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Additivity of Certain Functionals and the Construction of Invariant Integrals

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1. Introduction

Throughout the paper X denotes a locally compact Hausdorff space and $C_{00}(X)$ ($C_{00}^{+}(X)$) denotes the set of all complex-valued (nonnegative) continuous functions on X having compact support. The support of a function f is denoted by $\mathrm{supp}(f)$. Our main result is the following:

Theorem 1. *Let I be a mapping from $C_{00}^{+}(X)$ into $[0, \infty)$ and suppose that for all $f, g \in C_{00}^{+}(X)$ the following relations hold:*

(i) $I(f) \leqq I(g)$ *whenever* $f \leqq g$ *(I is monotone);*

(ii) $I(pf) = pI(f)$ *if* $p \in R$ *and* $p \geqq 0$ *(I is homogeneous);*

(iii) $I(f+g) \leqq I(f) + I(g)$ *(I is subadditive);*

(iv) $I(f+g) = I(f) + I(g)$ *whenever* $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$.

Then I is additive, i.e., $I(f+g) = I(f) + I(g)$ for all $f, g \in C_{00}^{+}(X)$.

We shall prove Theorem 1 in Sect. 3. This theorem enables us to give a short proof of a general result concerning the existence of invariant integrals in $C_{00}^{+}(X)$ (Theorem 2). For the proof of the existence we use Weil's method which is usually applied for construction of the Haar integral on locally compact groups [2, 3].

Theorem 2 is the analogue of a result of Banach [1, p. 239]. Banach considered an equivalence relation $\cong$ between subsets of X and showed that if $\cong$ satisfies certain conditions then there exists a regular measure μ on X such that $\mu(A) = \mu(B)$ whenever $A \cong B$. Our concern will be with the existence of an integral I on $C_{00}^{+}(X)$ for which $I(f) = I(g)$ whenever $f \cong g$, where $\cong$ is a certain equivalence relation in $C_{00}^{+}(X)$. The Haar integral on locally compact groups will be obtained as a special case.

We remark that Theorem 1 remains true in a more general setting (see Remark in Sect. 3). An application of this Remark will be given in Sect. 4.

2. Existence of Invariant Integrals

A mapping $I: C_{00}^{+}(X) \to [0, \infty)$ will be called an *integral* on $C_{00}^{+}(X)$ if I is additive, homogeneous and not identically zero. It is well known that for any such I there exists a nonnegative regular measure μ on X for which the equality $I(f)=\int_X f\,d\mu$ holds for all $f\in C_{00}^+(X)$ [3].

We shall use the notation $\|f\|=\operatorname{supp}\{f(x):x\in X\}$, $f\in C_{00}^+(X)$. Let $\cong$ be an *equivalence relation* in $C_{00}^+(X)$, i.e., $f\cong f$; $f\cong g$ if and only if $g\cong f$; $f\cong g$ and $g\cong h$ imply $f\cong h$. An equivalence relation $\cong$ is called a *congruence relation* if the following conditions (2.1–2.3) are satisfied:

$$f\cong g \quad \text{implies} \quad \|f\|=\|g\|; \tag{2.1}$$

if $f, g\in C_{00}^+(X)$ and $g\neq 0$ then there exist a positive integer n, real numbers $c_i>0$, functions $g_i\in C_{00}^+(X)$ such that $g_i\cong g$ $(i=1,\dots,n)$ and $f\leqq\sum_1^n c_i g_i$; (2.2)

if $f\cong g$ and $f\leqq\sum_1^n c_i f_i$ $(c_i>0)$ then there exist $g_i\in C_{00}^+(X)$ such that $g_i\cong f_i$ and $g\leqq\sum_1^n c_i g_i$. (2.3)

Theorem 2. *Let $\cong$ be a congruence relation in $C_{00}^+(X)$ and let $\{f_\alpha:\alpha\in\Gamma\}\subset C_{00}^+(X)$ be a net of nonzero functions satisfying the following condition:*

if f, $g\in C_{00}^+(X)$ and $\operatorname{supp}(f)\cap\operatorname{supp}(g)=\emptyset$ then there exists $\beta(f,g)\in\Gamma$ such that for any $\alpha\geqq\beta$ the relations $h\cong f_\alpha$, $\operatorname{supp}(h)\cap\operatorname{supp}(f)\neq\emptyset$ imply $\operatorname{supp}(h)\cap\operatorname{supp}(g)=\emptyset$. (2.4)

Then there exists an integral I on $C_{00}^+(X)$ for which $I(f)=I(g)$ if $f\cong g$ and $I(f)>0$ whenever $f\neq 0$.

Proof. Choose a fixed nonzero function $f_0\in C_{00}^+(X)$. For every $f\in C_{00}^+(X)$ we set

$$J_\alpha(f):=\inf\left\{\sum_1^n c_i : f\leqq\sum_1^n c_i f_i,\ f_i\cong f_\alpha,\ c_i>0,\ i=1,\dots,n\right\}.$$

If $f\neq 0$, then $J_\alpha(f)>0$. Indeed, if $f\leqq\sum_1^n c_i f_i$ and $f_i\cong f_\alpha$ then by (2.1) we have $\|f\|\leqq\left(\sum_1^n c_i\right)\|f_\alpha\|$ and hence $J_\alpha(f)\geqq\dfrac{\|f\|}{\|f_\alpha\|}$. Let

$$I_\alpha(f):=\frac{J_\alpha(f)}{J_\alpha(f_0)},\qquad f\in C_{00}^+(X).$$

It is immediate that I_α is monotone, homogeneous and subadditive and that $I_\alpha(f_0)=1$. Moreover, we have:

$$\text{if}\quad f\cong g\quad\text{then}\quad I_\alpha(f)=I_\alpha(g); \tag{2.5}$$

$$\begin{gathered} I_\alpha(f+g)=I_\alpha(f)+I_\alpha(g)\quad\text{whenever}\\ \operatorname{supp}(f)\cap\operatorname{supp}(g)=\emptyset\quad\text{and}\quad\alpha\geqq\beta(f,g); \end{gathered} \tag{2.6}$$

if $f\neq 0$ then there exist positive numbers $c(f)$, $C(f)$ for which $c(f)\leqq I_\alpha(f)\leqq C(f)$, $\alpha\in\Gamma$. (2.7)

Property (2.5) follows from (2.3) while (2.6) is a consequence of (2.4). In order to show (2.7) choose $f_i, f_j' \in C_{00}^+(X)$, $c_i, c_j' > 0$ $(i=1,\dots,n;\ j=1,\dots,n')$ so that $f_i \cong f_0$, $f_j' \cong f$ and

$$f_0 \leqq \sum_1^{n'} c_j' f_j', \quad f \leqq \sum_1^{n} c_i f_i.$$

Using the properties of I_α we get

$$I_\alpha(f_0) = 1 \leqq \left(\sum_1^{n'} c_j'\right) I_\alpha(f) \quad \text{and} \quad I_\alpha(f) \leqq \left(\sum_1^{n} c_i\right),$$

from which (2.7) follows.

The set of all mappings $J: C_{00}^+(X) \to [0,\infty)$ with $J(0)=0$ and $c(f) \leqq J(f) \leqq C(f)$ for nonzero f is compact in the topology of pointwise convergence (by Tikhonov's theorem). Consequently there exists a subnet of $\{I_\alpha\}$ converging to a mapping I. It is clear that I is monotone, subadditive, homogeneous and $I(f)=I(g)$ whenever $f \cong g$. Moreover, (2.6) implies that $I(f+g)=I(f)+I(g)$ if $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$. It follows from Theorem 1 that I is additive. This completes the proof.

Examples. Let $x_0 \in X$ and let $\{W_\alpha : \alpha \in \Gamma\}$ be a neighbourhood basis at x_0. We introduce a partial ordering $\geqq$ in Γ by setting $\alpha \geqq \beta$ if $W_\alpha \subset W_\beta$. For each $\alpha \in \Gamma$ we choose a nonzero function f_α such that $\mathrm{supp}(f_\alpha) \subset W_\alpha$. Suppose that H is a group of homeomorphisms of X onto itself satisfying the following conditions:

$$\bigcup_{h \in H} hV = X \quad \text{for every nonvoid open set} \quad V \subset X; \tag{2.8}$$

for each pair of compact disjoint sets F_1 and F_2, there exists $\beta(F_1, F_2) \in \Gamma$ such that $hW_\alpha \cap F_1 \neq \emptyset$ implies $hW_\alpha \cap F_2 = \emptyset$ for all $h \in H$ and $\alpha \geqq \beta(F_1, F_2)$. (2.9)

Setting $f \cong g$ if $f(x) = g(h(x))$ for some $h \in H$, we see that $\cong$ is a congruence relation in $C_{00}^+(X)$ such that the net $\{f_\alpha\}$ satisfies condition (2.4). [Relation (2.2) follows from the fact that, in view of (2.8), the set $\mathrm{supp}(f)$ can be covered by a finite number of sets of the form hV $(h \in H)$ where $V = \left\{x \in X : g(x) > \frac{\|g\|}{2}\right\}$.]

Note that (2.8) is always satisfied if H is transitive, i.e., for any $x, y \in X$ there exists $h \in H$ with $h(x) = y$.

Plainly condition (2.9) is satisfied in the following cases:

a) X is metrizable, $x_0 \in X$ is arbitrary, and H is the group of all isometric homeomorphisms of X onto itself; or more generally

b) X is a uniformly locally compact uniform space, $x_0 \in X$ is arbitrary, and H is a uniformly equicontinuous group of uniformisms of X.

Thus, as a special case, we obtain a theorem of Segal [4, p. 187] (see also the remarks of Goetz in [1, p. 352]).

There are classical cases where conditions (2.8, 2.9) are satisfied. The symbol X denotes a locally compact group and x_0 is always the identity of X. We list these examples:

c) H is the group of left translations $x \to yx$ $(x, y \in X)$;

d) X is compact or commutative and H is generated by the mappings $x \to x^{-1}$, $x \to yx$ and $x \to xy$ $(x, y \in X)$;

e) G is a compact subgroup of X and H is generated by the mappings $x \to yx$ $(x, y \in X)$ and $x \to xg$ $(x \in X,\ g \in G)$;

f) X admits a left invariant metric ϱ and H is the group of all invertible isometric mappings $h: X \to X$. If ϱ is also right (inverse) invariant then the corresponding (Haar) integral will be right (inverse, respectively) invariant as well.

3. Proof of Theorem 1

To illustrate our method, we consider the case where X is discrete. Then any function $f \in C_{00}^{+}(X)$ can be written as $f = \sum_{x \in X} f(x)\delta_x$ (finite sum) where $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for $y \in X$, $y \neq x$. Using (ii) and (iv) in Theorem 1 we get

$$\begin{aligned} I(f+g) &= I\Big(\sum_{x \in X} (f(x)+g(x))\delta_x\Big) = \sum_{x \in X} (f(x)+g(x))I(\delta_x) \\ &= \sum_{x \in X} f(x)I(\delta_x) + \sum_{x \in X} g(x)I(\delta_x) \\ &= I\Big(\sum_{x \in X} f(x)\delta_x\Big) + I\Big(\sum_{x \in X} g(x)\delta_x\Big) \\ &= I(f) + I(g). \end{aligned}$$

In the general case, we will construct continuous functions which will play the role of the functions δ_x. We will write

$$C_{00}^{1}(X) = \{h \in C_{00}^{+}(X) : h(X) \subset [0,1]\}.$$

We write $C_{00}^{1}([0,1])$ as C^1.

The proof. Let $f \in C_{00}^{1}(X)$ and denote by B_f the set of all $t \in [0,1]$ with the following property: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $I(fh) < \varepsilon$ whenever $h \in C_{00}^{1}(X)$ and $\operatorname{supp}(h) \subset f^{-1}([t-\delta, t+\delta])$.

We show first that the set $D_f := [0,1] \setminus B_f$ is denumerable, so that B_f is everywhere dense in $[0,1]$. Assume that D_f is nondenumerable. Then there exist a positive integer m, mutually distinct numbers $t_i \in D_f$ and $\varepsilon_i > \frac{1}{m}$ $(i = 1, 2, \ldots)$ such that for every $\delta > 0$ we can find functions $h_i \in C_{00}^{1}(X)$ with $\operatorname{supp}(h_i) \subset f^{-1}([t_i - \delta, t_i + \delta])$ and $I(fh_i) \geqq \varepsilon_i > \frac{1}{m}$. Let N be a positive integer with $N > mI(f)$ and let $\delta = \delta(N)$ be a positive number for which

$$[t_i - \delta, t_i + \delta] \cap [t_j - \delta, t_j + \delta] = \emptyset \quad (i \neq j;\ i, j = 1, \ldots, N).$$

Choose $h_i \in C_{00}^{1}(X)$ so that $\operatorname{supp}(h_i) \subset f^{-1}([t_i - \delta,\ t_i + \delta])$ and $I(fh_i) \geqq \varepsilon_i > \frac{1}{m}$. Then $\operatorname{supp}(h_i) \cap \operatorname{supp}(h_j) = \emptyset$ $(i \neq j)$ and hence

$$I(f) \geqq I\left(\sum_{1}^{N} fh_i\right) = \sum_{1}^{N} I(fh_i) \geqq \frac{N}{m},$$

contradicting the choice of N. Thus D_f is denumerable. Now suppose that $f\in C^1_{00}(X)$ and $\|f\|<1$. We will show that $I(f)$ can be approximated by sums of the form $\sum_1^n t_i I(h_i)$ where $h_i\in C^1_{00}(X)$ and $\operatorname{supp}(h_i)\cap\operatorname{supp}(h_j)=\emptyset$ $(i\neq j)$.

Let $\varepsilon>0$ be arbitrary and choose $t_1,\dots,t_n\in B_f$ so that

$$0=:t_0<t_1\dots<t_n<t_{n+1}:=1 \quad\text{and}\quad \max_{i=0,\dots,n}(t_{i+1}-t_i)<\varepsilon.$$

Note that trivially $1\in B_f$ since $\|f\|<1$. As $t_i\in B_f$, there is a $\delta>0$ for which $\delta<\frac{1}{2}\min_{i=0,\dots,n}(t_{i+1}-t_i)$ and

$$I(f\tilde{h}_i)<\frac{\varepsilon}{n+1} \tag{3.1}$$

for every $\tilde{h}_i\in C^1_{00}(X)$ with $\operatorname{supp}(\tilde{h}_i)\subset f^{-1}([t_i-\delta,\ t_i+\delta])$, $i=1,\dots,n+1$. Choose $g_i\in C^1$ such that

$$g_i([t_i+\delta,t_{i+1}-\delta])=1 \quad\text{and}\quad g_i([0,1]\setminus[t_i,t_{i+1}])=0,$$

$i=0,\dots,n$, and let $\tilde{g}_i\in C^1$ be the functions which are uniquely determined by the relations

$$\operatorname{supp}(\tilde{g}_i)\subset[t_i-\delta,t_i+\delta],\quad i=0,\dots,n+1,\quad\text{and}\quad \sum_0^n g_i+\sum_0^{n+1}\tilde{g}_i=1.$$

We set $h_i:=g_i(f)$, $i=0,\dots,n$; $\tilde{h}_i:=\tilde{g}_i(f)$, $i=1,\dots,n+1$, and $\tilde{h}_0:=\chi\tilde{g}_0(f)$, where χ is a function in $C^1_{00}(X)$ for which $\chi(\operatorname{supp}(f))=1$. Write $\tilde{h}=\sum_0^{n+1}\tilde{h}_i$. Then we have $\tilde{h}$, h_i, $\tilde{h}_j\in C^1_{00}(X)$ $(i=0,\dots,n;\ j=0,\dots,n+1)$. These functions have the following properties:

$$\operatorname{supp}(h_i)\cap\operatorname{supp}(h_j)=\emptyset\qquad(i\neq j;\, i,j=0,\dots,n);$$

$$\operatorname{supp}(\tilde{h}_i)\subset f^{-1}([t_i-\delta,t_i+\delta]);\qquad f=f\left(\sum_1^n h_i+\tilde{h}\right);$$

$$\sum_1^n t_ih_i\leqq f<\chi;\qquad f\tilde{h}_0\leqq\varepsilon\chi;\qquad \sum_0^n h_i\leqq\chi;$$

and

$$0\leqq fh_i-t_ih_i\leqq\varepsilon h_i\qquad(i=0,\dots,n).$$

The last inequality implies that $I(fh_i)\leqq(t_i+\varepsilon)I(h_i)$, $i=0,\dots,n$. From this and properties of the functions h_i and $\tilde{h}_i$, we get

$$\begin{aligned}\sum_1^n t_iI(h_i)=I\left(\sum_1^n t_ih_i\right)&\leqq I(f)=I\left(f\left(\sum_0^n h_i+\tilde{h}\right)\right)\\ &\leqq\sum_0^n I(fh_i)+I(f\tilde{h})\leqq\sum_0^n(t_i+\varepsilon)I(h_i)+I(f\tilde{h})\\ &=\sum_1^n t_iI(h_i)+\varepsilon I\left(\sum_0^n h_i\right)+I(f\tilde{h})\\ &\leqq\sum_1^n t_iI(h_i)+\varepsilon I(\chi)+I(f\tilde{h}).\end{aligned}\tag{3.2}$$

In view of (3.1) we have

$$I(f\tilde{h}) \leqq \sum_{1}^{n+1} I(f\tilde{h}_i) + I(f\tilde{h}_0) < (n+1)\frac{\varepsilon}{n+1} + \varepsilon I(\chi) = \varepsilon(1+I(\chi)).$$

Putting this in (3.2) we obtain

$$0 \leqq I(f) - \sum_{1}^{n} t_i I(h_i) \leqq (1+2I(\chi)).$$

We will now prove that I is additive. Since I is homogeneous, it suffices to show that $I(f+g) = I(f) + I(g)$ for all $f, g \in C_{00}^1(X)$ with $\|f\| < 1$ and $\|g\| < 1$.

Suppose first that $cf \leqq g \leqq Cf$, where c and C are positive numbers, and choose $\chi \in C_{00}^1(X)$ for which $\chi(\operatorname{supp}(f+g)) = 1$. Let $\varepsilon > 0$ be arbitrary. It follows from the facts just proved that there exist functions h_i, h'_j, $\tilde{h}$, $\tilde{h}' \in C_{00}^1(X)$ and numbers $t_i, t'_j \in [0,1]$ $(i=0,\dots,n;\, j=0,\dots,m)$ such that

$$0 \leqq I(f) - \sum_{1}^{n} t_i I(h_i) \leqq \varepsilon, \quad \sum_{0}^{n} h_i + \tilde{h} = 1 \quad \text{on} \quad \operatorname{supp}(f+g),$$

$$I(f\tilde{h}) \leqq \varepsilon, \quad \sum_{1}^{n} t_i h_i \leqq f,$$

$$\operatorname{supp}(h_i) \cap \operatorname{supp}(h_j) = \emptyset \quad (i \neq j;\, i,j = 0,\dots,n),$$

and g, h'_j, $\tilde{h}'$, and t'_j satisfy the same relations with m instead of n. We have

$$\begin{aligned}
I(f) + I(g) &\leqq \sum_{1}^{n} t_i I(h_i) + \sum_{1}^{m} t'_j I(h'_j) + 2\varepsilon \\
&= \sum_{1}^{n} t_i I\left(h_i\left(\sum_{1}^{m} h'_j + \tilde{h}'\right)\right) + \sum_{1}^{m} t'_j I\left(h'_j\left(\sum_{1}^{n} h_i + \tilde{h}\right)\right) + 2\varepsilon \\
&\leqq \sum_{1}^{n} t_i I\left(h_i\left(\sum_{1}^{m} h'_j\right)\right) + \sum_{1}^{m} t'_j I\left(h'_j\left(\sum_{1}^{n} h_i\right)\right) \\
&\quad + I\left(\sum_{1}^{n} t_i h_i \tilde{h}'\right) + I\left(\sum_{1}^{m} t'_j h'_j \tilde{h}\right) + 2\varepsilon.
\end{aligned}$$

The sum of the first two term on the right is equal to

$$S_1 := I\left(\sum_{i=1}^{n} \sum_{j=1}^{m} (t_i + t'_j) h_i h'_j\right).$$

Since

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (t_i + t'_j) h_i h'_j \leqq f + g,$$

we get $S_1 \leqq I(f+g)$. Let S_2 denote the sum of the third and fourth terms. From the inequalities $\sum_{1}^{n} t_i h_i \leqq f$ and $\sum_{1}^{m} t'_j h'_j \leqq g$, we obtain

$$S_2 \leqq I(f\tilde{h}') + I(g\tilde{h}) \leqq I\left(\frac{g}{c}\tilde{h}'\right) + I(Cf\tilde{h}) \leqq \left(\frac{1}{c} + C\right)\varepsilon,$$

and so

$$I(f)+I(g)\leqq I(f+g)+\left(2+\frac{1}{c}+C\right)\varepsilon.$$

This being true for all positive ε, it follows that $I(f+g)=I(f)+I(g)$.

Let now $f, g\in C_{00}^{+}(X)$ be arbitrary. For $\varepsilon>0$, we set $F:=f+\varepsilon(f+g)$ and $G:=g+\varepsilon(f+g)$. There exist positive numbers c and C such that $cF\leqq G\leqq CF$ and hence the relaiton

$$\begin{aligned} I(f+g)+2\varepsilon I(f+g) &= I(f+g+2\varepsilon(f+g))=I(F+G)\\ &= I(F)+I(G)\geqq I(f)+I(g) \end{aligned}$$

holds for every $\varepsilon>0$. That is, we have $I(f+g)=I(f)+I(g)$. The proof is complete.

Remark. Theorem 1 is true for an arbitrary topological space X and for an arbitrary family $\mathscr{F}$ of bounded nonnegative continuous functions on X having the following properties:

(ii) fg, $af+bg\in\mathscr{F}$ whenever $f, g\in\mathscr{F}$ and $a, b\geqq 0$;

(ii) if $f\in\mathscr{F}$, $\|f\|\leqq 1$ and $h\in C^1$ then there exists $\chi\in\mathscr{F}$ such that $\chi\geqq 1$ on $\operatorname{supp}(f)$ and $\chi h(f)\in\mathscr{F}$.

Actually, it is unnecessary to require (ii) for every $h\in C^1$. It suffices for example to require (ii) for infinitely differentiable $h\in C^1$. Consequently Theorem 1 holds for the family of compactly supported nonnegative infinitely differentiable functions on R^n.

4. Additivity of the Upper Integral

Let I be an integral on $C_{00}^{+}(X)$ and denote by $\mathscr{M}^{+}$ the set of upper semicontinuous functions $f:X\to[0,\infty]$. The characteristic function of a set $A\subset X$ will be denoted by χ_A. Define the nonnegative functional $\bar{I}$ on $\mathscr{M}^{+}$ by

$$\bar{I}(f):=\sup\{I(g):g\in C_{00}^{+}(X),\ g\in f\}.$$

The functional $\bar{I}$ is monotone, homogeneous and additive on $\mathscr{M}^{+}$ [3, (11.12) and (11.14)]. Let now $h:X\to[0,\infty]$ be arbitrary and define $\bar{\bar{I}}(h)$ be setting

$$\bar{\bar{I}}(h):=\inf\{\bar{I}(g):g\in\mathscr{M}^{+},g\geqq h\}.$$

This functional is not additive in general but it is monotone, homogeneous and subadditive [3, (11.17)]. Moreover, $\bar{\bar{I}}\left(\lim_n h_n\right)=\lim_n \bar{\bar{I}}(h_n)$ whenever $\{h_n\}$ is an increasing sequence of nonnegative functions [3, (11.18)]. The set function $\iota(A):=\bar{\bar{I}}(\chi_A)$, $A\subset X$, is an outer measure, which is regular on the σ-algebra of ι-measurable sets [3, (11.34)].

Let $\mathscr{F}_m$ denote the set of ι-measurable functions $f:X\to[0,\infty]$. An important property of $\mathscr{F}_m$ is that $\bar{\bar{I}}$ is additive on $\mathscr{F}_m$. This can be proved for example by showing first additivity for step functions and then approximating measurable functions by step functions. To illustrate the Remark in Sect. 3, we give another proof.

Theorem 3. *$\bar{\bar{I}}$ is additive on $\mathscr{F}_m$.*

Proof. Let $\mathscr{F}$ denote the set of all bounded functions $f \in \mathscr{F}_m$ with $\bar{\bar{I}}(f) < \infty$ and $\iota(\operatorname{supp}(f)) < \infty$. It is easy to see that $\mathscr{F}$ satisfies conditions (i) and (ii) of the Remark if X is replaced by X_d, the discrete version of X. Consequently, the additivity of $\bar{\bar{I}}$ on $\mathscr{F}$ will follow from

$$\bar{\bar{I}}(f_1+f_2)=\bar{\bar{I}}(f_1)+\bar{\bar{I}}(f_2), \quad \text{whenever} \quad A_1 \cap A_2 = \emptyset, \tag{4.1}$$

where $A_i = \{x \in X : f_i(x) > 0\}$, $f_i \in \mathscr{F}$ $(i=1,2)$.

To prove (4.1), let $\varepsilon > 0$ be arbitrary and choose compact sets $F_i \subset A_i$ such that $\|f_i\| \iota(A_i \backslash F_i) < \varepsilon$ $(i=1,2)$. Then we have

$$\bar{\bar{I}}(f_i) = \bar{\bar{I}}((\chi_{F_i} + \chi_{(A_i \backslash F_i)}) f_i) \leqq \bar{\bar{I}}(\chi_{F_i} f_i) + \varepsilon .$$

Since $F_1 \cap F_2 = \emptyset$, there exist disjoint open sets U_1 and U_2, for which $U_1 \supset F_1$ and $U_2 \supset F_2$. Choose a function $h \in \mathscr{M}^+$ such that $h \geqq f_1 + f_2$ and $\bar{\bar{I}}(f_1+f_2) > \bar{I}(h) - \varepsilon$. We then have $\chi_{U_i} h \in \mathscr{M}^+$ and $\chi_{U_i} h \geqq \chi_{F_i} f_i$. Using the properties of $\bar{I}$ and $\bar{\bar{I}}$, we get

$$\begin{aligned} \bar{\bar{I}}(f_1+f_2) &> \bar{I}(h) - \varepsilon \geqq \bar{I}((\chi_{U_1} + \chi_{U_2})h) - \varepsilon = \bar{I}(\chi_{U_1} h) + \bar{I}(\chi_{U_2} h) - \varepsilon \\ &\geqq \bar{\bar{I}}(\chi_{F_1} f_1) + \bar{\bar{I}}(\chi_{F_2} f_2) - \varepsilon > \bar{\bar{I}}(f_1) + \bar{\bar{I}}(f_2) - 3\varepsilon , \end{aligned}$$

and hence $\bar{\bar{I}}(f_1+f_2) \geqq \bar{\bar{I}}(f_1) + \bar{\bar{I}}(f_2)$. Relation (4.1) follows now from the subadditivity of $\bar{\bar{I}}$.

Finally, consider arbitrary $f_1, f_2 \in \mathscr{F}_m$. If $\bar{\bar{I}}(f_1) = \infty$ or $I(f_2) = \infty$ then $\bar{\bar{I}}(f_1+f_2) = \bar{\bar{I}}(f_1) + \bar{\bar{I}}(f_2)$. If $\bar{\bar{I}}(f_i) < \infty$ then $\iota(\{x \in X : f_i(x) = \infty\}) = 0$ and hence we may suppose that $f_i(x) < \infty$ for all $x \in X$ $(i=1,2)$. For each nonnegative integer n, we set $A_n^{(i)} := \left\{x \in X : \frac{1}{n} \leqq f_i(x) \leqq n\right\}$ and $f_n^{(i)} := \chi_{A_n^{(i)}} f_i$ $(i=1,2)$. Then we have $f_n^{(i)}(x) \to f_i(x)$ $(x \in X)$. Using the additivity of $\bar{\bar{I}}$ on $\mathscr{F}$ we obtain

$$\begin{aligned} \bar{\bar{I}}(f_1+f_2) &= \bar{\bar{I}}\left(\lim_n \,(f_n^{(1)} + f_n^{(2)})\right) = \lim_n \bar{\bar{I}}(f_n^{(1)} + f_n^{(2)}) \\ &= \lim_n \bar{\bar{I}}(f_n^{(1)}) + \lim_n \bar{\bar{I}}(f_n^{(2)}) = \bar{\bar{I}}(f_1) + \bar{\bar{I}}(f_2) . \end{aligned}$$

This completes the proof.

Addendum. We would like to thank the managing editor for bringing to our attention the following results of J. R. Baxter and R. V. Chacon, related to our Theorem 1.

Let M be a metric space and denote by $C^r(M)$ the set of continuous real valued functions on M. Let $\Phi : C^r(M) \to R$ be a functional such that:

(i) $\lim_{\|f\| \to 0} \Phi(f) = 0$;
(ii) $\Phi(f+g) = \Phi(f) + \Phi(g)$; if $fg = 0$;
(iii) $\Phi(f+\alpha) = \Phi(f) + \Phi(\alpha)$ for all $f \in C^r(M)$, $\alpha \in R$.

It has been shown in [J. R. Baxter, R. V. Chacon: Functionals on continuous functions, Pac. J. Math. 51, 355–362 (1974)] that if M has dimension no greater than one, Φ must be linear. In view of Theorem 1 it is quite surprising that if $M = [0,1] \times [0,1]$ then there exist nonlinear functionals on $C^r(M)$ which are bounded, continuous, monotone, and satisfy conditions (ii) and (iii) [J. R. Baxter, R. V. Chacon: Nonlinear functionals on $C([0,1] \times [0,1])$, Pac. J. Math. 48, 347–353 (1973)].

References

1. Banach, S.: Oeuvres. I. Warszawa: Éditions Scientifiques de Pologne 1967
2. Bourbaki, N.: Éléments de mathématique VI, Première partie. Integration. Paris: Hermann 1959
3. Hewitt, E., Ross, K.A.: Abstract harmonic analysis. I. Berlin Heidelberg New York: Springer 1963
4. Segal, I.E., Kunze, R.A.: Integrals and operators. Berlin Heidelberg New York: Springer 1978

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