|\ \|_{\mathbf{A}}^{-1/2}) Z_P(\omega; w, t, \tilde{I}_{PQ} f')$$

where

$$\tilde{I}_{PQ} = \frac{L(\omega^6\alpha^{-3}\chi^{-2}\tilde{\chi}^{-2}\|\ \|_{\mathbf{A}}^{3/2})}{L(\omega^6\alpha^{-3}\chi^{-2}\tilde{\chi}^{-2}\|\ \|_{\mathbf{A}}^{1/2})} I_{PQ}.$$

This follows from Proposition 3.1 and the results recalled in Sect. 2.

We shall next derive alternative expressions for Z_P and Z_Q as Euler products. For this we need some preparations.

Recall that if we represent W as $\otimes W_v$ then each W_v has a unique Whittaker model; thus we can represent $\Lambda_w(\otimes w_v)$ as $\prod_v \Lambda_{w,v}(w_v)$ where $\Lambda_{w,v}$ is a Whittaker functional of W_v and the product is over all places of k.

Consider next the function $P_{\mathbf{A}} \to \mathbf{C}$

$$g \mapsto \int_{U(P)_k\backslash U(P)_{\mathbf{A}}} L_\theta(ngt) dn.$$

This is P_k^*-invariant and is an automorphic form belonging to a automorphic representation θ of $P_{\mathbf{A}}$. As θ is given as the residue of Eisenstein series [10, Theorem II.2.1] it is immediate that θ_P is too. It follows from studying the constant term that θ_P is irreducible. By [10, Theorem II.2.5] each local factor has a unique Whittaker model and the global Whittaker functional is non-trivial as the representation is genuine. Thus if $\theta = \otimes\theta_v$ and $t = \otimes t_v$ it follows that

$$\int_{(N\cap M(P))_k\backslash(N\cap M(P))_{\mathbf{A}}} \int_{U(P)_k\backslash U(P)_{\mathbf{A}}} L_\theta(ungt)du\, e(n)dn$$

can be factorized as $\prod_v \Lambda_{\theta,v}^{10}(g_v t_v)$. The functional $\Lambda_{\theta,v}^{10}$ is determined up to a scalar multiple by its transformation property under N_v.

As V is exceptional we can represent V as $\otimes V_v$. One has again that

$$\int_{(N\cap M(P))_k\backslash(N\cap M(P))_{\mathbf{A}}} L(nf)e(n)dn$$

with

$$f = \otimes f_v$$

can be expressed as $\prod_v \Lambda_v(f_v)$. Note that $f \in \mathbf{F}_P(\omega)$ can, as usual, be represented as a finite sum of such primitive elements and that at almost all places the factor f_v is $v_v^0(\omega)$. One demands that $\Lambda_v(v_v^0(\omega)) = 1$ for almost all v.

One now has:

Proposition 3.3. *With the notations above*

$$Z_P(\omega;\, \otimes w_v, \otimes t_v, \otimes f_v)$$

is equal to

$$\prod_v \left(L_v(\omega^6\alpha^3\chi^2\tilde{\chi}^2 \|\ \|_{\mathbf{A}}^{3/2}) \int_{N_v Z_v\backslash GL_3(k_v)} \Lambda_{w,v}(gw_v)\Lambda_{\theta,v}^{01}(gt_v)\cdot \Lambda_{P,v}(f_v(g,\omega))dg\right)$$

where $L_v(\phi)$ is the v^{th} factor of $L(\phi)$, and $\Lambda_{P,v}$ is the extension of Λ_v to $\mathbf{F}_{P,v}(\omega_v)$. Likewise

$$Z_Q(\omega;\, \otimes w_v, \otimes t_v, \otimes f_v)$$

is equal to

$$\prod_v \left(L_v(\omega^6\alpha^{-3}\chi^{-2}\tilde{\chi}^{-2} \|\ \|_{\mathbf{A}}^{1/2}) \int_{N_v Z_v\backslash GL_3(k_v)} \Lambda_{w,v}(gw_v)\cdot \Lambda_{\theta,v}^{10}(gt_v)\Lambda_{Q,v}(f_v(g,\omega))dg\right)$$

where $\Lambda_{Q,v}$ is the extension of Λ_v to $\mathbf{F}_{Q,v}(\omega_v)$.

This is immediate from Proposition 3.2.

4. The Generic Case

In Sect. 3 we have shown how the functionals $Z_P(\omega;\, w, t, f)$ and $Z_Q(\omega;\, w, t, f)$ can be represented as Euler products. In this section we shall develop the local theory at a generic place. We shall consider a non-archimedean local field F with odd residual characteristic. Let $V(\omega)$ be a principal series representation of $GL_3(F)$ with unramified quasicharacter ω. Let w_0 be a K-invariant vector of $V(\omega)$ where K is the standard maximal compact subgroup of $GL_3(F)$. Let $e: N(F)\to\mathbf{C}$ be an unramified non-degenerate character. Let $\Lambda: V(\omega)\to\mathbf{C}$ be the Whittaker functional for e such that $\Lambda(w_0)=1$.

Next let $V_0(\tilde{\omega})$ be an exceptional representation of $\widetilde{GL}_3(F)$ in the sense of [10, II.1]. Suppose that $\tilde{\omega}$ is unramified so that $V_0(\tilde{\omega})$ has a K^*-invariant t_0. Let e_P (resp. e_Q) be the character of $N(F)$ so that

$$e_\Pi | N(F) \cap M(\Pi)(F) = 1, \quad e_\Pi | U(\Pi)(F) = e | U(\Pi)(F)$$

where $\Pi = P$ or Q. Let $\tilde{\Lambda}_P : V_0(\tilde{\omega}) \to \mathbf{C}$ be the unique linear form so that

$$\tilde{\Lambda}_P(nv) = \bar{e}_P(n)\tilde{\Lambda}_P(v) \qquad v \in V_0(\tilde{\omega}), n \in N(F)^*$$

and

$$\tilde{\Lambda}_P(t_0) = 1 .$$

That is the first condition defines a one-dimensional space of functionals follows from [10, Theorem I.3.6] since $\tilde{\Lambda}_P$ factors through the Jacquet map. That the normalization is possible then follows from [10, Theorem I.4.2]. Henceforth we shall write N for $N(F)$ etc.

Analogously to $\tilde{\Lambda}_P$ we can define Λ_Q. Next let $V_0(\Omega)$ be an exceptional representation of $GL_2(F)$ where Ω is again unramified. Let $v_0 \neq 0$ be a K^*-invariant vector in $V_0(\omega)$, where K now refers to the standard maximal compact subgroup of $GL_2(F)$. We define the character χ of the centre of $GL_3(F)$ to be $(\omega | \tilde{Z}(F))^{-1} \cdot (\tilde{\omega} | \tilde{Z}(F))^{-1}$. Then we can regard $V_0(\Omega)$ as a $\tilde{P}$ or a $\tilde{Q}$ representation with central quasicharacter χ. Define the function $f_0^P : \widetilde{GL}_3(F) \to V_0(\Omega)$ by

$$f_0^P(gug') = g f_0^P(g')\mu_P(g) \qquad g' \in \widetilde{GL}_3(F), g \in \tilde{P}, u \in U(P)^* ,$$

and

$$f_0^P(k) = v_0 .$$

We can define f_0^Q analogously.

There is a unique Whittaker functional on $V_0(\Omega)$ with respect to $e | (N \cap N(Q))$, taking the value 1 on v_0. Denote this by Λ_P (resp. Λ_Q).

Our main result is then:

Proposition 4.1. *If* $\Omega\left(\mathbf{s}\begin{pmatrix}\pi^2 & 0\\ 0 & 1\end{pmatrix}\right)$ *is small enough* (π *being a uniformizer of* F) *then*

$$\int_{NZ \backslash \widetilde{GL}_3(F)} \Lambda(g w_0)\tilde{\Lambda}_P(g t_0)\Lambda_P(f_0^P(g))dg$$

converges. We suppose the measure on $NZ\backslash GL_3(F)$ *to be right-invariant measure giving the open subset* $NZ\backslash NZK$ *measure* 1. *Let*

$$\omega_1 = \omega\begin{pmatrix}\pi & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \omega_2 = \omega\begin{pmatrix}1 & 0 & 0\\ 0 & \pi & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \omega_3 = \omega\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \pi\end{pmatrix}$$

be the Satake parameters of $V(\omega)$. *Let*

$$X = \tilde{\omega}\left(\mathbf{s}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \pi^2\end{bmatrix}\right) \cdot \Omega\left(\mathbf{s}\begin{pmatrix}\pi^2 & 0\\ 0 & 1\end{pmatrix}\right)|\pi| .$$

Then the integral is equal to

$$(1-(\omega_1\omega_2\omega_3)^2X^3)/(1-\omega_1^2X)(1-\omega_2^2X)(1-\omega_3^2X)(1-\omega_1\omega_2X)\\ \times(1-\omega_2\omega_3X)(1-\omega_3\omega_1X).$$

One can give a very similar evaluation of the corresponding integral with Q instead of P but we shall not formulate this explicitly.

Proof. The convergence of the integral will become clear during the proof. By splitting the integral into a sum over right K-cosets we see that it is equal to

$$\sum_{\eta\in H/Z(H\cap K)} \Lambda(\eta\omega_0)\tilde{\Lambda}_P(\eta t_0)\Lambda_P(f_0^P(\eta))\mu(\eta)^{-2}$$

where H is the diagonal subgroup of $GL_3(F)$. There are explicit formulae for each of the terms here, as we shall now explain; the proposition will then follow from carrying out the summation. We shall take η to be of the form

$$\begin{pmatrix} \pi^{f_1} & 0 & 0 \\ 0 & \pi^{f_2} & 0 \\ 0 & 0 & \pi^{f_3} \end{pmatrix}.$$

As the summand is zero unless $f_1 \geqq f_2 \geqq f_3$ we shall assume this henceforth. By Shintani's theorem, [18], we have

$$\Lambda(\eta\omega_0)\mu(\eta)^{-1} = \det\begin{pmatrix} \omega_1^{f_1+2} & \omega_1^{f_2+1} & \omega_1^{f_3} \\ \omega_2^{f_1+2} & \omega_2^{f_2+1} & \omega_2^{f_3} \\ \omega_3^{f_1+2} & \omega_3^{f_2} & \omega_3^{f_3} \end{pmatrix}/(\omega_1-\omega_2)(\omega_2-\omega_3)(\omega_1-\omega_3).$$

Next $\tilde{\Lambda}_P$ factors through the Jacquet map for $\tilde{Q}$ and so we see that

$$\begin{aligned} \mu(\eta)^{-1}\Lambda_P(\eta t_0) &= \tilde{\omega}\left(\mathbf{s}\begin{bmatrix} \pi^{f_3} & 0 & 0 \\ 0 & \pi^{f_2} & 0 \\ 0 & 0 & \pi^{f_1} \end{bmatrix}\right) && \text{if} \quad f_2 \equiv f_3 \pmod 2 \\ &= 0 && \text{otherwise}. \end{aligned}$$

This follows from [10, Theorem I.4.2].

The same result shows that

$$\begin{aligned} \Lambda_P(f_0^P(\eta))\mu(\eta)^{-1} &= \chi(\mathbf{s}(\pi^{f_3}I))\cdot\Omega\left(\mathbf{s}\begin{pmatrix} \pi^{f_2-f_3} & 0 \\ 0 & \pi^{f_1-f_3} \end{pmatrix}\right) && \text{if} \quad f_1 \equiv f_2 \pmod 2 \\ &= 0 && \text{otherwise}. \end{aligned}$$

Next observe that as ω is exceptional

$$\tilde{\omega}\left(\mathbf{s}\begin{bmatrix} \pi^{f_3} & 0 & 0 \\ 0 & \pi^{f_2} & 0 \\ 0 & 0 & \pi^{f_1} \end{bmatrix}\right) = |\pi|^{f_3+f_2/2}\tilde{\omega}\left(\mathbf{s}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi^2 \end{bmatrix}\right)^{(f_1+f_2+f_3)/2}$$

and as Ω is exceptional

$$\Omega\left(\mathbf{s}\begin{pmatrix}\pi^{f_2-f_3} & 0\\ 0 & \pi^{f_1-f_3}\end{pmatrix}\right)=|\pi|^{-(f_1-f_2)/2}\,\Omega\left(\mathbf{s}\begin{pmatrix}\pi^2 & 0\\ 0 & 1\end{pmatrix}\right)^{(f_1+f_2-2f_3)/2}$$

Let us take $f_2=0$ to fix the representative modulo Z.

The integral then becomes

$$\sum_{\substack{f_1\geqq f_2\geqq 0\\ f_1,f_2\equiv 0(\mathrm{mod.}\,2)}} X^{(f_1+f_2)/2}\det\begin{pmatrix}\omega_1^{f_1+2} & \omega_1^{f_2+1} & 1\\ \omega_2^{f_1+2} & \omega_2^{f_2+1} & 1\\ \omega_3^{f_1+2} & \omega_3^{f_2+1} & 1\end{pmatrix}\cdot\Delta^{-1}$$

where $\Delta=(\omega_1-\omega_2)(\omega_2-\omega_3)(\omega_1-\omega_3)$. This we can write formally as

$$\sum_{\substack{f_1\geqq f_2\geqq 0\\ f_1,f_2\equiv 0(\mathrm{mod.}\,2)}} det\begin{pmatrix}(X^{1/2}\omega_1)^{f_1}\cdot\omega_1^2 & (X^{1/2}\omega_1)^{f_2}\cdot\omega_1 & 1\\ (X^{1/2}\omega_2)^{f_1}\cdot\omega_2^2 & (X^{1/2}\omega_2)^{f_2}\cdot\omega_2 & 1\\ (X^{1/2}\omega_3)^{f_1}\cdot\omega_3^2 & (X^{1/2}\omega_3)^{f_3}\cdot\omega_3 & 1\end{pmatrix}\Delta^{-1}$$

The summation over f_1 can be carried out; it yields

$$\sum_{\substack{f_2\geqq 0\\ f_2\equiv 0(\mathrm{mod.}\,2)}} \det\begin{pmatrix}\omega_1^2(X^{1/2}\omega_1)^{f_2}/(1-X\omega_1^2) & \omega_1(X^{1/2}\omega_1)^{f_2} & 1\\ \omega_2^2(X^{1/2}\omega_2)^{f_2}/(1-X\omega_2^2) & \omega_2(X^{1/2}\omega_2)^{f_2} & 1\\ \omega_3^2(X^{1/2}\omega_3)^{f_2}/(1-X\omega_3^2) & \omega_3(X^{1/2}\omega_3)^{f_2} & 1\end{pmatrix}\Delta^{-1}.$$

On multiplying this out and evaluating the sum over f_2 we see this is equal to Δ^{-1} times the alternating sum over all permutations of ω_1, ω_2, ω_3 of

$$\omega_1^2\omega_2/(1-X\omega_1^2)(1-X^2\omega_1^2\omega_2^2).$$

It is now an exercise involving the invariant theory of the symmetric group of order 6 to simplify this sum. After a rather long calculation one finds the result quoted in the statement of the proposition.

This can now be applied to the Z_P and Z_Q. We find:

Corollary 4.2. *Let, in the notations of Sect. 3, v be a place so that*

i) *v is non-archimedean with odd residual characteristic,*

ii) *the character e is unramified at v,*

iii) *ω_v is unramified.*

then the v^{th} favtor of $Z_P(\omega;\otimes w_u,\otimes t_u,\otimes f_u)$ with w_v, t_v, f_v the standard unramified vectors is equal to

$$\prod_{1\leqq i\leqq j\leqq 3}(1-\omega_{i,v}\omega_{j,v}\omega_v(\pi_v)^2\alpha_v(\pi_v)\tilde{\omega}_{2,v}^2(\pi_v)\,|\pi_v|_v^{1/2})^{-1}$$

where ω_2^2 is the Größencharacter

$$\tilde{\omega}_2^2(x)=\tilde{\omega}\left(\mathbf{s}\begin{bmatrix}1 & 0 & 0\\ 0 & x^2 & 0\\ 0 & 0 & 1\end{bmatrix}\right)$$

and $\omega_{1,v}$, $\omega_{2,v}$, $\omega_{3,v}$ *are the Satake parameters of* W_v. *Also the* v^{th} *factor of* $Z_Q(\omega; \otimes w_u, \otimes t_u, \otimes f_u)$ *under the same assumptions is*

$$\prod_{1 \leqq i \leqq j < 3} (1-(\omega_{i,v}\omega_{j,v})^{-1} \cdot \omega_v(\pi_v)^2 \cdot \alpha_v(\pi_v)^{-1} \tilde{\omega}_{2,v}^2(\pi_v)^{-1} |\pi_v|_v^{1/2})^{-1}$$

Proof. The formulae here are direct consequences of Proposition 4.1 when one compares the corresponding notations. The one point which one needs to verify is that

$$\Omega_v \begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix} = |\pi_v|_v^{1/2} \omega_v(x^2) \alpha_v(x).$$

In this we are comparing an exceptional representation as constructed in [10, I.2] with a Weil representation. The equation above follows on comparing the Whittaker models, i.e. [10, Theorem I.4.2] with [5, Proposition 2.3.3].

Before leaving this result we remark that

$$\tilde{\chi}^2(x) = \tilde{\omega}_2^2(x)^3 \varepsilon(-1, x)_{\mathbf{A}}$$

where $(\,,\,)_{\mathbf{A}}$ is the adelic Hilbert symbol of order 2 and $\varepsilon: \mu_2(k) \to \mathbf{C}^\times$ is the unique injective homomorphism.

5. The Local Functional Equation

The results we have already proved allow us to prove the local functional equation for our situation. Let F be a local field of characteristic $\neq 2$. Let W be a irreducible admissible representation of $GL_3(F)$. Let $e: N \to \mathbf{C}$ be a non-degenerate character and let $\Lambda: W \to \mathbf{C}$ be a non-trivial Whittaker functional. Let θ be an exceptional representation of $\widetilde{GL}_3(F)$ as constructed in [10, I.2]. Let $\Lambda^{01}: \theta \to \mathbf{C}$ be a linear form so that $\Lambda^{01}(nt) = \bar{e}_P(n) \Lambda^{01}(t)\,(n \in N)$ where $e_P: N \to \mathbf{C}$ is that character for which $e_P | M(P) \cap N = 1$ and $e_P | U(P) = e | U(P)$. Likewise we choose Λ^{10} so that $\Lambda^{10}(nt) = \bar{e}_Q(n) \Lambda^{10}(t)$ and e_Q is defined as e_P but with Q replacing P throughout.

Let $\tilde{V}$ be an exceptional, possibly cuspidal, representation of $GL_2(F)$. Then let χ (resp. $\tilde{\chi}$) be the central quasicharacters of W (resp. θ). Let us extend $\tilde{V}$ to $\tilde{M}(P)$ (resp. $\tilde{M}(Q)$) by requiring that $\tilde{Z}$ acts through $(\chi\tilde{\chi})^{-1}$. Let Ω_P (resp. Ω_Q) be the complex manifold of quasicharacters of $M(P)$ (resp. $M(Q)$) trivial on Z. Let for $\omega \in \Omega_P$ (resp. $\omega \in \Omega_Q$) $\mathbf{F}_P(\omega)$ (resp. $\mathbf{F}_Q(\omega)$) be the space of locally constant functions $f(\gamma u g) = \omega(\gamma) \mu_P(\gamma) f(g)$, $\gamma \in \tilde{M}(P)$, $u \in U(P)$, $g \in \widetilde{GL}_3(F)$ (resp. $f(\gamma u g) = \omega(\gamma) \mu_Q(\gamma) f(g)$, $\gamma \in \tilde{M}(Q)$, $u \in U(Q)$, $g \in \widetilde{GL}_3(F)$). Then $\mathbf{F}_P$ and $\mathbf{F}_Q$ are holomorphic vector bundles over Ω_P and Ω_Q respectively.

Let us realize $\tilde{V}$ as r_α as in [5, Sect. 1]. Then we can construct intertwining operators $I_{PQ}: F_Q(\omega) \to F_P(\omega)$ and $I_{QP}: F_P(\omega) \to F_Q(\omega)$ where ω has the meaning ascribed to it in Sect. 3. Let us also write for $\omega \in \Omega_P$ (resp. $\omega \in \Omega_Q$)

$$\omega(x) = \omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \left(\text{resp. } \omega(x) = \omega \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \right) \right).$$

Then $(I_{QP}f)(g)$ is the regularized value of

$$\frac{L(\omega^6\alpha^{-3}(\chi\tilde{\chi})^{-2}|\,|_F^{3/2})}{L(\omega^6\alpha^{-3}(\chi\tilde{\chi})^{-2}|\,|_F^{-1/2})}\int_{U(Q)^*} f(w_{132}ng)dn.$$

Let $\tilde{\Lambda}_P: \tilde{V}\to\mathbf{C}$ be a non-trivial Whittaker functional for $\tilde{V}$ considered as a $\tilde{M}(P)$-representation with respect to $\bar{e}|M(P)$; we choose $\tilde{\Lambda}_Q$ analogously. Let for $w\in W$, $t\in\theta$, $f\in\Gamma_m(\mathbf{F}_P)$, $f'\in\Gamma_m(\mathbf{F}_Q)$

$$Z_P(\omega; w, t, f) = L(\omega^6\alpha^3\chi^2\tilde{\chi}^2|\,|_F^{3/2})\int_{NZ\backslash GL_3(F)} \Lambda(gw)\Lambda^{01}(gt)\tilde{\Lambda}_P(f(g,\omega))dg$$

and

$$Z_Q(\omega; w, t, f') = L(\omega^6\alpha^{-3}\chi^{-2}\tilde{\chi}^{-2}|\,|_F^{3/2})\int_{NZ\backslash GL_3(F)} \Lambda(gw)\Lambda^{10}(gt)\tilde{\Lambda}_Q(f'(g,\omega))dg.$$

Here if F is archimedean $\Gamma_m(\mathbf{F}_P)$ (resp. $\Gamma_m(\mathbf{F}_Q)$) denotes the space of meromorphic sections of $\mathbf{F}_P$ (resp. $\mathbf{F}_Q$). If F is non-archimedean then $\Gamma_m(\mathbf{F}_P)$ (resp. $\Gamma_m(\mathbf{F}_Q)$) denotes the space of rational sections of $\mathbf{F}_P$ (resp. $\mathbf{F}_Q$), where "rational" has the usual meaning in this context. Likewise we let $\Gamma(\mathcal{O},\Omega_P))$ (resp. $\Gamma(\mathcal{O},\Omega_Q)$) be the ring of holomorphic functions (if F is nonarchimedean) on Ω_P (resp. Ω_Q). Let $\Gamma_m(\mathcal{O},\Omega_P)$ and $\Gamma_m(\mathcal{O},\Omega_Q)$ denote the corresponding rings of meromorphic or "rational" functions.

Let us call an admissible irreducible representation W of $GL_3(F)$ *relevant* if there exists a global field k, a place v of k with $k_v\cong F$ and an irreducible cuspidal representation π of $GL_3(k_{\mathbf{A}})$ with $\pi_v\cong W$. Only relevant representations play a role in any global applications which may arise. Note that a square-integrable representation is relevant by [2].

Theorem 5.1. *Let F be a local field and let W be a relevant representation of $GL_3(F)$. Let θ, $\tilde{V}$ be as above. Then one has:*

i) *The integrals defining $Z_P(\omega; w, t, f)$ and $Z_Q(\omega; w, t, f')$ converge if $\sigma(\omega)$ is large enough and represent elements of $\Gamma_m(\mathcal{O},\Omega_P)$ and $\Gamma_m(\mathcal{O},\Omega_Q)$ respectively.*

ii) *There exists $\gamma(W,\theta,\tilde{V})\in\Gamma_m(\mathcal{O},\Omega_P)$ so that*

$$Z_P(\omega; w, t, f) = \gamma(\omega; W, \theta, \tilde{V})Z_Q(\tilde{\omega}; w, t, I_{QP}f).$$

iii) *There exists $\phi\in\Gamma(\mathcal{O},\Omega_P)^\times$ so that*

$$\tilde{I}_{PQ}\tilde{I}_{QP} = \phi\cdot Id.$$

Remarks. 1. Note that this does not allow us to compute the $\gamma(\omega; W,\theta, V)$; the theorem merely asserts their existence. The proof of the theorem will suggest the correct form of the $\gamma(\omega; W,\theta, V)$. Indeed it may be possible to actually prove this using an asymptotic analysis of Z_P and Z_Q (cf. [8, Sect. 5]) as ω becomes highly ramified. This we do not undertake here.

2. It is unfortunate that we have no local proof of Theorem 5.1(ii) even in the case of W a principal series representation. It would also be very desirable to have independent proofs in the archimedean case.

3. The results of Sect. 4 allow us to find $\gamma(W,\theta, V)$ and L_P, L_Q in the "generic" case.

Proof. (i). That the integrals converge if $\sigma(\omega)$ is large enough follows from the analysis of Sect. 2. That they represent "rational" functions when F is non-archimedean follows by a standard argument – cf. [8, Sect. 1]. An archimedean version of this, as in [8, Sect. 9] shows that the Z_P and Z_Q have analytic continuations as meromorphic functions. We shall not give these arguments in detail as they are now standard and fairly lengthy.

ii) Let us define for an admissible representation π of $GL_3(F)$ and a quasicharacter α of $F^\times$

$$L(\alpha, \pi, \mathrm{Sym}^2) = L(0, (\pi \otimes \alpha) \times \pi)/L(0, \check{\pi} \otimes \chi\alpha)$$

where χ is the central quasicharacter of π. Here $L(s, \pi_1 \times \pi_2)$ is as in [7, 9] and $L(s, \pi)$ is as in [8]. Likewise we define

$$\varepsilon(\alpha, \pi, \mathrm{Sym}^2, e) = \varepsilon(0, (\pi \otimes \alpha) \times \pi, e)/\varepsilon(0, \check{\pi} \otimes \chi\alpha, e)$$

where e is an additive character of F.

Note that if π is an automorphic representation of $GL_3(k_{\mathbf{A}})$ and if α is a Größencharakter then

$$L(\alpha, \pi, \mathrm{Sym}^2) = \prod L(\alpha_v, \pi_v, \mathrm{Sym}^2)$$

and

$$\varepsilon(\alpha, \pi, \mathrm{Sym}^2, e) = \prod \varepsilon(\alpha_v, \pi_v, \mathrm{Sym}^2, e_v)$$

exist, the first if $\sigma(\alpha)$ is large enough. The function $L(\alpha, \pi, \mathrm{Sym}^2)$ has an analytic continuation to the space of all Größencharaktere as a meromorphic function and satisfies the functional equation

$$L(\alpha, \pi, \mathrm{Sym}^2) = \varepsilon(\alpha, \pi, \mathrm{Sym}^2, e) L(\| \, \| \alpha^{-1}, \pi^v, \mathrm{Sym}^2)$$

– [8, 13.6], [9, 2.7]. Moreover in the product defining ε all but finitely many factors are 1 and $\varepsilon(\alpha, \pi, \mathrm{Sym}^2, e)$ is a monomial function.

Next we note that in the situation of Propositions 3.2 and 3.3 we have that if v is a place so that the conditions of Corollary 4.2 can be satisfied then

$$Z_P(\omega_v; w_v, t_v, f_v) = L(\omega_v^2, \alpha_v, \omega_{2,v}^2 | \, |_v^{1/2}, W_v, \mathrm{Sym}^2)$$

and

$$Z_Q(\omega_v; w_v, t_v, f_v') = L(\omega_v^2 \alpha_v^{-1} \omega_{2,v}^{-2} | \, |_v^{1/2}, \check{W}_v, \mathrm{Sym}^2)$$

where w_v, t_v, f_v, f_v' take their standard values.

Let S be a finite set of places of k so chosen that the conditions of Corollary 4.2 may be satisfied outside S. We choose then for $v \notin S$ the standard vectors as arguments. Thus we see that

$$\begin{aligned} &Z_P(\omega; \otimes w_v, \otimes t_v, \otimes f_v)/L(\omega^2 \alpha \tilde{\omega}_2^2 \| \, \|_{\mathbf{A}}^{1/2}, W, \mathrm{Sym}^2) \\ &= \prod_{v \in S} Z_P(\omega_v; w_v, t_v, f_v)/L(\omega_v^2 \alpha_v \tilde{\omega}_{2,v}^2 | \, |_v^{1/2}, \check{W}_v, \mathrm{Sym}^2). \end{aligned}$$

The right-hand sides of these expressions are finite products and therefore are convergent everywhere that the factors are finite. The left-hand sides of these are related by the functional equation. We thus obtain the following:

$$\prod_{v\in S}\frac{Z_P(\omega_v; w_v, t_v, f_v)}{L(\omega_v^2\alpha_v\tilde{\omega}_{2,v}^2|\,|_v^{1/2}, W, \mathrm{Sym}^2)}$$
$$=\frac{\varepsilon(\omega^6\alpha^3\chi^2\tilde{\chi}^2\|\,\|_{\mathbf{A}}^{-1/2}, e)}{\varepsilon(\omega^2\alpha\tilde{\omega}_3^2\|\,\|_{\mathbf{A}}^{1/2}, W, \mathrm{Sym}^2, e)}\prod_{v\in S}\frac{Z_Q(\tilde{\omega}_v; w_v, t_v, \tilde{I}_{QP}f)}{L(\omega_v^{-2}\alpha_v^{-1}\tilde{\omega}_{2,v}^{-2}|\,|_v^{1/2}, \check{W}_v, \mathrm{Sym}^2)}.$$

Consequently the trilinear functionals

$$Z_P(\omega_v; *, *, *) \quad \text{and} \quad Z_Q(\omega_v; *, *, I_{QP}*)$$

are proportional. We note here that for no W with a Whittaker model can one have that $Z_P(*, *, *, *)$ is identically zero by a density argument. This then shows that a function γ as asserted in (ii) exists. The argument used here also suggests strongly that $\gamma(\omega; W, \theta, \tilde{V})$ is of the form

$$(\text{monomial})\cdot L(\omega_v^2\alpha_v\tilde{\omega}_{2,v}^2|\,|_v^{1/2}, \mathrm{Sym}^2, W_v)/L(\omega_v^{-2}\alpha_v^{-1}\tilde{\omega}_{2,v}^{-2}|\,|_v^{1/2}, \mathrm{Sym}^2, \check{W}_v).$$

Note that it is immediate that if F is non-archimedean then $\gamma(W, \theta, \tilde{V})$ is "rational".

This reasoning is valid as long as W occurs in some cuspidal automorphic representation, that is, if W is relevant.

We shall now prove (iii). In the case that $\tilde{V}$ is a non-cuspidal exceptional representation this follows from [10, I.2] and the realization of $V_0(\omega)$ as a subrepresentation of $V({}^{w_0}\omega)$, [10, Theorem I.2.9]. We let k be a global field and let α be a Größencharakter. Let S be the set of places where $\alpha_v(-1)=-1$. Let $V^0(\alpha_v)$ be the representation denoted by r_{α_v} in [5]. Then we consider instead of the local field F the ring $k_S=\prod_{v\in S}k_v$. The same constructions can be made in this case. From the global theory (Sect. 2) and the results recalled above for the places outside S we see that the analogue of (iii) holds for

$$\tilde{V}=\bigotimes_{v\in S}V^0(\alpha_v) \quad (\text{tensor product over } \mathbf{C}[\mu_2(k)]).$$

Next note that if α_1 and α_2 satisfy $\alpha_1(-1)=-1$, $\alpha_2(-1)=-1$ then there exists β so that $\alpha_1=\alpha_2\beta^2$ and $V^0(\alpha_1)=V^0(\alpha_2)\otimes(\beta\circ\det)$ in the case of a local field. Thus we have only to prove (iii) over F for one quasicharacter α with $\alpha(-1)=-1$. The validity of (iii) is thus a property of F.

Consider first the case $k=\mathbf{Q}$ and p, q two rational primes. Then $\alpha(x)=(pq, x)_{2,\mathbf{A}}$ is as above with $S=\{p, q\}$ if $p, q>0$, $p, q\equiv 1 \pmod 4$. Note that $\tilde{I}_{PQ,v}\tilde{I}_{QP,v}=\phi_v\mathrm{Id}$ where ϕ_v is "rational in q_v^{-s}". We have just seen that $\prod_{v\in S}\phi_v$ is monomial. As $p\neq q$ it follows in this case that ϕ_p and ϕ_q are monomial. This proves (iii) for $\mathbf{Q}_p$, $p\equiv 1 \pmod 4$. Next let $\alpha(x)=(-p, x)_{2,\mathbf{A}}$, $p\equiv 1 \pmod 4$. In this case $S=\{\infty, p\}$ and we deduce the validity of (iii) for $F=\mathbf{R}$. Starting from $k=\mathbf{Q}(\sqrt{-1})$ we deduce (iii) also for $F=\mathbf{C}$. To prove it in general using the same method we have only to show that given F there exist a global field k, a place v of k with $k_v\cong F$, a Größencharakter α of $K_{\mathbf{A}}^\times$ so that $\alpha_v(-1)=-1$ with the property that if $S=\{w|\alpha_w(-1)=-1\}$ and $w\in S$, $w\neq v$ then the residual characteristic of w is different to that of v. This follows as we have shown that

$$\prod_{\substack{w\in S\\ w\,\text{non-arch.}}} \phi_w \quad \text{is monomial.}$$

To prove this fact let T be the set of places consisting of the archimedean places and those with the same residual characteristic as v (but $v\notin T$). Let $S=\{v\}\cup T$. Then let $U(S)$ be the group of S-units in $k_S^\times$. Let $x\in k_S^\times$ be that element so that $x_v=-1$, $x_w=1$ ($w\in T$). Then we form the character β of $U(S)\cup U(S)x$ by demanding that $\beta|U(S)=1$, $\beta(x)=-1$. This can be extended to a character β of $k_S^\times$ and hence to a Größencharakter β of $k_{\mathbb{A}}^\times$ unramified outside S. This β will serve as the sought for α.

In the case of F of finite characteristic we have to modify this slightly. Let v be a place of a global field k and let ϕ_v be as above. Find $m_0>1$ so that $\phi_v(\omega\,|\,{}^s_v)$ is not a rational function of q_v^{-ms} for $m\geqq m_0$. Let w be a place of k with a residue field of q_v^m elements, $m\geqq m_0$. Then, as above, we can find a Größencharakter α of $k_{\mathbb{A}}^\times$ unramified outside $\{v,w\}$ so that $\alpha_v(-1)=-1$, $\alpha_w(-1)=-1$. Hence $\phi_v\cdot\phi_w$ is monomial. As ϕ_w is "rational in q_v^{-ms}" it follows that both ϕ_v and ϕ_w are themselves monomial. This completes the proof of (iii).

As we have remarked above we shall only need the assertion of the theorem when W is relevant. We shall now extend this to cover the class of principal series representations.

Corollary 5.2. *Assertion* (ii) *of Theorem 5.1 is valid if W is a principal series representation.*

Proof. For a representation ψ of H let $W(\psi)$ be the corresponding principal series representation. The W form a holomorphic vector bundle over the set of all possible ψ with $\psi|Z=\chi$. Let T be the set of all ψ for which the assertion is false. If $\psi_1\in T$ then there would exist w_1, $w_2\in W(\psi_1)$ so that

$$\frac{Z_P(\omega;w_1,t,f)}{Z_Q(\omega;w_1,t,I_{QP}f)} \neq \frac{Z_P(\omega;w_2,t,f)}{Z_Q(\omega;w_2,t,I_{QP}f)}.$$

The set of ψ can be considered as the set of functions on an irreducible algebraic variety. The subset of relevant ψ is Zariski dense – see [1, pp. 69–72]. On the other hand the remark above shows that T is an open subset and by Theorem 5.1(ii) it contains no relevant ψ. It now follows that T is empty, and this is the assertion of the theorem.

We are indebted to the referee for pointing out this proof which yields a sharper statement than our original discussion.

Theorem 5.1 asserts the existence of the trilinear functionals Z_P and Z_Q but says nothing about the possible poles or zeros of these functions. In order to be able to derive information about $L(\omega,\pi,\mathrm{Sym}^2)$ from Proposition 3.2 we need a "non-vanishing" about Z_P. Proposition 5.3 below gives such a statement which suffices for out purposes although it is rather unsatisfactory from an aesthetic point of view.

Proposition 5.3. *Let F be a non-archimedean field of characteristic $\neq 2$; we retain the notations of Theorem 5.1. Let $\Omega_P^*\subset\Omega_P$ be a connected component of Ω_P. Then there exist $w\in W$, $t\in\theta$, $\tilde v\in\tilde V$ and a compact open subgroup K_1 of K^* so that if $w\in\Omega_P^*$*

we define $f(\omega)\in F_P(\omega)$ *by*

(i) $f(I,\omega)=\tilde{v}$
(ii) f *is* K_1*-invariant*
(iii) f *is supported on* $\tilde{P}K_1$

then

$$Z_P(\omega;\, w,t,f)=L(\omega^6\alpha^3\chi^2\tilde{\chi}^2|\,|_F^{3/2}).$$

Proof. We have to show that

$$\int \Lambda(gw)\Lambda^{01}(gt)\Lambda_P(f(g,\omega))dg=1\,.$$

The proof of this is now fairly standard, see, e.g. [8, Sect. 4], so we sketch only the main points. Since W has a Whittaker model it follows from the Gelfand-Kazhdan theorem that if $\phi: P(F)\to\mathbf{C}$, $\phi(ng)=e(n)\phi(g)$, $n\in N(F)$ and $\phi(zg)=\chi(z)\phi(g)$ $z\in Z(F)$, and if ϕ has compact support modulo $N(F)/Z(F)$ then there exists $w\in W$ so that

$$\Lambda(gw)=\phi(w)\,,\qquad g\in P(F)\,.$$

We choose a sufficiently small compact open subgroup K_2 so that $e|N(F)\cap K_2=1$, χ, $\tilde{\chi}|Z(F)\cap K_2=1$ and let ϕ be that function supported on $N(F)Z(F)K_2$ satisfying the conditions above and $\phi|K_2=1$. We then find $t\in\theta$ so that $\Lambda^{01}(t)\neq 0$, and $\tilde{v}\in\tilde{V}$ so that $\Lambda_P(\tilde{v})\neq 0$. We choose K_1 so that $K_1\subset K$ and w, t are K_1-invariant and $f(\omega)$ as in the statement of the proposition is well-defined. It is then obvious that

$$\int \Lambda(gw)\Lambda^{01}(gt)\Lambda_P(f(g,\omega))dg$$

is equal to $\Lambda^{01}(t)\cdot\Lambda_P(f(I,\omega))\cdot\mathrm{meas}(K_1)$. Since $\Lambda_P(f(I,\omega))$ does not depend on ω the assertion follows directly. This proves the proposition.

6. Global Results

We can now apply the results of the previous section to the study of the global L-function. For the convenience of the reader we recall the definitions. Let k be a global field and let π be an irreducible cuspidal automorphic representation of $GL_3(k_{\mathbf{A}})$. Let χ be the central quasicharacter of π. Let α be a Größencharakter of k, i.e. a quasicharacter of $k_{\mathbf{A}}^\times$ trivial on $k^\times$. Let

$$L(\alpha,\pi,\mathrm{Sym}^2)=L(\mathcal{O},(\pi\otimes\alpha)\times\pi)/L(\mathcal{O},\check{\pi}\otimes\chi\alpha)\,.$$

This is an Euler product of analogously defined local factors; it converges if $\sigma(\alpha)$ is large enough. Likewise we define the monomial function

$$\varepsilon(\alpha,\pi,\mathrm{Sym}^2)=\varepsilon(\mathcal{O},(\pi\otimes\alpha)\times\pi)/\varepsilon(\mathcal{O},\check{\pi}\otimes(\chi\alpha))$$

which, given a choice of additive character, can be expressed as a product of monomials over the set of places of k. All but finitely many are equal to 1. Then, as we have already pointed out it is known that as a function of $\alpha L(\alpha,\pi,\mathrm{Sym}^2)$ has an analytic continuation as a meromorphic function to the complex manifold of all Größencharaktere. Moreover one has the functional equation

$$L(\alpha,\pi,\mathrm{Sym}^2)=\varepsilon(\alpha,\pi,\mathrm{Sym}^2)L(\|\;\|_{\mathbf{A}}\alpha^{-1},\check{\pi},\mathrm{Sym}^2)\,.$$

We now have:

Theorem 6.1. *Suppose k is of characteristic >2. Then $L(\alpha, \pi, \mathrm{Sym}^2)$ has a pole at α_0 only if*

$$\pi \otimes \alpha_0 \cong \check{\pi} \quad or \quad \pi \otimes (\alpha_0 \| \|_{\mathbf{A}}^{-1}) \cong \check{\pi}.$$

The corresponding pole is simple.

Remarks. 1. α_0 satisfies $\alpha_0^3 = \chi^{-2}$ or $\alpha_0^3 = \chi^{-2} \| \|_{\mathbf{A}}^3$. If one such α_0 exists satisfying $\alpha_0^3 = \chi^{-2}$ then $\pi \otimes (\alpha_0 \chi)$ has trivial central character.

2. The restriction that the characteristic be $\neq 2$ is natural as the symmetric square behaves quite differently in this characteristic.

3. That the case when the characteristic is 0 is not covered is a consequence that no analogue of Proposition 5.3 is available in the archimedean cases. In fact, as we shall see it would suffice to be able to show that when $W, \theta, \tilde{V}$ are all unitary then there is no ω_1 satisfying $\sigma(\omega_1) \geqq \frac{1}{2}$ for which

$$Z_P(\omega_1; w, t, f) = 0$$

for all w, t, f. Although this is very probably true we do not have a proof.

Proof. By renormalizing the representations $W, \theta, \tilde{V}$ we may assume that they are unitary. Here W will be a representation which realizes the class π.

The fact that $L(0, (\pi \otimes \alpha) \times \pi)$ has a pole at the stated points follows from [9, p. 368]. Since π is cuspidal $L(0, \pi \otimes \chi\alpha)$ is finite and non-zero for such α. Hence these poles exist, and there are no further poles of $L(\alpha, \pi, \mathrm{Sym}^2)$ when $\sigma(\alpha) \geqq 1$ or $\sigma(\alpha) \leqq 0$. Thus it suffices to show that there are no poles α satisfying

$$\tfrac{1}{2} \leqq \sigma(\alpha) \leqq 1.$$

If there are none then there are none in the region $0 < \sigma(\alpha) \leqq \frac{1}{2}$ by the functional equation and so we would have proved our assertion.

We have next, with the notations above for

$$\omega = \otimes \omega_v, \qquad t = \otimes t_v, \qquad f = \otimes f_v$$

$$Z_P(\omega; w, t, f) = L(\omega^2 \alpha \tilde{\omega}_2^2 \| \|_{\mathbf{A}}^{1/2}, \pi, \mathrm{Sym}^2) \times \prod_v Z_{P,v}(\omega_v; w_v, t_v, f_v) / L(\omega_v^2 \alpha_v \tilde{\omega}_{2,v}^2 | |_v^{1/2}, \pi_v, \mathrm{Sym}^2).$$

As we have already seen the second factor on the right-hand side is such that almost all factors are 1. The left-hand side is holomorphic if $0 \leqq \sigma(\omega) < \frac{1}{4}$ by Proposition 3.2. It suffices therefore to verify that each factor

$$Z_{P,v}(\omega_v; w_v, t_v, f_v) / L(\omega_v^2 \alpha_v \tilde{\omega}_{2,v}^2 | |_v^{1/2}, \pi_v, \mathrm{Sym}^2),$$

is, for a suitable choice of w_v, t_v, f_v, non-zero for any such ω.

That this is so for $Z_{P,v}(\omega_v; w_v, t_v f_v)$ follows immediately from Proposition 5.3. Write $\beta_v = \omega_v^2 \alpha_v \omega_{v,v}^2 | |_v^{1/2}$. Then the factor

$$1/L(\beta_v, \pi_v, \mathrm{Sym}^2) = L(\mathcal{O}, \pi_v \otimes \chi_v \beta_v) / L(\mathcal{O}, (\pi_v \otimes \beta_v) \times \pi_v).$$

Thus it remains to show that if $\sigma(\beta_v)$ satisfies $\frac{1}{2} \leqq \sigma(\beta) < 1$ then $L(0, (\pi_v \otimes \beta) \times \pi_v)^{-1} \neq 0$ when π_v is unitary, irreducible and generic. If π_v is tempered then this follows from [9, Proposition 8.4]. If π_v is not tempered then it is a member of the complementary series (see [8, Sects. 6.1–6.4]) and the computation of [9, Proposition 9.4] (note the misprint!) shows that $L(0, (\pi_v \otimes \beta) \times \pi_v) \neq 0$ in this case as well. The fact that this argument does not hold when $\frac{1}{2}$ is replaced by any smaller number is the reason why we use the functional equation to make this deduction.

This now completes the proof of the theorem.

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Received December 21, 1987