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Nonlinear Elliptic Equations with Singular Boundary Conditions and Stochastic Control with State Constraints

1. The Model Problem

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Contents

I. Introduction

I.1. General Introduction

One of the primary goals of this paper is to study various models of stochastic control problems involving constraints on the state of the system (state constraints). And by following the dynamic programming approach this is equivalent to study some nonlinear second-order elliptic equations. Then, the state constraints lead to highly singular boundary conditions. A typical example would be: let Ω be a bounded, smooth domain in $\mathbb{R}^N$, we look for a solution $u \in C^2(\Omega)$ of

$$-\Delta u + |\nabla u|^p + \lambda u = f \quad \text{in} \quad \Omega \tag{1}$$

where $p > 1$, $\lambda > 0$, f is a given smooth function in Ω, and the boundary condition is given by

$$u(x) \to +\infty \quad \text{as} \quad \operatorname{dist}(x, \partial\Omega) \to 0\,, \tag{2}$$

(in fact, this boundary condition will correspond to the case $1 < p \leqq 2$).

We will show in this paper how such nonlinear, boundary value problems (1)–(2) can be solved and we will thus obtain existence, uniqueness and comparison results leading also to a complete solution of the stochastic control problem we are considering. It turns out that many cases have to be investigated and the results differ somewhat from one case to the other: typical behaviours involve the cases $1<p\leqq 2$, $p>2$, f blowing up near $\partial\Omega$, λ going to 0. Let us also mention that the methods introduced below allow us to treat more general nonlinear second-order elliptic equations, like more general quasilinear elliptic equations, Hamilton-Jacobi-Bellman equations, semilinear elliptic equations, first-order Hamilton-Jacobi equations, Monge-Ampère equations: in all those examples, singular boundary conditions may be encountered (and are even of a fundamental use) and we refer to Brézis [6], Crandall and Lions [7], Urbas [30], Simon [26–28], where such boundary conditions appear. And, our methods enable us to treat these equations with boundary conditions like (2).

I.2. Description of the Stochastic Control Problems

The basic model problem we are considering is a stochastic control problem where the *state of the controlled system is a diffusion process* and a typical example is the solution of the following *stochastic differential equation*

$$dX_t = a_t dt + dB_t, \qquad X_0 = x \in \mathbb{R}^N, \tag{3}$$

where B_t is a standard Brownian motion [in some probability space (Ω, F, F_t, P)...] and where a_t is the control process i.e. a progressively measurable stochastic process that we may choose as we wish (taking possibly into account restrictions on the controls such that, for instance, a_t takes its values into a given set A...). A very important, particular class of controls is given by the so-called feedback controls i.e., given a function $a(\cdot)$, one looks for a solution of

$$dX_t = a(X_t)dt + dB_t, \qquad X_0 = x. \tag{4}$$

This general class of problems occur in many contexts: however, depending on the particular examples of applications, it is possible to classify those problems in the following categories. For some problems, the state process X_t may take its value in $\mathbb{R}^N$ without any restriction while in other problems the state X_t should remain in some given region $\bar{\Omega}$. In the latter case, the model is to be complemented with boundary prescriptions in case the process reaches or crosses the boundary $\partial\Omega$. Let us immediately mention that if Ω is bounded, and a_t or $a(\cdot)$ are bounded then *for all* $t>0$ $P(X_t \in \partial\Omega)>0$. The most usual models in stochastic control theory yield the following boundary prescriptions: in the case of the so-called *exit problems*, one considers the first exit time of X_t from $\bar{\Omega}$ (or the first hitting time of X_t on $\partial\Omega$) and the process is stopped at this time. The other standard model consists in a boundary mechanism which prevents the state process from escaping from $\bar{\Omega}$; the simplest of which is the *reflecting boundary condition.*

Now, at least for deterministic problems, it is well-known that another way to enforce state constraints (i.e. $X_t \in \bar{\Omega}$) is simply to restrict our attention to controls such that X_t remains in $\bar{\Omega}$ or in Ω. In the case of (nondegenerate) stochastic models like (3) or (4), this possibility does not seem to have been explored. And this is precisely the type of problems we have in mind. In view of a remark made above, it is clear enough that in order to constrain a Brownian motion in a bounded domain Ω we need to use unbounded drifts a_t or $a(\cdot)$: in other words, we will have to choose feedbacks controls which, roughly speaking, push back the state process inside Ω when it gets near $\partial\Omega$ and with an intensity which blows up at the boundary. To be more specific, we will consider the class $\mathscr{A}$ of feedback controls i.e. of, say, continuous functions on Ω, $a(\cdot)$ such that the solution X_t of (4) stays in Ω with probability 1 for all $t \geqq 0$, (and for all initial points $x \in \Omega$).

Once, admissible control processes and thus state processes have been specified we may now describe a typical example of the optimal control problems we want to study. For each $a \in \mathscr{A}$, we will consider for example the *following cost function*

$$J(x,a) = E \int_0^\infty \left\{ f(X_t) + \frac{1}{q} |a_t|^q \right\} e^{-\lambda t} dt, \qquad \forall x \in \Omega, \tag{5}$$

where $q > 1$, f is a given function on Ω say bounded from below and $\lambda > 0$ is a given parameter – the so-called *discount factor*, and where we denote by $a_t = a(X_t)$. Let us emphasize that this particular form of the *running cost* $g(x,a) = f(x) + \frac{1}{q} |a|^q$ is by no means essential for the analysis which follows: it just provides a simple but general enough model problem. Let us also mention that this choice of cost functions corresponds to the so-called *infinite horizon* problems and that other cases are considered in this paper.

Finally, we wish to minimize J i.e. we want to determine the *value function* (or Bellman function):

$$u(x) = \inf_{a \in \mathscr{A}} J(x,a), \qquad \forall x \in \Omega \tag{6}$$

and optimal (feedback) controls $\underset{\sim}{a}$ such that $u(x) = J(x, \underset{\sim}{a})$.

I.3. Description of the Associated Boundary Value Problem

We want now, in this section, to follow the heuristic dynamic programming approach to such optimal stochastic control problems: the dynamic programming argument (which can be viewed as a modern, extended version of Hamilton-Jacobi-Carathéodory theories for problems in the calculus of variations), leads to a nonlinear partial differential equation. More precisely, the dynamic programming principle, due to R. Bellman, indicates that the value function u given by (6) should satisfy the following second-order, quasilinear, elliptic equation

$$-\frac{1}{2} \Delta u + \frac{1}{p} |\nabla u|^p + \lambda u = f \quad \text{in} \quad \Omega, \tag{7}$$

where p is the conjugate exponent of q i.e. $p = \dfrac{q}{q-1}$. In fact, such a claim, even if we forget the heuristic aspect of Bellman's derivation of (7) is by no means obvious here, in view of the restriction to feedback controls and of the state constraints. But nevertheless (7) is to be expected for the value function u. This equation is a very particular case of the so-called Hamilton-Jacobi-Bellman equations. And at least for problems like exit problems or the ones corresponding to reflecting boundary conditions (as described in the preceding section), a rigorous derivation of the

Hamilton-Jacobi-Bellman equation and the analysis of such nonlinear p.d.e. are now available: see Fleming and Rishel [8]; Bensoussan and Lions [2, 3]; Krylov [11, 12]; Lions [16–18]; Lions and Trudinger [24, 25] and the bibliography therein.

Let us add to this general description that the exit problems lead to Dirichlet type boundary conditions like

$$u=\varphi \quad \text{on} \quad \Omega, \tag{8}$$

where φ is the exit cost i.e. the price to be paid for hitting the boundary at a point x of $\partial\Omega$. On the other hand, reflecting type boundary conditions lead to Neumann (or oblique derivative) type boundary conditions like for instance

$$\frac{\partial u}{\partial n}=\psi \quad \text{on} \quad \partial\Omega, \tag{9}$$

where n is the unit outward normal to $\partial\Omega$ and ψ is the reflection cost i.e. the price to be paid for reflecting on the boundary $\partial\Omega$ at the point x.

Finally, let us mention that another aspect of Bellman's dynamic programming argument is a rule for finding an optimal feedback control which in the case of (7) reduces to the choice

$$a(x)=-|\nabla u|^{p-2}\nabla u(x) \quad \text{for} \quad x\in\Omega. \tag{10}$$

Now, we go back to the state-constraints problem described in the preceding section and we ask ourselves the following question: what is the boundary condition (or any other characterization at $\partial\Omega$) we may expect for the value function u given by (6)? From the above considerations it is tempting to say that to discourage hitting the boundary we should impose an infinite exit cost or reflection cost i.e.

$$u(x)\to+\infty \quad \text{as} \quad \operatorname{dist}(x,\partial\Omega\to 0 \tag{11}$$

or

$$\frac{\partial u}{\partial n}(x)\to+\infty \quad \text{as} \quad \operatorname{dist}(x,\partial\Omega)\to 0 \tag{12}$$

[where $n(x)$ is defined near $\partial\Omega$ by $-\nabla(\operatorname{dist}(x,\partial\Omega))$]. More sophisticated formulations, which are also very natural from the control viewpoint, are: u is the maximum solution (or even subsolution) of (7); or: u is the upperenvelope of bounded solutions of (7).... Finally, for readers experienced with viscosity solutions, a possible form of the boundary condition could be

$$u-\varphi \quad \text{achieves its minimum over} \quad \Omega \tag{13}$$

for all $\varphi\in C^2(\bar\Omega)$ [or $C^{1,1}(\bar\Omega)$, or $C^1(\bar\Omega)$, or even $C^{0,1}(\bar\Omega)$]: this "viscosity formulation" will be explained below in Sect. IVV, see also Lions [17, 29] for the deterministic case.

It turns out, and the precise results are given in the next section, that if the latter formulations are always true, the choice between the boundary conditions (11) or (12) requires some careful analysis and will in fact depend on the behaviour of f

near $\partial\Omega$ and on q. This can easily be "justified" by a vague economical argument: if f blows up fast enough near $\partial\Omega$ of if q is large (remember that a has to blow up near $\partial\Omega$) then the cost functions will blow up at $\partial\Omega$ and so will u. Then, we should expect (11). On the other hand, if f, say, is bounded and if q is near 1 then it does not cost much to drive the state off $\partial\Omega$ and we may expect now u to be bounded on Ω. On the other hand, recalling Bellman's rule (10) for the optimal control and the fact that a cannot remain bounded if we want X_t to stay in Ω, we should expect that some condition like (12) holds. Of course, the reason for which we insist on conditions like (11) or (12) compared to a "maximum solution" characterization is because of the specific information contained in those formulations (which could turn out to be crucial for numerical purposes). Finally, note that $p=1$ is excluded in the p.d.e. results (see next Sect. I.4). This corresponds to the fact that it is impossible to force state-constraints with bounded controls. All these heuristic considerations will find their mathematical counter parts in the results presented in the next section.

I.4. Short Review of the Results

In this section, we present some of the results obtained in this paper on the simple example of the model equation (1) [equivalent to (7) after an obvious scaling]. In doing so, we follow the order of the sections below. To simplify the presentation we will always assume at least that $f \in C^1(\Omega)$, is bounded *from below*. We will denote by $d(x)=\operatorname{dist}(x,\partial\Omega)$ for all $x \in \bar{\Omega}$.

We begin with the case when the running cost f is not too large, while the other term in the cost function is quite large since we will assume $1<p\leqq 2$ i.e. $q\geqq 2$.

Theorem I.1. *Assume that* $1<p\leqq 2$ *and that* f *satisfies*

$$\lim\{f(x)d(x)^q/d(x)\to 0_+\}=C_1\geqq 0. \tag{14}$$

Then, there is a unique solution $u\in C^2(\Omega)$ *of* (1) *such that* $u(x)\to+\infty$ *as* $d(x)\to 0_+$. *In addition, any solution* $v\in C^2(\Omega)$ *of* (1) *satisfies:* $u\geqq v$ *on* Ω. *Finally, if* C_0 *is the unique positive root of* $\left(\frac{2-p}{p-1}\right)^p C_0^p-\frac{2-p}{(p-1)^2}C_0-C_1=0$ *if* $p<2$, $C_0^2-C_0-C_1=0$ *if* $p=2$, *then* u *satisfies*

$$\left.\begin{aligned}&\lim\left\{u(x)d(x)^{\frac{2-p}{p-1}}/d(x)\to 0_+\right\}=C_0 \quad \text{if} \quad p<2\\ &\lim\{u(x)\,|\operatorname{Log} d(x)|^{-1}/d(x)\to O_+\}=C_0 \quad \text{if} \quad p=2.\end{aligned}\right\} \quad \square \tag{15}$$

We now turn to the case when both terms in the running cost are not too large: in particular we assume that $p>2$ i.e. $1<q\leqq 2$.

Theorem I.2. *Assume that* $p>2$ *and that* f *satisfies*

$$\lim\{f(x)d(x)/d(x)\to 0_+\}=0, \quad \textit{for some} \quad \beta\in(0,p). \tag{16}$$

Then, all solutions $v\in C^2(\Omega)$ *of* (1) *bounded from below are bounded and may be extended continuously to* $\bar{\Omega}$. *And there exists a maximum solution* $u\in C^2(\Omega)$ *of* (1). *In* *addition, u satisfies*

$$\liminf_{y\in\Omega,\, y\to x} \{u(y)-u(x)\}\,|y-x|^{-\alpha}<0, \quad \textit{for all} \quad x\in\partial\Omega \tag{17}$$

where $\alpha=(p-2)/(p-1)$.

Furthermore, if $\liminf\{f(x)d(x)^{\gamma}/d(x)+0_+\}>0$ *for some* $\gamma\in(q,\beta)$, *then* (17) *holds with* $\alpha=1-\gamma/p$. □

Also, if additional assumptions on Ω or f are made, we are able to sharpen (17) or prove (12) [or even sharper estimates then (17) and (12)...].

The next case concerns the situation when the running cost f is blowing up near the boundary very fast. We have the

Theorem I.3. *Assume that f satisfies*

$$\liminf\{f(x)d(x)^{\beta}/d(x)\to 0_+\}>0, \quad \textit{for some} \quad \beta\geqq\max(p,q). \tag{18}$$

Then, any solution $v\in C^2(\Omega)$ *of* (1) *bounded from below converges to* $+\infty$ *as* $d(x)$ *goes to* 0. *In addition, such a solution is unique if* (18) *is replaced by*

$$\lim\{f(x)d(x)^{\beta}/d(x)\to 0_+\}=C_1>0, \quad \textit{for some} \quad \beta\geqq\max(p,q) \tag{18'}$$

and this solution, denoted by u, satisfies

$$\lim\{u(x)d(x)^{\alpha}/d(x)\to 0_+\}=C_0, \tag{19}$$

where $d(x)^{\alpha}$ *is replaced by* $|\mathrm{Log}\, d(x)|^{-1}$ *if* $\beta=p\geqq q$; $\alpha=\dfrac{\beta}{p}-1$ *and* $C_0=\left(\dfrac{C_1}{\alpha}\right)^{1/p}$ *if* $\beta>\max(p,q)$; $C_0=C_1^{1/p}$ *if* $\beta=p>2$; $C_0=(1+C_1)^{1/2}$ *if* $\beta=p=2$. □

Roughly speaking, the combination of Theorems I.1–I.3 cover all possible situations. One way of unifying the above results is by the use of the viscosity formulation of the various boundary conditions encountered above namely

$$u-\varphi \quad \text{achieves its minimum over } \Omega, \quad \text{for all} \quad \varphi\in C^2(\bar{\Omega}). \tag{20}$$

Theorem I.4. *Assume that* $p>1$ *and* $\beta>0$ *and that either f is bounded or* $f(x)d(x)^{\beta}$ *converges to a positive constant as* $d(x)$ *converges to* 0_+. *Then, there is a unique* $u\in C^2(\Omega)$ *solution of* (1) *satisfying* (20). □

This is a nonexhaustive list of results since we will consider below many related questions like the stochastic interpretation of the above solutions, the existence of optimal controls, the ergodic problem, i.e. $\lambda\to 0_+$, the approximation of such solutions, extensions to more general data f or Hamiltonians. Finally, we will also briefly explain how the techniques we introduce allow us to treat similar boundary conditions for other types of nonlinear equations.

I.5. Organization of the Paper

As usual in stochastic control problems, various strategies are possible. One can use p.d.e. methods to derive the existence of a smooth solution of the associated HJB equation – here a second order quasilinear elliptic equation with strong nonlinearities in the gradient and singular boundary conditions. The uniqueness question may be solved directly by p.d.e. methods or by checking that any solution is the value function. Finally, one builds an optimal control using, whenever it is possible, the solution of the HJB equation. This is why some sections below deal with purely p.d.e. questions while others are concerned with the stochastic interpretation. Another distinction is made below between what we call the model problem (1) and more general equations. This artificial distinction is made only to simplify the exposition. In fact, in all sections below, we adopt a layered presentation with gradual generalizations where we just explain the required modifications of proofs.

II. Subquadratic Hamiltonians

We will be dealing here with (1) in the case when $1<p\leqq 2$.

II.1. Bounded Data

We begin with the case of bounded data i.e. we assume that $f\in L^\infty(\Omega)$.

Theorem II.1. *There is a unique solution* $u\in W^{2,r}(\Omega)$ $(\forall r<\infty)$ *of* (1) *such that* $u(x)\to+\infty$ *as* $d(x)\to 0_+$. *In addition, if* $C_0=(p-1)^{\frac{p-2}{p-1}}(2-p)^{-1}$ *when* $p<2$, $C_0=1$ *when* $p=2$, *then* (15) *holds. Finally, let* $v\in L^1_{\mathrm{loc}}(\Omega)$ *satisfy*

$$-\Delta v+p|\xi|^{p-2}\xi\cdot\nabla v+\lambda v\leqq f+(p-1)|\xi|^p \quad in \quad \mathscr{D}'(\Omega), \qquad \forall\xi\in\mathbb{R}^n \tag{21}$$

then $v\leqq u$ *a.e. in* Ω; *in other words, u is the maximum* L^1_{loc} *subsolution.* □

Corollary II.1. *Let* $f_1, f_2\in L^\infty(\Omega)$ *and let* u_1, u_2 *be the corresponding solutions of* (1) *which go to* $+\infty$ *on* $\partial\Omega$. *Then, we have*

$$\sup_\Omega\,(u_1-u_2)^+\leqq\frac{1}{\lambda}\sup_\Omega\,(f_1-f_2)^+ . \quad \square$$

Proof of Corollary II.1. $u_1-\frac{1}{\lambda}\sup\limits_\Omega\,(f_1-f_2)^+$ is a subsolution of (1) with f replaced by f_2 so by Theorem II.1 $u_1\leqq u_2+\frac{1}{\lambda}\sup\limits_\Omega\,(f_1-f_2)^+$. □

The proof of Theorem II.1 is unfortunately a bit longer and we split it into several parts. First (step 1), we compute the explosion rate of such a solution and this trivial computation leads to families of super and subsolutions. Next (step 2), we build a minimum and a maximum "explosive" solution which have the same leading behaviour near the boundary. Then (step 3), we prove the uniqueness and (15). Finally (step 4), we prove the "maximal subsolution" property.

Step 1. It is reasonable to try to obtain the leading term in an expansion of a solution of (1) blowing up at the boundary by the following ansatz near the boundary: $u(x)\simeq C_0 d(x)^{-\alpha}$. The most explosive term in $[-\Delta u+|\nabla u|^p+\lambda u-f]$ is then

$$-C_0\alpha(\alpha+1)d^{-\alpha-2}+C_0^p\alpha^p d^{-(\alpha+1)p},$$

where we used (twice) the fact that $|\nabla d|=1$ near the boundary (in fact, as it is well-known: $|\nabla d|=1$ at each differentiability point of d, and d is smooth near the boundary if Ω is smooth). This leads to the choices

$$\alpha=\frac{2-p}{p-1}, \quad C_0=\alpha^{-1}(\alpha+1)^{1/(p-1)} \quad \text{if} \quad p<2.$$

Of course, if $p=2$ one replaces $C_0 d^{-\alpha}$ by $-C_0 \operatorname{Log} d$ and one finds $C_0=1$.

In order to use in a meaningful way the above formal consideration, we build two families of "approximations of $C_0 d^{-\alpha}$", each of which is a two-parameter family, where we first denote by d any smooth function, say $C^2(\bar{\Omega})$, on $\bar{\Omega}$ equal to $\operatorname{dist}(x, \partial\Omega)$ near the boundary, say for $\operatorname{dist}(x,\partial\Omega)\leqq\delta_0$ with $\delta_0>0$. Then, we introduce for $\varepsilon, \delta\geqq 0$

$$\left.\begin{aligned}\bar{w}_{\varepsilon,\delta}&=(C_0+\varepsilon)(d-\delta)^{-\alpha}+C_\varepsilon\\ \underline{w}_{\varepsilon,\delta}&=(C_0-\varepsilon)(d+\delta)^{-\alpha}-C_\varepsilon\end{aligned}\right\} \tag{22}$$

for some large constant C_ε to be determined. Of course, if $p=2$ then $(d\pm\delta)^{-\alpha}$ is replaced by $-\operatorname{Log}(d\pm\delta)$. Notice also that if $\underline{w}_{\varepsilon,\delta}$ is defined and smooth on Ω, $\bar{w}_{\varepsilon,\delta}$ is only defined on $\Omega_\delta=\{x\in\Omega, \operatorname{dist}(x,\partial\Omega)>\delta\}$ at least for $\delta<\delta_0$ (δ_0 to be choosen small enough; $0\leqq\delta<\delta_0$ will always be assumed in this proof). In fact, it will be handy to consider d as a smooth function on $\mathbb{R}^N$, say $C^2(\mathbb{R}^N)$, such that: $d(x)=\operatorname{dist}(x,\partial\Omega)$ if $x\in\bar{\Omega}$, $\operatorname{dist}(x,\partial\Omega)\leqq\delta_0$; $d(x)\geqq\delta_0$ if $\operatorname{dist}(x,\partial\Omega)\geqq\delta_0$, $x\in\Omega$; $d(x)=-\operatorname{dist}(x,\partial\Omega)$ if $x\notin\Omega$ and $\operatorname{dist}(x,\partial\Omega)\leqq\delta_0$; $d(x)\leqq-\delta_0$ if $\operatorname{dist}(x,\partial\Omega)\geqq\delta_0$, $x\notin\bar{\Omega}$. Observe of course that $|\nabla d|=1$ in $\{\operatorname{dist}(x,\partial\Omega)\leqq\delta_0\}$ and that $d(x)=-\delta=\operatorname{dist}(x,\partial\Omega_\delta)$ if $\operatorname{dist}(x,\partial\Omega)\leqq\delta_0$ while $d(x)+\delta=\operatorname{dist}(x,\partial\Omega^\delta)$ if $\operatorname{dist}(x,\partial\Omega)\leqq\delta_0$, where

$$\Omega^\delta=\{x\in\mathbb{R}^N, \operatorname{dist}(x,\bar{\Omega})\leqq\delta\}=\{x\in\mathbb{R}^N/d(x)\geqq-\delta\}.$$

So that, we may consider $\underline{w}_{\varepsilon,\delta}$ to be defined on Ω^δ. (Notice that such a function d exists as soon as Ω is open bounded and has a C^2-regular boundary $\partial\Omega$.)

We conclude these preliminaries with the following computations

$$\begin{aligned}&-\Delta\bar{w}_{\varepsilon,\delta}+|\nabla\bar{w}_{\varepsilon,\delta}|^p+\lambda\bar{w}_{\varepsilon,\delta}-f\\ &\quad=-\alpha(\alpha+1)(C_0+\varepsilon)(d-\delta)^{-\alpha-2}|\nabla d|^2+\alpha(C_0+\varepsilon)(d-\delta)^{-\alpha-1}\Delta d\\ &\qquad+\alpha^p(C_0+\varepsilon)^p(d-\delta)^{-p(\alpha+1)}|\nabla d|^p+\lambda(C_0+\varepsilon)(d-\delta)^{-\alpha}+\lambda C_\varepsilon-f.\end{aligned}$$

Recalling that $\alpha+2=(\alpha+1)p$ and $\alpha^p C_0^p=\alpha(\alpha+1)C_0$, we deduce easily for $\varepsilon\leqq1$, $\delta\leqq\delta_0$

$$-\Delta\bar{w}_{\varepsilon,\delta}+|\nabla\bar{w}_{\varepsilon,\delta}|^p+\lambda\bar{w}_{\varepsilon,\delta}-f\geqq\nu\varepsilon(d-\delta)^{-\alpha-2}+\lambda C-C(1+(d-\delta)^{-\alpha-1})$$

for some $\nu>0$, $C\geqq0$. And we can choose C_ε large enough in order to find

$$-\Delta\bar{w}_{\varepsilon,\delta}+|\nabla\bar{w}_{\varepsilon,\delta}|^p+\lambda\bar{w}_{\varepsilon,\delta}\geqq f \quad \text{in} \quad \Omega_\delta. \tag{23}$$

Similarly, one shows that C_ε can be choosen large enough to have:

$$-\Delta\underline{w}_{\varepsilon,\delta}+|\nabla\underline{w}_{\varepsilon,\delta}|^p+\lambda\underline{w}_{\varepsilon,\delta}\leqq f \quad \text{in} \quad \Omega^\delta. \tag{24}$$

Step 2. Building a minimum "explosive" solution is easy in the subquadratic case. Indeed, one solves

$$-\Delta u_R + |\nabla u_R|^p + \lambda u_R = f \quad \text{in} \quad \Omega, \qquad u_R \in W^{2,r}(\Omega) \quad (\forall r < \infty) \tag{25}$$

with boundary conditions going to infinity (as $R \to \infty$) like for instance

$$u_R = R \quad \text{on} \quad \partial\Omega \tag{26}$$

or

$$u_R = \underline{w}_{\varepsilon, 1/R} \quad \text{on} \quad \partial\Omega, \quad \text{for any fixed} \quad \varepsilon > 0. \tag{27}$$

Since $p \leqq 2$, the existence follows from standard results on subquadratic quasilinear equations (see for example Amann and Crandall [1]). In view of the maximum principle (we have to use here the slightly more general form of maximum principle in Sobolev spaces – see for example Bony [5] and Lions [23]) we deduce in the case of (27) for example

$$\underline{w}_{\varepsilon, 1/R} \leqq u_R \leqq u_{R'} \leqq \bar{w}_{\varepsilon'} \quad \text{if} \quad 0 < R < R', \quad \forall \varepsilon' > 0$$

and where $\bar{w}_\varepsilon = \bar{w}_{\varepsilon, 0}$. The last inequality of this string comes from the maximum principle provided we observe that $u_{R'} < \bar{w}_\varepsilon$ near $\partial\Omega$ since $\bar{w}_\varepsilon$ blows up at the boundary.

Hence, u_R is bounded in L^∞_{loc}. This combined with (25) implies that u_R is bounded in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r < \infty)$: this can be deduced either from [1] using again the fact that we are dealing with a subquadratic Hamiltonian or by using the gradient estimates of the appendix (see Lions [16, 19]) which yield bounds in $W^{1,\infty}_{\text{loc}}(\Omega)$ and then using (25). Anyway, u_R converges (as $R \to \infty$) to a solution $\underline{u}$ of (1) in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r < \infty)$ which also satisfies $\underline{w}_\varepsilon \leqq \underline{u} \leqq \bar{w}_{\varepsilon'}$ $\forall \varepsilon' > 0$. Next, we claim that $\underline{u} \geqq \underline{w}_{\varepsilon'}$ for all $\varepsilon' > 0$. Indeed for any $R' > 0$, we can find R such that $\underline{w}_{\varepsilon', 1/R'} \leqq \underline{w}_{\varepsilon', 1/R}$ and letting R' go to $+\infty$, we conclude easily.

We now claim that $\underline{u}$ is the minimum "explosive" solution of (1). Indeed, let u be another solution of (1) in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r < \infty)$ such that $u \to +\infty$ as $d(x) \to 0_+$ then by maximum principle $u \geqq u_R$ in Ω and thus passing to the limit we obtain $u \geqq \underline{u}$ in Ω.

To build a maximum explosive solution, we consider the preceding minimum explosive solution $\underline{u}_\delta$ in Ω_δ and we let δ go to 0. Recall that we have

$$(C_0 - \varepsilon)(d + \delta)^{-\alpha} - C_\varepsilon \leqq \underline{u}_\delta \leqq \bar{w}_{\varepsilon, \delta}, \qquad \forall \varepsilon > 0$$

and clearly enough $\underline{u}_\delta \geqq \underline{u}_{\delta'}$ if $0 < \delta' < \delta$. Therefore, passing to the limit, exactly as above we find a solution $\bar{u}$ of (1) such that

$$\underline{w}_\varepsilon \leqq \bar{u} \leqq \bar{w}_\varepsilon \quad \text{in} \quad \Omega.$$

The fact that $\bar{u}$ is the maximum explosive solution is proved by using again the maximum principle to show (with the above notations)

$$u \leqq \underline{u}_\delta \to \bar{u} \quad \text{as} \quad \delta \text{ goes to } 0.$$

In conclusion, we found solutions $\underline{u}$, $\bar{u} \in W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r < \infty)$ of (1) such that

$$\underline{w}_\varepsilon \leqq \underline{u} \leqq u \leqq \bar{u} \leqq \bar{w}_\varepsilon \quad \text{in} \quad \Omega, \quad \text{for all} \quad \varepsilon > 0, \tag{28}$$

where u is any solution of (1) in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r < \infty)$ such that $u \to \infty$ as $d \to 0_+$.

Step 3: Uniqueness. It is of course enough to show that $\underline{u} \equiv \bar{u}$ in Ω. We first observe that (28) implies that $\bar{u}(x)(\underline{u}(x))^{-1}$ converges to 1 as $d(x) \to 0_+$. Therefore, if we denote by $m = \inf_{\Omega} f(x)$, we deduce that for all $\theta \in (0,1)$

$$\underline{u}(x) > \theta\bar{u}(x) + (1-\theta)m/\lambda \quad \text{in a neighbourhood of} \quad \partial\Omega .$$

In addition, $w = \theta\bar{u} + (1-\theta)m/\lambda$ satisfies in Ω

$$-\Delta w + |\nabla w|^p + \lambda w \leqq \theta f + (1-\theta)m \leqq f .$$

Therefore, we deduce easily by the maximum principle

$$w \leqq \underline{u} \quad \text{in} \quad \Omega$$

and we conclude letting θ go to 1.

Step 4. We wish to prove that the unique explosive solution $\underline{u}$ of (1) that we built above is also the maximum L^1_{loc} subsolution. Let $v \in L^1_{\mathrm{loc}}(\Omega)$ satisfy (21). In order to avoid some rather unpleasant technicalities, we begin with the case $f \in C(\bar{\Omega})$: in that case, we smooth v by convolution i.e. we consider $v_n = v * \varrho_n$ where $\varrho \in \mathcal{D}(\mathbb{R}^N)$, $0 \leqq \varrho$, $\int_{\mathbb{R}^N} \varrho dx = 1$, $\operatorname{supp}(\varrho) \subset B_1$ and $\varrho_n = n^N \varrho(n\,\cdot)$. Then, if $\delta > \frac{1}{n}$, we find easily

$$-\Delta v_n + |\nabla v_n|^p + \lambda v_n \leqq f * \varrho_n \quad \text{in} \quad \Omega_\delta$$

and $f * \varrho_n \leqq f + \varepsilon_n$ where $\varepsilon_n \underset{n}{\to} 0$. Therefore, we deduce

$$\left(v_n - \frac{\varepsilon_n}{\lambda_n}\right) \leqq \underline{u}_\delta \quad \text{if} \quad \delta > \frac{1}{n}$$

and we conclude letting n go to $+\infty$ and then δ go to 0_+.

If $f \in L^\infty(\Omega)$, we obtain by the above proof that $v_n \leqq \underline{u}^n_\delta$ where $\underline{u}^n_\delta$ is the explosive solution in Ω_δ corresponding to $f * \varrho_n$ $\left(\text{still if } \delta > \frac{1}{n}\right)$. In addition, the proof made above also shows that $\underline{u}^n_\delta$ is bounded in $L^\infty_{\mathrm{loc}}(\Omega_\delta)$ and thus in $W^{2,r}_{\mathrm{loc}}(\Omega_\delta)$ $(\forall r < \infty)$ since $f * \varrho_n$ is bounded in $L^\infty(\Omega)$: in fact, one may even choose C_ε such that

$$(C_0 - \varepsilon)(d-\delta)^{-\alpha} - C_\varepsilon \leqq \underline{u}^n_\delta \leqq (C_0 + \varepsilon)(d+\delta)^{-\alpha} + C_\varepsilon$$

(with the usual modifications if $p=2$). Then, we may pass to the limit as n goes to $+\infty$ and $\underline{u}^n_\delta$ (or subsequences) converges to a solution of (1) in Ω_δ thus below $\underline{u}_\delta$ (in fact it is $\underline{u}_\delta$ because the above inequality shows it blows up at $\partial\Omega_\delta$). Therefore, $v \leqq \underline{u}_\delta$ in Ω_δ and we conclude letting δ go to 0. □

Remark II.1. One may deduce from the above arguments the "continuity" of the explosive solution with respect to Ω, p or f (for the weak $L^\infty *$ topology).

Remark II.2. By a convenient (and technical) variation of the above method one can show that it is possible to replace $f \in L^\infty(\Omega)$ by $f \in L^p_{\mathrm{loc}}(\Omega)$ $\left(p > \frac{N}{2}\right)$, f bounded from below and f bounded near $\partial\Omega$.

II.2. General Data

We now wish to allow some data f which may not be bounded near $\partial\Omega$.

Theorem II.2. *Let $f \in L^\infty_{loc}(\Omega)$, assume that f is bounded from below and that f satisfies* (14). *Then, Theorem II.1 still holds provided one replaces C_0 by the unique positive solution of the equation $\left(\frac{2-p}{p-1}\right)^p C_0^p - \frac{2-p}{(p-1)^2} C_0 - C_1 = 0$ if $p<2$, $C_0^2 - C_0 - C_1 = 0$ if $p=2$.*

Proof. We only present the main modifications in the preceding proof. With the above new value of C_0, one builds exactly as in the proof of Theorem II.1 a maximum explosive solution $\bar{u}$ of (1) such that

$$(C_0-\varepsilon)d^{-\alpha} - C_\varepsilon \leqq \bar{u} \leqq (C_0+\varepsilon)d^{-\alpha} + C_\varepsilon \quad \text{in} \quad \Omega, \quad \forall \varepsilon > 0. \tag{29}$$

The above equation for C_0 comes into the picture when making the formal computations of Step 1 and balancing the various leading terms in $d^{-\alpha-2} = d^{-q}$. The only modification in the proof of Theorem II.1 consists in proving that there exists a minimum explosive solution $\underline{u}$ which also satisfies (29). To this end, we observe that $\underline{w}_{\varepsilon,\delta}$ is a subsolution of (1) when Ω is replaced by Ω^δ and f is replaced by

$$f_\delta = \min(f, C_2 + C_3(d+\delta)^{-q}) \quad \text{in} \quad \Omega, \quad = C_2 + C_3(d+\delta)^{-q} \quad \text{in} \quad \Omega^\delta - \Omega,$$

where C_3, C_2 are positive constants such that $C_3 > C_1$, $C_2 + C_3 d^{-q} > f$ in Ω. Obviously, $f_\delta \in L^\infty(\Omega)$. Therefore, by Theorem II.1 and its proof, there exists a unique explosive solution u_δ of (1) with f replaced by f_δ, [obtained by an increasing limit of solutions of (1) with finite boundary values] and $u_\delta \geqq \underline{w}_{\varepsilon,\delta}$. Since $f \geqq f_\delta$, any explosive solution of (1) is above u [use the maximum principle with the approximating bounded solutions of (1)] and thus in particular $\bar{u} \geqq u_\delta$. From this, we deduce easily letting δ go to 0 the existence of a minimum explosive solution of (1) $\underline{u}$ satisfying (29).

Remark II.3. The analogues of Remarks II.1–II.2 still hold: notice only that the stability with respect to f holds with respect to the weak $*$ L^∞ topology provided the data f are uniformly bounded from below and, satisfy (14) with C_1 bounded and $f(x) \leqq C d^{-q} + C$ for some $C \geqq 0$.

Remark II.4. The proof also shows that if $\bar{w}$ is a supersolution of (1) which blows up on $\partial\Omega$ i.e.

$$-\Delta\bar{w} + |\nabla\bar{w}|^p + \lambda\bar{w} \geqq f \quad \text{in} \quad \Omega, \qquad \bar{w} \to 0 \quad \text{as} \quad d(x) \to 0_+$$

then $\bar{w} \geqq u$.

Remark II.5. If we allow f to go to $-\infty$ near $\partial\Omega$ (or some points of $\partial\Omega$) then the situation is a bit more complex. Let $f \in L^\infty_{loc}(\Omega)$, if we assume (14) with $C_1 = 0$ then the above result is no longer true. In that case, there still exists a maximum explosive solution which behaves as $C_0 d^{-\alpha}$ and is the unique solution going to $+\infty$ as $C_0 d^{-\alpha}$. However, in general, there may exist other solutions going to $+\infty$

less rapidly: indeed, consider

$$f(x)=\frac{\Delta d}{d}-\frac{|\nabla d|^2}{d^2}+\frac{|\nabla d|^p}{d^p}-\lambda\,\mathrm{Log}\,d.$$

If $1<p<2$, f behaves like $-\frac{1}{d^2}$ near $\partial\Omega$ and thus satisfies (14). And notice that $u(x)=-\mathrm{Log}\,d(x)$ is then a solution of (1) which goes to $+\infty$ as $d(x)$ goes to 0.

If we assume (14) and $C_1>0$, then f is bounded from below and Theorem II.2 applies. Now, if we assume (14) and $C_1<0$, then there are two positive solutions C_0 of the equation stated in Theorem II.2 say $0<C_0^-<C_0^+$ and $C_0^-\to 0$, $C_0^+\to C_0$ as $C_1\to 0_-$. Again, there exists a maximum explosive solution of (1) behaving near $\partial\Omega$ as $C_0^+d^{-\alpha}$ and it is the unique such solution. But there also exists in general another explosive solution of (1) behaving near $\partial\Omega$ as $C_0^-d^{-\alpha}$: for instance, consider $f=-\Delta w+|\nabla w|^p+\lambda w$ where $w=C_0^-d^{-\alpha}$. □

II.3. Asymptotic Expansions Near the Boundary

In this section, we want to precise a bit the behaviour near the boundary of solutions which blow up at the boundary. Even if we will not present a complete asymptotic expansion near the boundary (which should include $[q]-1$ singular terms plus a bounded term where $[q]$ denotes the integer part of q), the methods we use should give it and we leave the awful computations to a courageous reader. We will only prove the

Theorem II.3. *Let $f\in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below and assume that*

$$\lim\{f(x)d(x)^{q-1}/d(x)\to 0_+\}=0. \tag{30}$$

We denote by u the unique solution of (1) *in $W^{2,r}(\Omega)\,(\forall r<\infty)$ which goes to $+\infty$ on $\partial\Omega$. Then, if $p\in(\frac{3}{2},2]$ i.e. $q\in[2,3)$, $u-\frac{C_0}{d^\alpha}$ is bounded on Ω when $p<2$ while $u+\log d$ is bounded on Ω when $p=2$. Next, if $p\in(1,\frac{3}{2}]$ we set*

$$\begin{aligned} C_1(x)&=-\frac{1}{2}\frac{\alpha}{\alpha-1}C_0\Delta d(x) \quad \text{if} \quad p<\frac{3}{2},\\ C_1(x)&=-\frac{1}{2}C_0\Delta d(x) \qquad\quad \text{if} \quad p=\frac{3}{2}, \end{aligned} \tag{31}$$

and we have

$$\left.\begin{aligned} &\left\{u-\frac{C_0}{d^\alpha}\right\}d^{\alpha-1}\to C_1 \quad \text{as} \quad d\to 0_+ \quad \text{if} \quad p<\frac{3}{2},\\ &\left\{u-\frac{C_0}{d}\right\}|\log d|^{-1}\to C_1 \quad \text{as} \quad d\to 0_+ \quad \text{if} \quad p=\frac{3}{2}. \end{aligned}\right\} \tag{32}$$

Proof. We begin with the case $1<p\leq\frac{3}{2}$. In view of the results of the previous sections, it is enough build appropriate sub and supersolutions which blow up

near $\partial\Omega$. To this end, we consider

$$\left.\begin{array}{lll} w_\varepsilon^+ = \dfrac{C_0}{d^\alpha} + \dfrac{(C_1+\varepsilon)}{d^{\alpha-1}} + C_\varepsilon, & w_\varepsilon^- = \dfrac{C_0}{d^\alpha} + \dfrac{(C_1-\varepsilon)}{d^{\alpha-1}} - C_\varepsilon & \text{if} \quad p<\dfrac{3}{2}, \\ w_\varepsilon^- = \dfrac{C_0}{d} - (C_1+\varepsilon)\operatorname{Log} d + C_\varepsilon, & w_\varepsilon^- = \dfrac{C_0}{d} - (C_1-\varepsilon)\operatorname{Log} d - C_\varepsilon & \text{if} \quad p=\dfrac{3}{2}, \end{array}\right\} \tag{33}$$

where C_ε is a positive constant to be determined. Tedious computations show that, provided C_1 is given by (31) and C_ε is large enough, w_ε^+ (resp. w_ε^-) is a supersolution of (1) [resp. subsolution of (1)]. Therefore, $w_\varepsilon^- \leqq u \leqq w_\varepsilon^+$ in Ω for all $\varepsilon>0$ and (32) is proved.

Next, if $\frac{3}{2}<p\leqq 2$, we also want to build convenient sub and supersolutions. However, in this case, the choices are not straightforward as above. Indeed, recalling that $\alpha = \dfrac{2-p}{p-1}$ we choose

$$w_\varepsilon^+ = \frac{C_0}{d^\alpha} - (C_1+\varepsilon)d^{1-\alpha} + C_\varepsilon, \qquad w_\varepsilon^- = \frac{C_0}{d^\alpha} - (C_1-\varepsilon)d^{1-\alpha} - C_\varepsilon, \tag{34}$$

where

$$C_1 = -\frac{1}{2}\frac{\alpha}{1-\alpha} C_0 \Delta d \quad \text{if} \quad p<2, \qquad C_1 = -\frac{1}{2}\Delta d \quad \text{if} \quad p=2.$$

Again, one can check that w_ε^+, w_ε^- for conveniently large C_ε are sub and supersolutions of (1) and since they go to $+\infty$ at $\partial\Omega$ we deduce that $w_\varepsilon^- \leqq u \leqq w_\varepsilon^+$ in Ω and we conclude. □

Remark II.6. In the various bounds on the behaviour of explosive solutions near the boundary, it may seem strange that the leading terms are not continuous with respect to p (as p goes to 2 for example). Similarly, in (34) the term $d^{1-\alpha}$ vanishes and could seem to be irrelevant. However – and this fits well with the stochastic control interpretation – these questions disappear if we look for formal expansions of the gradient obtained by differentiating these expansions for the solution: indeed, in Theorem II.1, u behaves like

$$(p-1)^{\frac{p-2}{p-1}} \frac{1}{2-p} d(x)^{-\frac{2-p}{p-1}}$$

so $\nabla u(x)$ should behave like $-(p-1)^{-1/(p-1)}\nabla d(x) d^{-1/(p-1)}$ and when p goes to 2 this quantity goes to $-\nabla d(x) d^{-1}$ which is precisely the gradient of $-\log d$. A similar explanation holds for (34).

III. Infinite Boundary Conditions and Blowing up Data

In this section, we consider the case of data f blowing up at the boundary fast enough to force solutions of (1) bounded from below to blow up at the boundary. This also will yield some uniqueness results. The results of this section correspond to Theorem I.3.

III.1. Forced Infinite Boundary Conditions

Theorem III.1. *Assume that* $f \in L^\infty_{\text{loc}}(\Omega)$ *satisfies* (18). *Then, any solution* u *of* (1) $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ *which is bounded from below converges to* $+\infty$ *as* $d(x)$ *goes to* 0.

Remark III.1. The proof below may be adapted to treat the case of $f \in L^r_{\text{loc}}(\Omega)$ satisfying (18) with $r>N$.

Remark III.2. In general, there may exist solutions of (1) which are not bounded from below. For instance, take

$$f(x)=-\frac{\alpha C_0 \Delta d}{d^{\alpha+1}}+\frac{C_0\alpha(\alpha+1)}{d^{\alpha+2}}|\nabla d|^2+C_0^p\alpha^p|\nabla d|^p d^{-(\alpha+1)p}-\lambda C_0 d^{-\alpha}$$

with α, $C_0>0$, $u(x)=-\frac{C_0}{d^\alpha}$ is obviously a solution of (1) and f satisfies (18) with $\beta=\max((\alpha+1)p,\alpha+2)$. And it is easy to check that any $\beta>\max(p,q)$ can be reached with a convenient α [in fact even $\beta=\max(p,q)$] may be reached provided we replace $-C_0 d^{-\alpha}$ by $C_0\log d$ for $\beta=p\geqq q$. It is also worth noticing that such solutions may exist for linear equations like

$$-\Delta u+u=f \quad \text{in} \quad \Omega$$

provided f behaves like $\frac{C_1}{d^\beta}$ with $\beta\geqq 2$ near the boundary.

Proof of Theorem III.1. Even if the arguments are very much similar, we will have to consider two different cases namely $\beta\geqq p>2$ and $\beta\geqq q\geqq p$. In both cases, the strategy of proof consists in picking a point x_0 at a distance $2r$ of the boundary, working in the ball $B(x_0,r)$ rescaling the equation conveniently in order to deduce that $\liminf\{u(x)\mid d(x)\to 0_+\}$ is more than a fixed constant K_0 and then reiterating the argument to show that $\liminf\{u(x)\mid d(x)\to 0_+\}\geqq nK_0$ for all $n\geqq 1$.

Without loss of generality (add a large constant to u) we may assume that $u\geqq 0$ in Ω and that $f\geqq C_2 d^{-\beta}$ for some $C_2>0$, with $\beta=\max(p,q)$. Next, let $r>0$ and let x_0 be any point in Ω such that $d(x_0)=2r$. Clearly, we have

$$-\Delta u+|\nabla u|^p+\lambda u\geqq C_3 r^{-\beta} \quad \text{in} \quad B(x_0,r), \qquad u|_{\partial B(x_0,r)}\geqq 0, \tag{35}$$

where $C_3=C_2 2^{-\beta}$. Using the existence results of Lions [16], we deduce that $u\geqq \tilde{u}_r(x-x_0)$ in $B(x_0,r)$ where $\tilde{u}_r\in C^2(\overline{B(0,r)})$ solves

$$-\Delta\tilde{u}_r+|\nabla\tilde{u}_r|^p+\lambda\tilde{u}_r=C_3 r^{-\beta} \quad \text{in} \quad B(0,r), \qquad \tilde{u}|_{\partial B(0,r)}=0. \tag{36}$$

Next, in the case when $1<p\leqq 2\leqq q=\beta$, we introduce $u_r(x)=r^\alpha\tilde{u}_r(rx)$ for $x\in B(0,1)$ where $\alpha=(2-p)/(p-1)$ so that u_r solves

$$-\Delta u_r+|\nabla u_r|^p+\lambda r^2 u_r=C_3 \quad \text{in} \quad B(0,1), \qquad u_r|_{\partial B(0,1)}=0. \tag{37}$$

And using the estimates of [21], one checks easily that u_r, as r goes to 0, converges uniformly to the solution u_0 of

$$-\Delta u_0+|\nabla u_0|^p=C_3 \quad \text{in} \quad B(0,1), \qquad u_0|_{\partial B(0,1)}=0. \tag{38}$$

Observing that $u_0>0$ in $B(0,1)$ (strong maximum principle) and so $u_0(0)>0$, we deduce easily that if $p<2<q=\beta$ then u blows up at $\partial\Omega$ and $\liminf\{u(x)d(x)^\alpha \mid d(x)\to 0_+\}>0$.

Now, if $p=2=q=\beta$, the above argument only shows

$$\liminf\{u(x) \mid d(x)\to 0_+\} \geqq K_0>0, \tag{39}$$

where $K_0=u_0(0)$.

In the other case i.e. $2<p=\beta$, we introduce $u_r(x)=\tilde{u}_r(rx)$ for $x\in B(0,1)$ so that $u_r\in C^2(\overline{B(0,1)})$ solves

$$-r^{p-2}\Delta u_r+|\nabla u_r|^p+\lambda r^p u_r=C_3 \quad \text{in} \quad B(0,1), \qquad u_r|_{\partial B(0,1)}=0.$$

And using the results of Lions [21], one sees that u_r converges uniformly to the unique viscosity solution u_0 in $C(\overline{B(0,1)})$ of

$$|\nabla u_0|^p=C_3 \quad \text{in} \quad B(0,1), \qquad u_0|_{\partial B(0,1)}=0$$

which is in fact explicitely given by

$$u_0(x)=C_3^{1/p}(1-|x|).$$

Therefore, in this case also, we prove that (39) holds with $K_0=C_3^{1/p}$.

In particular, for any $\varepsilon>0$, there exists $s_\varepsilon>0$ such that for $x\in\Omega$, $d(x)<s_\varepsilon$ then $u(x)\geqq K_0-\varepsilon$. Then, we go back to (35) replacing the boundary inequality by $u|_{\partial B(x_0,r)}\geqq K_0-\varepsilon$ if $r<s_\varepsilon/2$. And we go through the above proof to deduce finally

$$\liminf\{u(x) \mid d(x)\to 0_+\} \geqq K_0+K_0-\varepsilon=2K_0-\varepsilon$$

for all $\varepsilon>0$: indeed, the limit functions u_0' now satisfy the boundary conditions $u_0'=K_0-\varepsilon$ on $\partial B(0,1)$ i.e. $u_0'=u_0+K_0-\varepsilon$. Letting ε go to 0 and iterating the above argument, Theorem III.1 is proved. □

Remark III.3. Considering $w_{\varepsilon,\delta}(x)=-\varepsilon\log(d(x)+\delta)+\delta\log d(n)-C$, we see that $u\geqq w_{\varepsilon,\delta}$ near $d\Omega$ and this proves Theorem III.1 even if $\beta\geqq 2>\mathrm{p}>1$.

III.2. Uniqueness Results

Theorem III.2. *Let $f\in L^\infty_{\mathrm{loc}}(\Omega)$ satisfy* (18). *Then, there exists a maximum solution of* (1) *in $W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ which goes to $+\infty$ on $\partial\Omega$ and any $v\in L^1_{\mathrm{loc}}(\Omega)$ satisfying* (21) *satisfies $v\leqq u$ a.e. in Ω. Among all solutions of* (1) *in $W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ which go to $+\infty$ on $\partial\Omega$, or equivalently that are bounded from below on Ω, there exists a minimum one which is the increasing limit of sequence of subsolutions of* (1) *(i.e. satisfying* (21)*) in $W^{2,r}(\Omega)$ $(\forall r<\infty)$.*

If we impose further restrictions on f, when we have the

Theorem III.3. *Let $f\in L^\infty_{\mathrm{loc}}(\Omega)$ satisfy* (18′). *Then, there exists a unique solution of* (1) *in $W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ bounded from below. In addition, this solution satisfies* (19).

Proof of Theorem III.2. Let $C>0$ be a constant such that

$$f(x)\geqq Cd(x)^{-\beta}-C,$$

where $\beta=\max(p,q)$. Then, we set $\underline{w}_\delta=-M\,\mathrm{Log}(d+\delta)-K$ if $p\geqq 2\geqq q$, $\underline{w}=M(d+\delta)^{-\alpha}-K$ if $p<2<q$, where $\alpha=(q-p)/p$, M, K are positive constants

chosen in such a way that for δ small enough $\underline{w}_\delta$ is a subsolution of (1). In fact, we may find $R(\delta)\downarrow+\infty$ as $\delta\uparrow 0_+$ such that (with $a\wedge b=\inf(a,b)$):

$$+\Delta\underline{w}_\delta+|\nabla\underline{w}_\delta|^p+\lambda\underline{w}_\delta\leqq f\wedge R(\delta)\quad\text{in}\quad\Omega.$$

Then, using the existence results of Lions [16, 19] we deduce that there exists $\underline{u}_\delta\in W^{2,r}(\Omega)$ ($\forall r<\infty$) solution of

$$-\Delta\underline{u}_\delta+|\nabla\underline{u}_\delta|^p+\lambda\underline{u}_\delta=f\wedge R(\delta)\quad\text{in}\quad\Omega,\qquad \underline{u}_\delta=\underline{w}_\delta\quad\text{on}\quad\partial\Omega;$$

and by the maximum principle $\underline{u}_\delta\geqq\underline{w}_\delta$ in Ω.

The remainder of the proof consists in passing to the limit as δ goes to 0 in order to build the minimum solution. To do so we need local upper bounds on $\underline{u}_\delta$: we will achieve this by building a supersolution. We first observe that it is possible to find $\Phi\in C^1(0,\infty)$ such that $\Phi(t)\to+\infty$ as $t\to 0_+$, $\Phi'(t)<0$ if $t>0$, $\Phi(t)>0$ if $t>0$ and

$$(\Phi^{-1/q})'\to 0\quad\text{as}\quad t\to 0_+,\qquad f(x)\leqq\Phi(d(x))\qquad\text{a.e. in }\Omega.$$

Now let $R=\sup_{\bar\Omega} d$, $C_0=\sup_{[0,R]}(\Phi^{-1/q})'$. We denote by

$$\Psi_1=\mu\Phi^{1/p},\qquad \Psi(t)=\int_t^R \Psi_1(s)ds,$$

where μ is a positive constant to be determined. We finally set

$$\bar{w}(x)=\Psi(d)+K$$

where K is a positive constant to be determined. We claim next that for large μ and K, $\bar w$ is a supersolution of (1) which of course blows up at $\partial\Omega$. Indeed, we find, denoting by $C=\|\Delta d\|_\infty$, that if $d(x)\leqq\delta_0$

$$\begin{aligned}-\Delta\bar w+|\nabla\bar w|^p+\lambda\bar w&\geqq-\Psi''(d)-C|\Psi'(d)|+|\Psi'(d)|^p\\&=\frac{\mu}{p}\,\Phi^{\frac{1}{p}-1}\Phi'-C\mu\Phi^{1/p}+\mu^p\Phi\\&\geqq\mu^p\Phi-C\mu\Phi^{1/p}-\mu\frac{C_0}{p}\,\Phi^{\frac{1}{p}-1}\Phi^{\frac{1}{q}+1}\\&=\left(\mu^p-\mu\frac{C_0}{p}\right)\Phi-C\mu\Phi^{1/p}\geqq f,\end{aligned}$$

if μ is large enough, say $\mu\geqq\mu_0>0$. We then fix $\mu=\mu_0$ and we consider on the set $d(x)>\delta_0$

$$-\Delta\bar w+|\nabla\bar w|^p+\lambda\bar w\geqq -M+\lambda K$$

for some constant M, and choosing $K\geqq\frac{1}{\lambda}\left(M+\sup_{\Omega_{\delta_0}}|f|\right)$ we conclude.

In particular, we see that $\underline{u}_\delta\leqq\bar w$ and thus $\underline{u}_\delta$ is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$. Furthermore, by the bounds proved in the appendix, this implies that $\underline{u}_\delta$ is also bounded in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and thus in $W^{2,r}_{\mathrm{loc}}(\Omega)$ by elliptic regularity. And, letting δ go to 0, $\underline{u}_\delta$ increases to a solution of (1) $\underline{u}$ which is above $\underline{w}$. The fact that $\underline{u}$ is the minimum solution of (1) which goes to $+\infty$ on $\partial\Omega$ is an easy consequence of the fact that any such solution is above $\underline{u}_\delta$ by the maximum principle.

To prove the existence of a maximum solution of (1) going to $+\infty$ on $\partial\Omega$, we first observe that $\bar{w}_\delta = \Psi(d(x)-\delta)+K$ is also a supersolution of (1) with Ω replaced by Ω_δ. Therefore, by maximum principle, any solution of (1) is below $\bar{w}_\delta$ and, passing to the limit in δ, thus below $\bar{w}$.

To build the maximum solution, several arguments are possible. One way to do it consists in maximizing $u(x_0)$ for some fixed $x_0 \in \Omega$ among all solutions of (1) bounded from below on Ω (or equivalently going to $+\infty$ on $\partial\Omega$). Then, observe that if u_1, u_2 are two such solutions then there exists another one, say u_3, above u_1 and u_2: indeed $\max(u_1, u_2)$ is a subsolution of (1) and we may solve for

$$-\Delta u_3^\delta + |\nabla u_3^\delta|^p + \lambda u_3^\delta = f \quad \text{in} \quad \Omega_\delta, \qquad u_3^\delta = \max(u_1, u_2) \quad \text{on} \quad \partial\Omega_\delta$$

the existence follows from [19]. Then $u_3^\delta \leqq \bar{w}_\delta$ and thus is bounded in $W^{2,r}_{\mathrm{loc}}(\Omega)$ by arguments we already made several times. Using several times the maximum principle, we see that u_3^δ converges (and increases) to a solution u_3 of (1) which is above u_1 and u_2. This observation implies that there exists a maximizing sequence (u_n) of solutions of (1) which maximizes $u_n(x_0)$ and which is nondecreasing. Then, since $u_n \leqq \bar{w}$, u_n converges (use again the a priori estimates) to a solution $\bar{u}$ of (1) which is bounded from below on Ω and thus blows up at $\partial\Omega$. Furthermore, the above construction of u_3 shows that the fact that $\bar{u}$ maximizes $u(x_0)$ among all solutions implies in fact that $\bar{u}$ is the maximum solution of (1).

Proof of Theorem III.3. Using the results of Theorem III.2 and their proofs, it is now easy to mimick the proofs of Theorems II.1–II.2 in order to obtain the uniqueness. Indeed, if we use (18′), we may replace the functions $\bar{w}_\delta$, $\underline{w}_\delta$ built above by the ones given by (22) provided one takes the values for C_0, α which are given in Theorem I.3. Then, this implies that, by the same proof as above, the minimum solution $\underline{u}$ and the maximum solution $\bar{u}$ of (1) going to $+\infty$ on $\partial\Omega$ satisfy

$$(C_0-\varepsilon)d(x)^{-\alpha} - C_\varepsilon \leqq \underline{u}(x) \leqq \bar{u}(x) \leqq (C_0+\varepsilon)d(x)^{-\alpha} + C_\varepsilon \quad \text{in} \quad \Omega$$

and we may now conclude using the same proof as in Theorem I.1. □

We now conclude this section with an improved uniqueness result where however no precise behaviour of the solution is given.

Theorem III.4. *Let* $f \in L^\infty_{\mathrm{loc}}(\Omega)$ *satisfy*

$$C'd^{-\beta} - C' \leqq f \leqq Cd^{-\beta} + C \quad \text{for some} \quad C \geqq C' > 0, \quad \beta \geqq \max(p, q). \tag{40}$$

Then, there exists a unique solution of (1) *in* $W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r < \infty)$ *which is bounded from below. Denoting by u this solution, we have for some* $M \geqq 1$

$$\frac{1}{M} d^{-\alpha} - M \leqq u \leqq Md^{-\alpha} + M \quad \text{in} \quad \Omega,$$

where $\alpha = \dfrac{\beta}{p} - 1$ *if* $\beta > p$, *and* $d^{-\alpha}$ *is replaced by* $|\mathrm{Log}\, d|$ *if* $\beta = p \geqq q$.

Proof. By similar arguments to the ones given above, the maximum solution $\bar{u}$ and the minimum solution $\underline{u}$ satisfy for some $M \geqq 1$

$$\frac{1}{M} d^{-\alpha} - M \leqq u \leqq Md^{-\alpha} + M \quad \text{in} \quad \Omega.$$

Without loss of generality (adding a large constant to f, $\underline{u}$, $\bar{u}$) we may assume that $\bar{u}\geqq\underline{u}\geqq 1$, $f\geqq 1$ a.e. in Ω. Therefore, there exists $\theta\in(0,1)$ small enough such that $\underline{u}\geqq\theta\bar{u}$ in Ω. Let then $\theta_0=\sup\{\theta\in(0,1]/\underline{u}\geqq\theta\bar{u}$ in $\Omega\}$ – we follow a uniqueness argument which was introduced in a different context by Laetsch [14]. If $\theta_0=1$, we are done. We thus argue by contradiction and assume that $\theta_0<1$. Of course, we have $\underline{u}\geqq\theta_0\bar{u}$ in Ω. We then consider $z=\varepsilon d^{-\alpha}$ and we observe that z satisfies

$$-\Delta z+|\nabla z|^p+\lambda z\leqq\varepsilon^p d^{-\beta}+C_\varepsilon d^{-\beta+1}$$

and this is less than f for ε small enough say $\varepsilon\leqq\varepsilon_0$. We choose $\varepsilon=\varepsilon_0$. In fact $z_\delta=\varepsilon(d+\delta)^{-\alpha}$ also satisfies

$$-\Delta z_\delta+|\nabla z_\delta|^p+\lambda z_\delta\leqq f \quad \text{in} \quad \Omega.$$

And we consider $w_{\gamma,\delta}=(\theta_0-\gamma)\bar{u}+(1-\theta_0+\gamma)z_\delta$; $w_{\gamma,\delta}$ satisfies for $\gamma<\theta_0$

$$-\Delta w_{\gamma,\delta}+|\nabla w_{\gamma,\delta}|^p+\lambda w_{\gamma,\delta}\leqq(\theta_0-\gamma)f+(1-\theta_0+\gamma)f\equiv f \quad \text{in} \quad \Omega$$

and since $\underline{u}$, $\bar{u}$ blow up near the boundary we have $w_{\gamma,\delta}\leqq\theta_0\bar{u}\leqq\underline{u}$ near the boundary. Therefore, by the maximum principle, $w_{\gamma,\delta}\leqq\underline{u}$ in Ω. We now let γ go to 0_+ and then δ go to 0_+ to find

$$\theta_0\bar{u}+(1-\theta_0)z\leqq\underline{u} \quad \text{in} \quad \Omega$$

but we obviously have $z\geqq\nu\bar{u}$ for some $\nu>0$. Hence,

$$(\theta_0+(1-\theta_0)\nu)\bar{u}\leqq\underline{u} \quad \text{in} \quad \Omega$$

and this contradicts the definition of θ_0. □

IV. Superquadratic Hamiltonians

IV.1. Interior Gradient Bounds and Maximum Solutions

We begin with a result which gives interior gradient bounds for solutions of (1): similar bounds were first derived in [16, 19] and the proofs are recalled in the appendix. We only remark here that a sharper form of these bounds may be obtained by a simple scaling argument.

Theorem IV.1. *Let $f\in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below on Ω and satisfy*

$$|f(x)|\leqq C_1 d(x)^{-\beta} \quad \textit{for some} \quad \beta\geqq 0, \qquad C_1\geqq 0. \tag{41}$$

Let $u\in W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ be a solution of (1) satisfying

$$\lambda u\geqq -C_2 \quad \textit{for some} \quad C_2\geqq 0. \tag{42}$$

Then, we set $\gamma=\dfrac{1}{p-1}$ if $\beta\leqq q$, γ arbitrary in $\left(\dfrac{\beta}{p},1\right)$ if $\beta>q$ and $\gamma=\dfrac{\beta}{p}$ if $f\in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and $|\nabla f(x)|d(x)^{-\beta-1}\in L^\infty(\Omega)$. With these notations and assumptions we have

$$|\nabla u(x)|\leqq C_3 d(x)^{-\gamma} \quad \textit{in} \quad \Omega, \quad □ \tag{43}$$

where C_3 only depends on C_1, C_2, γ, β and the diameter of Ω.

Remark IV.1. The bound is optimal as it may be easily checked on simple examples like $\frac{C_0}{d^\alpha}$ if $p \leqq 2$ ($-\operatorname{Log} d$ if $p=2$) with C_0, α given as in Theorem I.1 or Theorem I.3, or $-C_0 d^\alpha$ if $p>2$ with $\alpha = \frac{p-2}{p-1}$ if $\beta \leqq q$, $\alpha = 1-\beta/p$ if $\beta < p$ and C_0 is a convenient positive constant. □

Exactly as in [19], this implies of course the following result

Corollary IV.1. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below on Ω and satisfy* (41). *Then, any solution $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$) of* (1) *which is bounded from below belongs to $W^{1,s}(\Omega)$ with $s < p-1$ if $p>2$ and $\beta \leqq q$, $s < p/\beta$ if $p > \beta > q$ (and thus $p>2$). In addition, any such solution may be extended continuously on $\bar{\Omega}$ and $u \in C^{0,\theta}(\bar{\Omega})$ with $\theta = (p-2)/(p-1)$ if $p>2$, $\beta \leqq q$; $\theta = 1-\beta'/p$ if $p > \beta > \beta' q$; and $\theta = 1-\beta/p$ if $p > \beta > q$ and $f \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ satisfies $|\nabla f| d^{-\beta-1} \in L^\infty(\Omega)$.* □

We now just sketch the *proof of Theorem IV.1*: let $x_0 \in \Omega$, set $r = \frac{1}{2} d(x_0)$ and consider $v(x) = r^{-(1-\gamma)} u(x_0 + rx)$ for $x \in B(0,1)$. One checks easily that v solves

$$-r^\sigma \Delta v + |\nabla v|^p + \lambda r^\nu u = r^{p\gamma} f(x_0 + rx) \quad \text{in} \quad B(0,1) \tag{44}$$

with $\sigma = (p-1)\gamma - 1$, $\nu = (p-1)\gamma + 1$. Next, observe that

$$|r^{p\gamma} f(x_0 + rx)| \leqq C_4 \quad \text{on} \quad B(0,1),$$

where C_4 depends only on C_R and β. And if $\beta = q$, then $\sigma = 0$, $\nu = 2$ while if $\beta > q$, $\nu = \sigma + 2$ and $\sigma > 0$. If $\beta \leqq q$ or if $\beta > q$ and $d \in W^{1,\infty}_{\mathrm{loc}}$, $|\nabla f| d^{-\beta-1} \in L^\infty(\Omega)$, interior estimates are available (see appendix) and we deduce from this

$$|\nabla v(0)| \leqq C_3$$

which of course yields (43). □

In the last case, we observe that

$$\|\Delta v\|_{L^m(0,0,k))} \leqq \frac{C'(m,k)}{r^\sigma} \quad \text{for all} \quad m \geqq 1, \quad k \in (0,1).$$

But then, recalling the following "standard" inequality for all $m > N$

$$\|\nabla v\|_{L^\infty(B(0,\frac{1}{2}))} \leqq C \|\nabla v\|^{1-\frac{N}{m}}_{L^m(B(0,\frac{1}{2}))} \{\|\Delta v\|_{L^m(B(0,\frac{1}{2}))} + \|\nabla v\|_{L^m(B(0,\frac{1}{2}))}\}^{\frac{N}{m}}$$

we finally obtain

$$|\nabla v(0)| \leqq C(m) r^{-N\sigma/m} \quad \text{for all} \quad m > N.$$

And this yields (43). □

Next, using these estimates and Corollary IV.1, we may now deduce easily the following

Corollary IV.2. *Let $p>2$, let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below on Ω and satisfy* (41) *with $\beta < p$. Then, there exist solutions $\underline{u}, \bar{u}$ of* (1) *in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$) bounded from below such that if v is a solution of* (1) *in $W^{2,r}(\Omega)$ ($\forall r < \infty$), respectively $W^{2,r}_{\mathrm{loc}}(\Omega)$*

($\forall r<\infty$), then $\underline{u}\geqq v$ in Ω, respectively $\bar{u}\geqq v$ in Ω. Furthermore, if $v\in L^1_{\mathrm{loc}}(\Omega)$ satisfies (21) *then $v\leqq\bar{u}$ a.e. in Ω. And if $\bar{u}_\delta$, $\underline{u}_\delta$ denote the corresponding maximum solutions of* (1) *with Ω replaced by Ω_δ then*

$$\bar{u}_{\delta'}\geqq\underline{u}_{\delta'}\geqq\bar{u}_\delta\geqq\underline{u}_\delta\geqq\bar{u}\geqq\underline{u} \quad \text{in} \quad \Omega_{\delta'} \qquad \text{for} \quad 0<\delta<\delta' \tag{45}$$

and $\bar{u}_\delta$ decreases to $\bar{u}$ as δ goes to 0_+. □

Remark IV.2. A consequence of the results we will prove in the following sections is the following: assume that $f\in L^\infty(\mathbb{R}^N)$ and denote by $\bar{u}^\delta$, $\underline{u}^\delta$ the corresponding maximum solution of (1) with Ω replaced by Ω^δ then

$$\underline{u}_{\delta'}\leqq\bar{u}_{\delta'}\leqq\underline{u}_\delta\leqq\bar{u}_\delta\leqq\underline{u} \quad \text{in} \quad \Omega \quad \text{for} \quad 0<\delta<\delta' \tag{46}$$

and $\underline{u}_\delta$ increases to $\underline{u}$ as δ goes to 0_+.

Remark IV.3. We will show in Sect. V that if f behaves like $C_1 d^{-\beta}$ near the boundary with $0\leqq\beta<p$ [$\beta=0$ means $f\in L^\infty(\Omega)$] then $\underline{u}=\bar{u}$ in Ω.

Proof of Corollary IV.2. The existence of the maximum solutions $\underline{u}$, $\bar{u}$ is exactly the same as in Theorem III.2. Next, the string of inequalities in (45) follows from the definitions of $\bar{u}$, $\underline{u}$. Finally, $\bar{u}_\delta$ decreases to a solution of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r<\infty$) in view of the estimates given by Theorem IV.1. Therefore, the limit is below $\bar{u}$. Since on the other hand, by (45), $\bar{u}_\delta\geqq\bar{u}$, we conclude easily. □

We conclude this section by a property of $\bar{u}$, $\underline{u}$ which will be useful later on.

Proposition IV.1. *Let ω be a bounded smooth domain such that $\bar{\omega}\subset\bar{\Omega}$. Let $v\in W^{2,r}(\omega)$ ($\forall r<\infty$) (resp. $W^{2,r}_{\mathrm{loc}}(\omega)\cap C(\bar{\omega})$ ($\forall r<\infty$)) be a subsolution of* (1) *with Ω replaced by ω. If $v\leqq\underline{u}$ (resp. $v\leqq\bar{u}$) on $\partial\omega\cap\Omega$ then $v\leqq\underline{u}$ (resp. $v\leqq\bar{u}$) in ω.* □

Proof. Let $\varepsilon>0$, $v_\varepsilon=v-\varepsilon$ satisfies the same properties than v. In one case, we just consider

$$w_\varepsilon=\bar{u} \quad \text{in} \quad \Omega-\omega, \qquad =\max(\bar{u},v-\varepsilon) \quad \text{in} \quad \omega$$

and we observe that w_ε is a subsolution of (1) [in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$]. Therefore, by Corollary IV.2, $w_\varepsilon\leqq\bar{u}$ and thus $v\leqq\bar{u}$ in ω by letting ε go to 0.

In the other case, the above construction has to be modified a bit since w_ε does not belong to $W^{2,r}(\Omega)$ ($\forall r<\infty$). We then consider $\beta_\varepsilon(t)=\varepsilon\beta\left(\frac{t}{\varepsilon}\right)$ where $\beta(t)=t$ if $t\geqq 0$, $\beta\in C^\infty(\mathbb{R})$, β is convex, $1\geqq\beta'(t)\geqq 0$ on $\mathbb{R}$, $\beta(t)\equiv-1$ if $t\leqq-2$. And we now introduce

$$z_\varepsilon=\bar{u} \quad \text{in} \quad \Omega-\omega, \qquad =\bar{u}+\beta_\varepsilon(v_\varepsilon-\bar{u}) \quad \text{in} \quad \omega.$$

Now, $z_\varepsilon\in W^{2,r}(\Omega)$ ($\forall r<\infty$) and we claim that z_ε is a subsolution of (1). We only have to check this claim inside ω where we find

$$\begin{aligned}\nabla z_\varepsilon&=\beta'_\varepsilon\nabla v_\varepsilon+(1-\beta'_\varepsilon)\nabla\bar{u},\\ -\Delta z_\varepsilon&=\beta'_\varepsilon(-\Delta v_\varepsilon)+(1-\beta'_\varepsilon)(-\Delta\bar{u})-\beta''_\varepsilon|\nabla v_\varepsilon-\nabla\bar{u}|^2\\ &\leqq\beta'_\varepsilon(-\Delta v_\varepsilon)+(1-\beta'_\varepsilon)(-\Delta\bar{u}),\\ z_\varepsilon&\leqq\bar{u}+\beta'_\varepsilon(v_\varepsilon-\bar{u})\end{aligned}$$

and our claim follows easily from these inequalities.

We may now complete the proof of Proposition IV.1 since, by definition, $z_\varepsilon \leqq \underline{u}$ in Ω and letting ε go to 0, remarking that β_ε converges uniformly to t^+, we obtain $v \leqq \underline{u}$ in ω. □

Remark IV.4. The expert reader will notice that this result is one form of the dynamic programming principle for the associated stochastic control problem!

IV.2. An Estimate on the Boundary Behaviour

We want to show in this section some properties of $\bar{u}$, $\underline{u}$ like (17). We will be always dealing with the case $p>2$, $f \in L^\infty_{\mathrm{loc}}(\Omega)$ bounded from below and satisfying (41) with $\beta<p$. Hence, Corollary IV.2 and Proposition IV.1 apply. In all the results which follow in this section and in Sect. IV.3, we will not recall these assumptions.

Theorem IV.2. *The maximum solutions $\bar{u}$, $\underline{u}$ satisfy* (17) *with $\alpha = -\dfrac{p-2}{p-1}$. In addition, if f satisfies*

$$\liminf \{ f(x) d(x)^\theta \mid d(x) \to 0_+ \} > 0 \quad \textit{for some} \quad \theta \in (q, \beta] \tag{47}$$

then $\bar{u}$, $\underline{u}$ satisfy (17) *with $\alpha = 1-\theta/p$.* □

Remark IV.5. Again, this result is rather optimal since if f satisfies (41) with $\beta \leqq q$, we already know that $u \in C^{0,\alpha}(\bar{\Omega})$ and $-C_0 d^\alpha$ gives a simple example (for the ad hoc $C_0>0$) which shows the sharpness of (17). Similarly if $f(x)$ behaves like $C_1 d(x)^{-\beta}$ for some $q<\beta<p$ then we already know that $u \in C^{0,\alpha}(\bar{\Omega})$ and again $-C_0 d^\alpha$ shows the sharpness of (17). The only improvement we could think of would be to show (and we were unable to do it)

$$\liminf_{y \in \Omega, y \to x} \{u(y)-u(x)\} |y-x|^{-\alpha} = -C_0, \quad \text{for all} \quad x \in \partial\Omega,$$

where $C_0 = (p-2)^{-1}(p-1)^{\frac{p-2}{p-1}}$ if $\beta<q$, solves $C_0^p \alpha^p - C_0 \alpha(1-\alpha) = C_1$ if $\beta = q$, $C_0 = \dfrac{1}{\alpha} C_1^{1/p}$ if $q<\beta<p$ at least when f behaves like $C_1 d^{-\beta}$ near the boundary.

Proof of Theorem IV.2. The proof is rather delicate so we will begin with a simpler claim than (17). But let us first give the idea of the proof: we just observe that (17) is equivalent to say that for all $x_0 \in \partial\Omega$, $u\,(=\underline{u}, \bar{u})$ $-\varepsilon|x-x_0|^\alpha$ cannot have a local minimum in $\bar{\Omega}$ at x_0 for ε small enough. To prove this fact, we will argue by contradiction and we will do so by building a subsolution on a neighbourhood of x_0 such that on the boundary of the neighbourhood it is below u while it is above u at x_0. This will contradict Proposition IV.1 proving thus our claim.

To explain how this strategy works, we will begin proving that if $\varphi \in C^{1,1}(\bar{\Omega})$ then $u-\varphi$ cannot have a local minimum on $\bar{\Omega}$ at $x_0 \in \partial\Omega$ where $u = \underline{u}$ or $\bar{u}$. Assume by way of contradiction that x_0 is a local minimum of $u-\varphi$ for some $\varphi \in C^{1,1}(\bar{\Omega})$. Then, denoting by $\xi_0 = \nabla\varphi(x_0)$, there exists $C \geqq 0$ such that

$$u(x) \geqq u(x_0) + (\xi_0, x-x_0) - C|x-x_0|^2 \quad \text{for all} \quad x \in \bar{\Omega}. \tag{48}$$

We then consider the following function defined on $\bar{\omega}$ where $\omega=\{x\in\Omega, d(x)<\delta\}$ where $\delta>0$ will be determined later on

$$w(x)=u(x_0)+(\xi_0, x-x_0)-C|x-x_0|^2+\mu(\delta^\alpha-d^\alpha), \qquad \forall x\in\bar{\omega} \tag{49}$$

with $\alpha=\dfrac{p-2}{p-1}$, for some $\mu>0$ to be determined. In view of (48) and (49), we have

$$w\leqq u \quad \text{on} \quad \partial\omega\cap\Omega, \qquad w(x_0)>u(x_0). \tag{50}$$

Hence, Proposition IV.1 will yield the desired contradiction if we show that w is a subsolution of (1) in ω. Therefore, we compute in ω

$$\begin{aligned}-\Delta w+|\nabla w|^p+\lambda w-f&=2NC+\alpha\mu d^{\alpha-1}\Delta d-\mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}}\\&\quad+\left|\xi_0-C(x-x_0)-\alpha\mu\frac{\nabla d}{d^{1-\alpha}}\right|^p+\lambda w-f\\&\leqq C\left(1+\frac{1}{d^{1-\alpha}}\right)-\mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}}+\left|\xi_0-C(x-x_0)-\alpha\mu\frac{\nabla d}{d^{1-\alpha}}\right|^p,\end{aligned}$$

where C denotes various constants independent of δ. Recalling that $|\nabla d|=1$, $(1-\alpha)p=2-\alpha$, we see that if δ is small enough and $(\alpha\mu)^{p-1}<1-\alpha$ [depending only on μ, $|\xi_0|$, C in (49), a lower bound on f and Ω] w is a subsolution of (1) in ω.

We now show (17): it is enough to show that the following inequality cannot hold

$$u(x)\geqq u(x_0)-\varepsilon_0|x-x_0|^\alpha-C|x-x_0|^2, \qquad \forall x\in\bar{\Omega} \tag{51}$$

for small ε_0, $\delta>0$ and for some $C\geqq 0$, where $\alpha=\dfrac{p-2}{p-1}$ or $\alpha=1-\theta/p$ if f satisfies (47). Indeed, if (51) holds, then we introduce

$$w(x)=u(x_0)-\varepsilon_0\beta_\varepsilon(|x-x_0|)-C|x-x_0|^2+\mu(\delta^\alpha-d^\alpha) \quad \text{in} \quad \omega$$

where $\omega=\{x\in\Omega/d(x)<\delta\}$, $\beta_\varepsilon(t)$ is the function defined by

$$\beta_\varepsilon(t)=\frac{\alpha}{2}\frac{|t|^2}{\varepsilon}+\frac{2-\alpha}{2}\varepsilon^{\frac{\alpha}{2-\alpha}} \quad \text{if} \quad |t|\leqq\varepsilon^{\frac{1}{2-\alpha}}, \qquad =|t|^\alpha \quad \text{if} \quad |t|\geqq\varepsilon^{\frac{1}{2-\alpha}}.$$

In view of (51), (50) will hold if

$$\mu\delta^\alpha>\frac{2-\alpha}{2}\varepsilon_0\varepsilon^{\frac{\alpha}{2-\alpha}}. \tag{52}$$

Next, we compute for x in ω the following quantity

$$\begin{aligned}-\Delta w+|\nabla w|^p+\lambda w-f&=2NC+(N-1)\varepsilon_0\beta'\frac{1}{|x-x_0|}+\varepsilon_0\beta''-\mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}}\\&\quad+\alpha\mu\frac{1}{d^{1-\alpha}}\Delta d+\left|-\varepsilon_0\beta'\frac{x-x_0}{|x-x_0|}-2C(x-x_0)-\alpha\mu\frac{\nabla d}{d^{1-\alpha}}\right|^p+\lambda w-f\end{aligned}$$

(in fact this equality holds a.e. in ω), and this yields

$$-\Delta w+|\nabla w|^p+\lambda w-f\leqq C-f+\varepsilon_0 N\frac{\alpha}{\varepsilon}+C\mu\frac{1}{d^{1-\alpha}}-\mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}}$$
$$+\left|\varepsilon_0\beta'\frac{x-x_0}{|x-x_0|}+2C(x-x_0)+\alpha\mu\frac{\nabla d}{d^{1-\alpha}}\right|^p.$$

Now, if we begin by the case where we do not assume (47), then we just bound f by a constant C and we deduce

$$-\Delta w+|\nabla w|^p+\lambda w-f\leqq C+\varepsilon_0 N\frac{\alpha}{\varepsilon}+C\mu\frac{1}{d^{1-\alpha}}+\mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}}$$
$$+\left(\varepsilon_0\alpha\frac{1}{\varepsilon^{\frac{1-\alpha}{2-\alpha}}}+C+\alpha\mu\frac{1}{d^{1-\alpha}}\right)^p.$$

We then choose $\varepsilon=(t\mu\delta^\alpha\varepsilon_0^{-1})^{\frac{2-\alpha}{\alpha}}$ with $0<t<\dfrac{2}{2-\alpha}$ so that (52) holds and we obtain

$$-\Delta w+|\nabla w|^p+\lambda w-f\leqq C+C\mu\frac{1}{d^{1-\alpha}}+Nt^{-\frac{2-\alpha}{\alpha}}\alpha\mu^{-\frac{2-\alpha}{\alpha}}\delta^{-(2-\alpha)}\varepsilon_0^{2/\alpha}$$

$$+-\mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}}+\left(\alpha t^{-\frac{1-\alpha}{\alpha}}\delta^{-(1-\alpha)}\mu^{-\frac{1-\alpha}{\alpha}}\varepsilon_0^{1/\alpha}+C+\alpha\mu\frac{1}{d^{1-\alpha}}\right)^p.$$

Next, if we fix t in $\left(0,\dfrac{2}{2-\alpha}\right)$ and μ in $\left(0,(t\alpha)^{\frac{1}{p-1}}\alpha^{-1}\right)$, recalling that $d(x)<\delta$, we see that for ε_0 small enough (depending only on N, t, μ, α) we may bound the above terms by

$$C+C\mu\frac{1}{d^{1-\alpha}}-K\frac{1}{d^{2-\alpha}}$$

for some $K>0$, and then we conclude choosing δ small enough.

In the other case, that is when we assume (47), we obtain

$$-\Delta w+|\nabla w|^p+\lambda w-f\leqq C-\frac{\nu}{d^\theta}+N\varepsilon_0\frac{\alpha}{\varepsilon}+\left(\varepsilon_0\alpha\frac{1}{\varepsilon^{\frac{1-\alpha}{2-\alpha}}}+C+\alpha\mu\frac{1}{d^{1-\alpha}}\right)^p$$
$$\leqq C-\frac{\nu}{d^\theta}+N\varepsilon_0\frac{\alpha}{\varepsilon}+\left(C+\alpha\varepsilon_0\varepsilon^{\frac{\theta}{\theta+p}}+\alpha\mu d^{-\frac{\theta}{p}}\right)^p$$

and again writing $\varepsilon=(t\mu\delta^\alpha\varepsilon_0^{-1})^{\frac{2-\alpha}{\alpha}}$ with $0<t<\dfrac{2}{2-\alpha}$ so that (52) holds we deduce

$$-\Delta w+|\nabla w|^p+\lambda w-f\leqq C-\frac{\nu}{d^\theta}+N\alpha t^{-\frac{2-\alpha}{\alpha}}\delta^{-(2-\alpha)}\mu^{-\frac{2-\alpha}{\alpha}}\varepsilon_0^{\frac{2}{\alpha}}$$

$$+\left(C+\alpha t^{-\frac{1-\alpha}{\alpha}}\mu^{-\frac{1-\alpha}{\alpha}}\delta^{-\frac{\theta}{p}}\varepsilon_0^{\frac{1}{\alpha}}+\alpha\mu d^{-\frac{\theta}{p}}\right)^p.$$

And, if we choose t in $\left(0, \frac{2}{2-\alpha}\right)$, μ in $(0, v^{1/p}\alpha^{-1})$ we see that for ε_0 small enough the above terms may be bounded by

$$C - \frac{v}{d^\theta}$$

therefore w is a subsolution in ω for δ small enough and we conclude. □

We, in fact, proved the

Corollary IV.3. *The maximum solutions $\underline{u}$, $\bar{u}$ satisfy for all $x \in \partial\Omega$*

$$\liminf_{x \in \Omega, x \to x_0} \{u(x) - u(x_0)\} |x - x_0|^{-\alpha} \leqq -K < 0 \tag{53}$$

where $K = K(p, N)$ and $\alpha = 1 - 1/(p-1)$.

And if f satisfies (47), *then* (53) *holds with $\alpha = 1 - \frac{\theta}{p}$ and $K = K(p, \theta, N, C_1)$ where $C_1 = \liminf \{f(x)d(x)^\theta / d(x) \to 0_+\}$.*

IV.3. Infinite Neumann Conditions

Our goal in this section is to investigate the behaviour of the maximum solutions near the boundary. We suspect that the results given in Proposition IV.3 hold in full generality but we were unable to prove it.

We will first sketch the proof of

Proposition IV.2. *Let $f \in W^{1,\infty}(\Omega)$, $p > 2$.*

i) *If Ω is a ball (or if Ω is an half-space), the maximum solutions of* (1) *are Lipschitz tangentially i.e. if $\Omega = B_R$ then*

$$|u(y) - u(x)| \leqq C|y - x| \quad \forall y, x \in \bar{\Omega} \quad \text{with} \quad |y| = |x| \tag{54}$$

and if $\Omega = \{x_N > 0\}$ then

$$|u(y) - u(x)| \leqq C|y - x| \quad \forall y, x \in \bar{\Omega} \quad \text{with} \quad y_N = x_N \tag{55}$$

for some $C \geqq 0$, where $u = \underline{u}$ or $\bar{u}$.

ii) *If Ω is convex, then $u = \underline{u}$ or $\bar{u}$ satisfies*

$$|\nabla u - (\nabla u(x) \cdot n(x))n(x)| \leqq C d^{-1/2} \quad \text{in} \quad \Omega \tag{56}$$

for some $C \geqq 0$, where n is any smooth vector-field equal to the unit outward normal near $\partial\Omega$ (i.e. $n = -\nabla d$ near $\partial\Omega$). And if $2 < p < 3$, this yields

$$|u(x) - u(y)| \leqq C|x - y|^{2(p-2)/(3p-5)} \quad \forall y, x \in \bar{\Omega} \quad \text{with} \quad d(x) = d(y). \tag{57}$$

Remark IV.6. It is proved in Lasry and Lions [15] that if Ω is convex, f is convex ($\in C(\Omega)$) and satisfies (41) then $\underline{u}$ and $\bar{u}$ are convex. In addition, if (41) holds then $\underline{u}$, $\bar{u} \in C^{0,1-\gamma}(\bar{\Omega})$ with $\gamma = \frac{1}{p-1}$ if $\beta \leqq q$ and $\gamma = \frac{\beta}{p}$ if $\beta > q$. This Hölder continuity combined with the convexity then implies

$$|\nabla u-(\nabla u\cdot n)n|\leq Cd^{-\gamma/2} \quad \text{in} \quad \Omega. \tag{58}$$

This improved bound on the tangential gradient enables us, in that particular case, to follow the arguments given below. □

Proof of Proposition IV.2. i) In the case of the half-space, one simply remark that $u(\cdot+he_i)$ (for $1\leq i\leq N-1$) is the maximum solution of (1) with f replaced by $f(\cdot+he_i)$ hence, using as in Corollary II.1, the maximality

$$\|u(\cdot+he_i)-u(\cdot)\|_\infty\leq\frac{1}{\lambda}\|f(\cdot+he_i)-f(\cdot)\|_\infty\leq C|h|$$

and (55) is proved. One proves (54) similarly replacing the tangential translations by rotations.

ii) Let y be an arbitrary point in $\bar{\Omega}$, we set $u_t(x)=\frac{1}{t^\alpha}u(y+t(x-y))w$ for $0<t<1$, $x\in y+\frac{1}{t}(\Omega-y)=\Omega_t$ with $\alpha=\frac{p-2}{p-1}$. Observe that $\Omega\subset\Omega_t$ and that u_t solves

$$-\Delta u_t+|\nabla u_t|^p+\lambda t^2u_t=t^{2-\alpha}f(tx) \quad \text{in} \quad \Omega_t.$$

Therefore, we have for some $C\geq 0$

$$-\Delta u_t+|\nabla u_t|^p+\lambda u_t\leq f+C(1-t) \quad \text{in} \quad \Omega,$$

and $u_t-\frac{C}{\lambda}(1-t)$ is a subsolution of (1); hence $u_t\leq u+\frac{C}{\lambda}(1-t)$. But this inequality immediately implies

$$(x-y,\nabla u(x))\geq -C, \qquad \forall (x,y)\in\Omega\times\bar{\Omega}, \tag{59}$$

which in turn yields (56) and (57). □

The improved Hölder continuity of $\underline{u}$, $\bar{u}$ in the tangential directions enables us to obtain the

Proposition IV.3. *Let $f\in W^{1,\infty}(\Omega)$, $p>2$. Assume that either Ω is a ball, or Ω is an half-space, or Ω is convex and $p<3$, then $\alpha=\frac{p-2}{p-1}$ the maximum solutions $\underline{u}$, $\bar{u}$ satisfy*

$$t^{-\alpha}\{u(x_0-tn(x_0))-u(x_0)\}\to -C_0 \quad \textit{as} \quad t\to 0_+, \quad \textit{uniformly in } x_0\in\partial\Omega \tag{60}$$

$$\nabla u(x)d(x)^{1-\alpha}\to C_0\alpha n \quad \textit{as} \quad d(x)\to 0_+, \tag{61}$$

where $C_0=(1-\alpha)^{1/(p-1)}\alpha^{-1}$.

Proof. We just sketch it. Let $x_0\in\partial\Omega$, we introduce the blown-up-functions u_t defined by $u_t(x)=t^{-\alpha}\{u(x_0+tx)-u(x_0)\}$ defined on $Q_t=(\Omega-x_0)/t$. We want to let t go to 0_+. We first observe that by Theorem IV.1 and Corollary IV.1 u_t is bounded in $L^\infty(Q_t\cap B_R)$ ($\forall R<\infty$). In addition, u_t solves

$$-\Delta u_t+|\nabla u_t|^p=t^{2-\alpha}\{f(x_0+tx)-\lambda u(x_0+tx)\} \quad \text{in} \quad Q_t.$$

And we obtain easily a priori bounds from the interior gradient estimates: therefore u_t is relatively compact and any convergent subsequence u_{t_n} converges uniformly on compact sets, as t_n goes to 0, to a solution $v \in W^{2,r}_{\mathrm{loc}}(\Pi)$ $(\forall r < \infty)$ of

$$-\Delta v + |\nabla v|^p = 0 \quad \text{in} \quad \Pi, \qquad v(x_0) = 0, \qquad |v(x)| \leqq C|x|^\alpha \quad \text{in} \quad \Pi,$$

where $\Pi = \{x \in \mathbb{R}^N / n(x_0) \cdot x < 0\}$. In addition, using Proposition IV.1, we deduce that v is above any function $w \in W^{2,r}_{\mathrm{loc}}(\omega) \cap C(\bar{\omega})$ [resp. $W^{2,r}(\omega)$ $\forall r < \infty$ if we are dealing with $\underline{u}$] satisfying

$$-\Delta w + |\nabla w|^p \leqq 0 \quad \text{in} \quad \omega, \qquad w \leqq v \quad \text{on} \quad \partial\omega \cap \Pi.$$

Finally, the estimates (54), (55) or (57) imply that v depends only on the variable $x \cdot n(x_0)$ i.e. $v(x) = \varphi(-x \cdot n(x_0))$ where φ solves

$$-\varphi'' + |\varphi'|^p = 0 \quad \text{for} \quad t > 0, \qquad \varphi(0) = 0, \qquad \varphi \in C([0, \infty)) \cap C^2(0, \infty).$$

Hence, $\varphi(t) \leqq 0$ or $\varphi(t) = C_0 \lambda^\alpha - C_0(t + \lambda)^\alpha$ on $\mathbb{R}_+$ for some $\lambda \geqq 0$. But, since v is "a maximum solution" we deduce that $\varphi \geqq \psi$ on $[0, L]$ for all $L > 0$, $\psi \in C^2([0, L])$ satisfying

$$\psi(L) \leqq \varphi(L), \qquad -\psi'' + |\psi'|^p \leqq 0 \quad \text{in} \quad (0, L).$$

And this implies by Theorem IV.2 and its proof that

$$\liminf_{t \to 0_+} \varphi(t) t^{-\alpha} < 0$$

therefore $\varphi(t) \equiv -C_0 t^\alpha$ and we conclude easily. $\square$

We conjecture that if $p > 2$, $f \in L^\infty(\Omega)$ then (60) and (61) always hold. Of course, in view of the preceding argument, it would be enough to prove that $|u(x) - u(y)| \leqq C|x - y|^\theta$ for some $\theta < \alpha$ if $x, y \in \Omega$, $d(x) = d(y)$ but this type of estimate seems rather difficult to obtain in general.

V. Viscosity Formulation of the Boundary Conditions

V.1. Uniqueness Results

If we accept the stochastic control interpretation of the solutions built in the preceding sections, one is led (see [20] for more details) to the following formulation of the boundary condition

$$\text{for all} \quad \varphi \in C^2(\bar{\Omega}), \qquad u - \varphi \quad \text{achieves its minimum over} \quad \Omega. \tag{62}$$

Or course, an equivalent formulation in the case when $u \in C(\bar{\Omega})$ is to impose that $u - \varphi$ never has a local minimum on $\bar{\Omega}$ at a point $x_0 \in \partial\Omega$ for all $\varphi \in C^2(\bar{\Omega})$. It is quite clear that solutions considered in Sects. II and III satisfy (62), even with $\varphi \in C(\bar{\Omega})$, since they blow up at $\partial\Omega$. Similarly, the maximum solutions built in Sect. IV also satisfy (62), even with $\varphi \in C^{0,\theta}(\bar{\Omega})$ for $\theta > \alpha$, since they satisfy (17): indeed, assume for instance that $\bar{u} - \varphi$ does not achieve its minimum over Ω. Since $u, \varphi \in C(\bar{\Omega})$, there is a minimum point x_0 over $\bar{\Omega}$ of $u - \varphi$ and $x_0 \in \partial\Omega$. Then, we have for $x \in \Omega$

$$u(x) - u(x_0) \geqq \varphi(x) - \varphi(x_0)$$

therefore

$$\liminf_{x\in\Omega, x\to x_0} \{u(x)-u(x_0)\}\,|x-x_0|^{-\alpha} \geqq \liminf_{x\in\Omega, x\to x_0} \{\varphi(x)-\varphi(x_0)\}\,|x-x_0|^{-\alpha}$$

and the right-hand side is 0 since φ is smooth and $\alpha\in(0,1)$. And we reach a contradiction with (17). In other words, any solution of (1) satisfying (17) does satisfy the boundary condition in "viscosity form" given by (62).

Our goal in this section is to prove, under quite general assumptions, that there is a unique solution of (1) satisfying (62). In particular, when this holds, this will imply that with the notations of Sect. IV the maximum solutions $\bar{u}$ and $\underline{u}$ are equal.

We may now state our main result.

Theorem V.I. *There exists a unique solution of* (1) *in* $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r<\infty$) *satisfying* (62) *under one of the following three sets of assumptions*

i) $1<p\leqq 2$, $f\in L^\infty_{\mathrm{loc}}(\Omega)$ *satisfies* (14) *and is bounded from below.*

ii) $1<p$, $f\in L^\infty_{\mathrm{loc}}(\Omega)$ *satisfies* (40).

iii) $2<p$, $f\in L^\infty_{\mathrm{loc}}(\Omega)$ *satisfies* (41) *with* $\beta<p$ *and is bounded from below.*

Corollary V.1. *Let* $p>2$. *Let* $f\in L^\infty_{\mathrm{loc}}(\Omega)$ *satisfy* (41) *with* $\beta<p$ *and be bounded from below. Then, the maximum solutions built in Sect. IV are equal. Furthermore, if* $f\in C(\bar{\Omega})$ *or if* $f\in C(\Omega)$ *and* $f(x)d(x)^{-\theta}\to C_1$ *as* $d\to 0_+$ *with* $0<\theta<p$, *they coincide also with the envelope of all* $C^2(\bar{\Omega})$ *subsolutions of* (1).

Remark V.1. Actually, (62) can be proved to be equivalent to

$$u-\varphi \text{ achieves its minimum in } \Omega \text{ for all quadratic functions } \varphi. \qquad (62')$$

Indeed, suppose (62′) holds and let $\varphi\in C^2(\bar{\Omega})$. Let $x_n\in\Omega$ be a minimizing sequence for $u-\varphi$ converging to some $x_0\in\bar{\Omega}$. For C large enough we have $\psi(x)<\varphi(x)\ \forall x\neq x_0$, where ψ is defined by $\psi(x)=\varphi(x_0)+\nabla\varphi(x_0)\cdot(x-x_0)-C|x-x_0|^2\ \forall x\in\Omega$. Hence, ψ is quadratic and $u(x)-\psi(x)>u(x)-\varphi(x)=\min(u-\varphi)=\min(u-\psi)$ for all $x\neq x_0$. Hence from (62′) x_0 lies in Ω.

Proof of Corollary V.1. As we already said, $\bar{u}$, $\underline{u}$ satisfy (62) and so are equal by Theorem V.1. In addition, if we denote by $\tilde{u}$ the envelope of all $C^2(\bar{\Omega})$ subsolutions of (1); we first claim that by the same arguments as in Sects. III and IV u is a solution of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)\cap C(\bar{\Omega})$ ($\forall r<\infty$). If $f\in C^{0,\gamma}(\bar{\Omega})$ for some $\gamma>0$, the arguments indeed adapt without changes. If $f\in C(\bar{\Omega})$, we just approximate f by $f_n\in C^1(\bar{\Omega})$ such that $f_n\leqq f$, $f_n \underset{n}{\to} f$ uniformly on $\bar{\Omega}$. If $f\in C(\Omega)$ satisfies (63), we first observe that $g(x)=f(x)d(x)^\theta$ may be extended continuously to $\bar{\Omega}$ by giving it the value C_1 on $\partial\Omega$ then we approximate g by $g_n\in C^1(\Omega)$ such that $g_n\leqq g$, $g_n\underset{n}{\to} g$ uniformly on $\bar{\Omega}$, g_n is constant on $\partial\Omega$ and we consider $f_{n,R}=g_n(d(x)^{-\theta}\wedge R)$. And these approximations easily yield our claim on $\tilde{u}$.

Next, we observe that Proposition IV.1 and Theorem IV.2 may be applied or more precisely that their proofs are immediately adapted to the case of $\tilde{u}\leqq 0$ that $\tilde{u}$ satisfies (17). Hence, $\tilde{u}$ satisfies (62) and Corollary V.I is proved. □

V.2. Proofs

We begin with the *proof of Theorem V.1 in the case i).* We denote by $\bar{u}$ the unique solution of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r<\infty$) which blows up at (see Theorems II.1 and II.2) and we consider another solution u of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r<\infty$) satisfying (62). We obviously have $u \leqq \bar{u}$ by Theorems II.1 and II.2 and we want to show that the reverse inequality also holds. The strategy of proof is quite simple: we just observe that if $v \in C^2(\bar{\Omega})$ is a subsolution of (1) then $u-v$ achieves its minimum over Ω at some point x_0 because of (62) and by the maximum principle we deduce $(u-v)(x_0) \geqq 0$ hence $u \geqq v$. Therefore, if we are able to approximate $\bar{u}$ by $C^2(\bar{\Omega})$ subsolutions of (1), we complete the uniqueness proof. Now, if $f \in C^1(\bar{\Omega})$ such that $f_n \leqq f$, $f_n \geqq -C$ for some C independent of n and f_n converges uniformly to f on compact subsets of Ω. We next denote by $\bar{u}_n$ the corresponding unique solutions of (1) (with f replaced by f_n) which blow up near the boundary (Theorem II.1) and as remarked in Sect. II we know that $\bar{u}_n$ converges uniformly on compact subsets of Ω to $\bar{u}$ and of course $\bar{u}_n \leqq \bar{u}$. Since we know now by the proof of Theorem II.1 that $\bar{u}_n$ is an increasing limit of $C^2(\bar{\Omega})$ (use the smoothness of f_n) solutions of (1) (with f replaced by f_n), the desired sequence of subsolutions of (1) in $C^2(\bar{\Omega})$ is built.

However, this argument does not work as well if we only assume (as we did in Theorem V.1) that $f \in L^\infty_{\mathrm{loc}}(\Omega)$, satisfies (14) and is bounded from below. In this general case, we approximate f by f_n given by

$$f_n = f \quad \text{in} \quad \Omega_{1/n}, \qquad = -C_0 \quad \text{in} \quad \Omega - \Omega_{1/n}, \tag{63}$$

where $C_0 \geqq 0$ is any constant such that $f \geqq -C_0$ in Ω.

Again, we consider the solutions $\bar{u}_n$ of (1) (with f replaced by f_n) which blow up near $\partial\Omega$. We know from the proof of Theorem II.1 that there exists for each $n \geqq 1$, a sequence $(\bar{u}_{n,m})_{m \geqq 1}$ satisfying

$$\begin{aligned} -\Delta \bar{u}_{n,m} + |\nabla \bar{u}_{n,m}|^p + \lambda \bar{u}_{n,m} = f_n \quad \text{in} \quad \Omega, \qquad & \bar{u}_{n,m} \in W^{2,r}(\Omega) \, (\forall r < \infty), \\ & \bar{u}_{n,m} = m \quad \text{on} \quad \partial\Omega \end{aligned} \tag{64}$$

and $\bar{u}_{n,m} \underset{m}{\uparrow} \bar{u}_n$ uniformly on compact subsets of Ω.

Therefore, we obtain: $u \geqq \bar{u}_{n,m}$ and, passing to the limit in m, $u \geqq \bar{u}_n$. And we recall from the results and arguments of Sect. II that $\bar{u}_n$ increases to $\bar{u}$ and thus $u \geqq \bar{u}$, completing the proof of Theorem V.1 in case i).

The proof of case ii) is almost trivial: indeed, we apply (62) with $\varphi \equiv 0$ to deduce that any solution of (1) satisfying (62) is bounded from below on Ω. Thus, by Theorems III.3 and III.4, the uniqueness is proved.

Unfortunately, *the proof of case iii)* is much more complicated; in order to keep the ideas clear (or to try at least) we will begin with the case when $f \in C^1(\bar{\Omega})$ and Ω is starshaped (step 1), then we will treat the case when $f \in C^1(\bar{\Omega})$ but Ω is arbitrary (step 2) and we will conclude with the general case (step 3).

Step 1. $f \in C^1(\bar{\Omega})$, Ω is starshaped.

Without loss of generality we may assume that Ω is starshaped with respect to 0. Again, we denote by $\bar{u}$ the maximum solution of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r < \infty)$ or equivalently in $C^2(\Omega)$ in view of the smoothness of f – see Sect. IV. And we denote by u any other solution of (1) [in $C^2(\Omega)$] satisfying (62). Recall that $\bar{u} \in C(\bar{\Omega})$ (cf. Sect. IV) and observe that applying (62) with $\varphi = 0$, one deduces that u is bounded from below and thus u may be extended continuously to $\bar{\Omega}$ (cf. Sect. IV). Finally, $u \leqq \bar{u}$ and thus we want to show the reverse inequality.

We then introduce for $t \in (0,1)$

$$v_t(x) = t^{-\frac{p-2}{p-1}} u(tx) \quad \text{for} \quad x \in \Omega/t \supset \bar{\Omega}. \tag{65}$$

Obviously, v_t satisfies

$$-\Delta v_t + |\nabla v_t|^p + \lambda t^2 v_t = t^q f(tx) \quad \text{in} \quad \Omega/t, \qquad v_t \in C^2(\Omega/t) \cap C(\bar{\Omega}/t) \tag{66}$$

and thus in particular $v_t \in C^2(\bar{\Omega})$ and satisfies

$$-\Delta v_t + |\nabla v_t|^p + \lambda v_t \leqq f(x) + C(1-t) \quad \text{in} \quad \Omega \tag{67}$$

for some $C \geqq 0$ independent of t. In other words, $v_t - \frac{C}{\lambda}(1-t)$ is a $C^2(\bar{\Omega})$ subsolution of (1) and our strategy applies: $u \geqq v_t$ in $\bar{\Omega}$ and thus passing to the limit as t goes to 1 we conclude $u \geqq \bar{u}$ in $\bar{\Omega}$.

Step 2. $f \in C^1(\bar{\Omega})$, Ω arbitrary.

We first observe that by the maximum principle the minimum of $u - \bar{u}$ is achieved at the boundary. Furthermore, if $\theta \in (0,1)$, we may still assume that the minimum of $u - \theta\bar{u}$ is still achieved at the boundary. Indeed, if $u - \theta_n \bar{u}$ has an interior minimum over $\bar{\Omega}$ say at $x_n \in \Omega$ for some sequence $\theta_n \underset{n}{\to} 1$, then observing that $\theta_n \bar{u}$ satisfies

$$-\Delta(\theta_n \bar{u}) + |\nabla(\theta_n \bar{u})|^p + \lambda \theta_n \bar{u} \leqq \theta_n f \leqq f + C(1-\theta_n)$$

we deduce from the maximum principle

$$\min_{\bar{\Omega}} (u - \theta_n \bar{u}) \geqq -\frac{C}{\lambda}(1-\theta_n)$$

and we conclude letting n to to $+\infty$.

Therefore, let fix $\theta \in (0,1)$, we assume that $u - \theta\bar{u}$ has a minimum over $\bar{\Omega}$ at $x_0 \in \partial\Omega$. Then, we remark that $u - \theta\bar{u} + (1-\theta)|x - x_0|^2$ has a *unique* maximum over $\bar{\Omega}$ at $x_0 \in \partial\Omega$ and, denoting by $\tilde{u} = \theta\bar{u} + (1-\theta)|x - x_0|^2$, that $\tilde{u}$ satisfies

$$-\Delta\tilde{u} + |\nabla\tilde{u}|^p + \lambda\tilde{u} \leqq \theta f + C(1-\theta) \leqq f + C(1-\theta), \tag{68}$$

where C denotes various nonnegative constants independent of θ.

We next observe that for some small $\delta > 0$, the open set $Q = (x_0, \delta) \cap \Omega$ is starshaped with respect to a point that we denote by 0 such that $d(0) \geqq \gamma > 0$ where γ, δ are independent of x_0 and θ. We then consider as in step 1 the functions

$$v_t(x) = t^{-\frac{p-2}{p-1}} \tilde{u}(tx) \quad \text{for} \quad x \in Q/t, \qquad t \in (0,1)$$

and we obtain exactly as in step 1 using now (68) instead of (1)

$$-\Delta v_t + |\nabla v_t|^p + \lambda v_t \leqq f + C(1-\theta) + C(1-t) \quad \text{in} \quad Q, \qquad v_t \in C^2(\bar{\Omega}). \tag{69}$$

Let $\bar{x}$ be a minimum point of $u - v_t$ on $\bar{Q}$: because of (62), $\bar{x} \in Q$ or $\bar{x} \in \partial Q \cap \Omega$. If $\bar{x} \in Q$, we use maximum principle to deduce

$$\min_{\bar{\Omega}} (u - v_t) \geqq -\frac{C}{\lambda}(1-\theta) - \frac{C}{\lambda}(1-t)$$

and thus in particular

$$u(x_0) - t^{-\frac{p-2}{p-1}} \theta\bar{u}(x_0) \geqq -\frac{C}{\lambda}(1-\theta) - \frac{C}{\lambda}(1-t)$$

and we deduce letting t go to 1

$$\min_{\bar\Omega}(u-\theta\bar u)=(u-\theta\bar u)(x_0)\geqq -\frac{C}{\lambda}(1-\theta).$$

The conclusion follows upon letting θ go to 1.

In the other case i.e. if $\bar x\in\partial Q\cap\Omega$, we obtain letting t go to 1

$$(u-\tilde u)(x_0)\geqq \min_{\partial Q\cap\Omega}(u-\tilde u)$$

and this yields a contradiction with the fact that x_0 is the unique minimum point of $u-\tilde u$ over $\bar\Omega$.

Step 3. The general case.

We begin by observing that if $f\in C(\bar\Omega)$ or even if f is continuous near $\partial\Omega$ the above proof is easily adapted: the only difficulty lies with the fact that $\bar u$, $\tilde u$, v_t do not belong to C^2 in general. But this can be taken care of by observing that $v_t*\varrho_\delta=v_{t,\delta}\left[\text{where } \varrho_\delta=\frac{1}{\delta^N}\varrho\left(\frac{\cdot}{\delta}\right),\ \varrho\geqq 0,\ \varrho\in\mathscr{D}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}\varrho dx=1,\ \operatorname{Supp}\varrho\subset B(0,1)\right]$ satisfies

$$\begin{aligned}&-\Delta v_{t,\delta}+|\nabla v_{t,\delta}|^p+\lambda v_{t,\delta}\\&\leqq f*\varrho_\delta+C(1-\theta)+\omega(1-t)\quad\text{in}\quad \{x\in Q/t,\ \operatorname{dist}(x,\partial Q/t)>\delta\},\end{aligned}$$

where ω is a modulus of continuity of f near the boundary. Taking δ small enough, we find that $v_{t,\delta}\in C^2(\bar Q)$ and

$$-\Delta v_{t,\delta}+|\nabla v_{t,\delta}|^p+\lambda v_{t,\delta}\leqq f+C(1-\theta)+\omega(1-t)+\omega(\delta)\quad\text{in}\quad Q$$

and we conclude as before letting δ go to 0, then t go to 1 and then θ go to 1.

To obtain the uniqueness in the case of a general f, we approximate f by f_n given by (63) where $C_0\geqq 0$ is any constant such that $f\geqq -C_0$ in Ω. The above arguments show that $u\geqq u_n$ where u_n is the unique solution of (1) satisfying (62) (with f replaced by f_n). Obviously, $\bar u\geqq u_n$ and u_n increases to a solution in $W^{2,r}(\Omega)$ $(\forall r<\infty)\cap C(\bar\Omega)$ of (1) and we just have to show that $\bar u\leqq\hat u$, where $\hat u$ denotes the limit of u_n. To this end let $\theta\in(0,1)$, let $\gamma\in(\beta,p)$, $\sigma=1-\gamma/p$ and choose $K>0$, $C>0$ so that $w=-C-Kd^\sigma$ satisfies

$$-\Delta w+|\nabla w|^p+\lambda w\leqq -C_0-\nu d^{-\gamma}\quad\text{in}\quad\Omega,\quad\text{for some}\quad \nu>0. \tag{70}$$

Then, we remark that $z=\theta\bar u+(1-\theta)w$ satisfies

$$-\Delta z+|\nabla z|^p+\lambda z\leqq\theta f-(1-\theta)(C_0+\nu d^{-\gamma})=g.$$

But on $\Omega_{1/n}$,

$$g=f-(1-\theta)(f+C_0+\nu d^{-\gamma})\leqq f$$

while on $\Omega-\Omega_{1/n}$

$$g\leqq\theta C(1+d^{-\beta})-(1-\theta)(C_0+\nu d^{-\gamma})$$

and thus, $g\leqq f_n$ in Ω provided n is large enough say $n\geqq n_0(\theta)$. Hence,

$$\theta\bar u+(1-\theta)w\leqq u_n\quad\text{if}\quad n\geqq n_0(\theta)$$

and passing to the limit in n, we deduce

$$\theta \bar{u} + (1-\theta) u \leqq \hat{u}.$$

We may now conclude letting θ go to 1. $\square$

V.3. Applications

We want in this section to show that (17) is equivalent to (62) when $p>2$ (with appropriate conditions on f) and that (62) is stable under some passages to the limit. This together with the uniqueness proved in Theorem V.1 will yield a rather powerful stability result. We begin with the relations between (17) and (62). Recall that (17) implies trivially (62).

Theorem V.1. *Let $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$, let $p>2$, let $f \in L^{\infty}_{\mathrm{loc}}(\Omega)$ be bounded from below and let $x_0 \in \Omega$. Assume that $u \in C(\bar{\Omega})$ satisfies* (62) *and*

$$-\Delta u + |\nabla u|^p + \lambda u \geqq f \quad \text{in} \quad \Omega, \quad \text{for some} \quad \lambda, C \geqq 0. \tag{72}$$

Then, u satisfies

$$\liminf_{x \in \Omega, x \to x_0} \{u(x) - u(x_0)\} |x - x_0|^{-\alpha} < 0, \quad \text{where} \quad \alpha = (p-2)/(p-1). \tag{73}$$

Proof. The proof is almost the same as the one of Theorem IV.2: with the notations of Theorem IV.2, we just have to replace ω by w^n defined exactly as w with d replaced by $d + \frac{1}{n}$. Then, $w^n \in C^2(\bar{\omega})$ is a subsolution of (1) and $u \geqq w^n$ on $\partial\omega \cap \Omega$. Therefore, by maximum principle, $u - w^n$ achieves its minimum on $\partial\Omega$ and we reach a contradiction. $\square$

Remark V.1. Many variants of the above result and of its proof exist that we will skip here.

We now present a stability result.

Theorem V.2. *Let $(F_n)_n$ be a sequence of continuous functions on $\mathbb{M}^N \times \mathbb{R}^N \times \mathbb{R} \times \bar{\Omega}$ where $\mathbb{M}^N$ denotes the space of $N \times N$ symmetric matrices, let $u_n \in W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ satisfy for some $C \geqq 0$ independent of n*

$$F_n(D^2 u_n, Du_n, u_n, x) \geqq -C \quad a.e. \text{ in } \quad \Omega. \tag{74}$$

We assume that $F_n(A, \xi, t, x) \leqq F_n(B, \xi, t, x)$ for all $\xi \in \mathbb{R}^N$, $t \in \mathbb{R}$, $x \in \Omega$, $A \geqq B$ (in the sense of symmetric matrices), F_n converges uniformly on compact subsets to $-\mathrm{tr}(A) + |\xi|^p + \lambda t$, for some $p>2$, $\lambda \geqq 0$, u_n satisfies (62) *and converges uniformly on compact subsets of Ω to some function $u \in C(\bar{\Omega})$, and that $(u_n - u)^-$ converges uniformly to* 0 *on Ω. Then, u satisfies* (62).

Proof. Assume by way of contradiction that $u - \varphi$ admits a minimum at $x_0 \in \partial\Omega$ for some $\varphi \in C^2(\bar{\Omega})$, without loss of generality we may assume that x_0 is the unique minimum point of $u - \varphi$. By assumption, $u_n - \varphi$ achieves its minimum over Ω at some point $x_n \in \Omega$. We remark that

$$\min_{\bar{\Omega}} (u_n - \varphi) \leqq u_n(x) - \varphi(x) \xrightarrow[n]{} u(x) - \varphi(x) \quad \text{for all} \quad x \in \Omega$$

while

$$\min_{\Omega}(u_n-\varphi)\geqq \min_{\bar\Omega}(u-\varphi)-\|(u_n-u)^-\|_\infty,$$

hence $u_n(x_n)-\varphi(x_n)$ converges to $u(x_0)-\varphi(x_0)$. Now if x_n (or a subsequence) converges to $\bar x\in\bar\Omega$, then

$$u_n(x_n)-\varphi(x_n)\geqq u(x_n)-\varphi(x_n)-\|(u_n-u)^-\|_\infty \underset{n}{\to} u(\bar x)-\varphi(\bar x)$$

and thus $\bar x=x_0$ by the uniqueness of the minimum.

Next, by maximum principle, we have

$$F_n(D^2\varphi(x_n), D\varphi(x_n), u_n(x_n), x_n)\geqq -C$$

and passing to the limit we find

$$-\Delta\varphi(x_0)+|\nabla\varphi(x_0)|^p+\lambda u(x_0)\geqq -C. \tag{75}$$

and we observe that we may replace φ by $\varphi+c(\delta^\alpha-(d+\delta)^\alpha)$ where $\delta>0$, $\alpha=\frac{p-2}{p-1}$, $c>0$, since $u-\varphi+c((d+\delta)^\alpha-\delta^\alpha)$ admits also a unique minimum at x_0. Therefore, we deduce from (75)

$$-\Delta\varphi(x_0)+c\alpha\delta^{-(1-\alpha)}\Delta d(x_0)-c\alpha(1-\alpha)\delta^{-(2-\alpha)}+|\nabla\varphi(x_0)-c\alpha d^{1-\alpha}\nabla d|^p\geqq -C$$

and if we choose c in such a way that $(c\alpha)^{p-1}<(1-\alpha)$, we easily reach a contradiction letting δ go to 0. □

From this stability result, we deduce the

Corollary V.2. *Let $p>2$, let $f_n\in L^\infty_{\mathrm{loc}}(\Omega)$ satisfy*

$$f_n\geqq -C,\quad f_n\leqq Cd^{-\beta}\quad \text{a.e. in}\quad \Omega,\quad \text{for some}\quad C\geqq 0,\quad \beta\in(0,p). \tag{76}$$

We denote by u_n the unique solution in $W^{2,r}_{\mathrm{loc}}(\Omega)\cap C(\bar\Omega)$ ($\forall r<\infty$) of (1) *satisfying* (62) *and we assume that f_n converges to f weakly in $L^\infty_{\mathrm{loc}}-*$. We denote by u the unique solution in $W^{2,r}_{\mathrm{loc}}(\Omega)\cap C(\bar\Omega)$ ($\forall r<\infty$) of* (1) *satisfying* (62). *Then, u_n converges uniformly on $\bar\Omega$ to u.*

Proof. By Theorem IV.1, u_n is bounded in $C^{0,\gamma}(\bar\Omega)$ for some $\gamma>0$ and in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r<\infty$). Then, we may assume (up to subsequences) that u_n converges uniformly on $\bar\Omega$ to a solution u of (1) [in $W^{2,r}_{\mathrm{loc}}(\Omega)\cap C(\bar\Omega)$ for all $r<\infty$]. By Theorem V.2, u satisfies (62) and thus $u\equiv\tilde u$ by Theorem V.1. □

VI. The Ergodic Problem

In this section, we want to study the questions associated with the so-called ergodic stochastic control problems with state constraints. From the p.d.e.'s viewpoint this amounts to study the behaviour of λu and u as λ goes to 0 where u is the solution of (1) considered in the preceding sections. We will perform such an analysis in the three different cases studied above. The typical result we will obtain is that λu, $u-u(x_0)$ converge uniformly on compact subsets of Ω to $u_0\in\mathbb{R}$, v solution of

$$-\Delta v+|\nabla v|^p+u_0=f\quad\text{in}\quad\Omega,\qquad v(x_0)=0 \tag{77}$$

with the same boundary conditions for v than for u. And these will uniquely determine (u_0, v). In the preceding statements and below, x_0 is any fixed point in Ω and we assume that Ω is connected.

VI.1. Subquadratic Hamiltonians

Whenever it exists, we will denote by u_λ the solution of (1) with appropriate boundary conditions and if x_0 is any fixed point in Ω we will denote by $v_\lambda(\cdot)=u_\lambda(\cdot)-u_\lambda(x_0)$. We assume throughout this section that $1<p<2$.

Theorem VI.1. *Let $f\in L^\infty_{\text{loc}}(\Omega)$ be bounded from below and satisfy*

$$\lim\{f(x)d(x)^{-q}/d(x)\to 0_+\}=0. \tag{78}$$

Let u_λ be the unique solution of (1) in $W^{2,r}_{\text{loc}}(\Omega)$ ($\forall r<\infty$) such that $u_\lambda\to+\infty$ as $d\to 0_+$. Then, ∇u_λ and λu_λ are bounded in $L^\infty_{\text{loc}}(\Omega)$ and λu_λ, v_λ converge uniformly on compact subsets of Ω to $u_0\in\mathbb{R}$, $v\in W^{2,r}_{\text{loc}}(\Omega)$ ($\forall r<\infty$) such that $v(x_0)=0$, v satisfies (15) and

$$-\Delta v+|\nabla v|^p+u_0=f \quad in \quad \Omega. \tag{79}$$

In addition, if $(\tilde{u}_0,\tilde{v})\in\mathbb{R}\times W^{2,r}_{\text{loc}}(\Omega)$ ($\forall r<\infty$) satisfies (79) and $\tilde{v}$ goes to $+\infty$ as d goes to 0_+, then $\tilde{u}_0=u_0$, $\tilde{v}=v+C$ for some $C\in\mathbb{R}$.

Proof. The proof involves several steps, we first obtain some bounds and we pass to the limit (Step 1). Then, we show that for any solution $(\tilde{u}_0,\tilde{v})$ as above $\tilde{v}$ blows up at the boundary like $C_0 d^{-\alpha}$ (Step 2). Next, we show the uniqueness of u_0 (Step 3). Finally, we conclude with the uniqueness (up to constants) of $\tilde{v}$ (Step 4).

Step 1. Going through the proofs of Theorems II.1 and II.2, we see that u_λ satisfies for all $\varepsilon>0$, $\lambda\in(0,1]$

$$\frac{C_0-\varepsilon}{d^\alpha}-\frac{C_\varepsilon}{\lambda}\leqq u_\lambda\leqq\frac{C_0+\varepsilon}{d^\alpha}+\frac{C_\varepsilon}{\lambda} \tag{80}$$

for some $C_\varepsilon\geqq 0$, with the usual modifications if $\alpha=0$ (i.e. $p=2$). In particular, λu_λ is bounded from below and in L^∞_{loc}. Then, using Theorem IV.1, we deduce that ∇u_λ is bounded from below. Therefore, v_λ is bounded in $W^{1,\infty}_{\text{loc}}$.

We next want to show that v_λ satisfies

$$\frac{C_1}{d^\alpha}-C\leqq v_\lambda \quad \text{in} \quad \Omega, \quad \text{for some} \quad C_1\in(0,C_0), \quad C\geqq 0. \tag{81}$$

Observe first that v_λ satisfies

$$-\Delta v_\lambda+|\nabla v_\lambda|^p+\lambda v_\lambda+\lambda u_\lambda(x_0)=f \quad \text{in} \quad \Omega.$$

And if we choose C_1 in $(0,C_0)$, we obtain denoting by $z=\dfrac{C_1}{d^\alpha}$

$$-\Delta z+|\nabla z|^p-\lambda z\leqq f-\lambda u_\lambda(x_0) \quad \text{on} \quad \Omega-\Omega_\delta$$

if δ is small enough, say $\delta\leqq\delta_0$. Now, there exists a constant $M\geqq 0$ such that

$$v_\lambda\geqq M \quad \text{on} \quad \Omega_{\delta_0}.$$

Hence, adapting the comparison results proved in Sect. II, we deduce

$$v_\lambda \geqq -M + \frac{C_1}{d^\alpha} \quad \text{on} \quad \Omega.$$

Extracting subsequences if necessary – the convergence of the whole sequence will follow from the uniqueness –, we may now pass to the limit $\lambda u_\lambda(x_0)$ converges to a constant u_0, v_λ converges to a solution v of (79) satisfying (81) and such that $v(x_0)=0$.

Step 2. Let $(\tilde{u}_0, \tilde{v}) \in \mathbb{R} \times W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ be a solution of (79) such that $\tilde{v}$ goes to $+\infty$ as d goes to 0_+. We want to prove that $\tilde{v}$ satisfies (15). To this end, we recall that $\bar{w}_{\varepsilon,\delta} = \dfrac{C_0+\varepsilon}{(d-\delta)^\alpha}$ satisfies

$$-\Delta \bar{w}_{\varepsilon,\delta} + |\nabla \bar{w}_{\varepsilon,\delta}|^p \geqq f - \tilde{u}_0 \quad \text{in} \quad \Omega_\delta - \Omega_{\delta_0} \quad \text{if} \quad 0<\delta<\delta_0=\delta_0(\varepsilon).$$

Then, let $M_\varepsilon = \sup\{|\tilde{v}(x)|/x\in\Omega,\ d(x)=\delta_0(\varepsilon)\}$, we deduce from the maximum principle that

$$\tilde{v} \leqq \bar{w}_{\varepsilon,\delta} + M_\varepsilon \quad \text{on} \quad \Omega_\delta - \Omega_{\delta_0}$$

and letting δ go to 0, we deduce

$$-C \leqq \tilde{v} \leqq (C_0+\varepsilon) d^{-\alpha} + M_\varepsilon \quad \text{on} \quad \Omega. \tag{82}$$

Next, we simply observe that $\tilde{v}$ satisfies

$$-\Delta\tilde{v} + |\nabla\tilde{v}|^p + \tilde{v} = g \quad \text{in} \quad \Omega, \qquad \tilde{v} \to +\infty \quad \text{as} \quad d \to 0_+$$

where $g = f - \tilde{u}_0 + \tilde{v} \in L^\infty_{\text{loc}}(\Omega)$ satisfies (78) because of (82). Therefore, Theorem II.2 yields the desired behaviour of $\tilde{v}$ near $\partial\Omega$.

Step 3. We first show that if (u_0, v), $(\tilde{u}_0, \tilde{v})$ are two solutions of (79) such that $v, \tilde{v} \to +\infty$ as $d \to 0_+$ then $u_0 = \tilde{u}_0$. To do so, we adapt an argument from Lions [16, 19]. Assume for instance that $u_0 < \tilde{u}_0$ and let $\varepsilon > 0$, $\theta \in (0,1)$. Obviously, we have

$$\begin{aligned} -\Delta(\theta\tilde{v}) + |\nabla(\theta\tilde{v})|^p + \varepsilon\theta\tilde{v} &\leqq \theta f + \varepsilon\theta\tilde{v} - \theta\tilde{u}_0 \\ &\leqq f + C(1-\theta) + \varepsilon\theta\tilde{v} - \theta\tilde{u}_0 . \end{aligned}$$

Next, since v, $\tilde{v}$ behave like $C_0 d^{-\alpha}$ near $\partial\Omega$, $\theta\tilde{v} \leqq v + C_\theta$ in Ω; hence

$$\begin{aligned} -\Delta(\theta\tilde{v}) + |\nabla(\theta\tilde{v})|^p + \varepsilon\theta\tilde{v} &\leqq f + C(1-\theta) + \varepsilon v + \varepsilon C_\theta - \theta\tilde{u}_0 \\ &\leqq f + \varepsilon v - u_0 + (u_0 - \theta\tilde{u}_0) + \varepsilon C_\theta + C(1-\theta) \end{aligned}$$

while v satisfies of course

$$-\Delta v + |\nabla v|^p + \varepsilon v = f + \varepsilon v - u_0 \quad \text{in} \quad \Omega.$$

But $u_0 < \tilde{u}_0$. Therefore, for θ near 1 and ε small enough (depending on θ) $\theta\tilde{v}$ is a subsolution of the equation satisfied by v. By Theorem II.2, this implies $\theta\tilde{v} \leqq v$. Letting θ go to 1, we find $\tilde{v} \leqq v$. But, $v + C_1$, $\tilde{v} + C_2$ satisfy the same problems for arbitrary constants C_1, C_2 and we reach a contradiction.

Step 4. Uniqueness of v_0 up to a constant.

Let $C_1 \in (0, C_0)$, again we observe that

$$-\Delta\left(\frac{C_1}{d^\alpha}\right) + \left|\nabla\left(\frac{C_1}{d^\alpha}\right)\right|^p \leqq f - u_0 \quad \text{in} \quad \Omega - \Omega_\delta$$

for some small enough $\delta > 0$. Therefore, if $\theta \in (0, 1)$, $w = \theta\tilde{v} + (1-\theta)\frac{C_1}{d^\alpha}$ satisfies

$$-\Delta w + |\nabla w|^p \leqq \theta(f - u_0) + (1-\theta)(f - u_0) = f \quad \text{in} \quad \Omega - \Omega_\delta .$$

And since v, $\tilde{v}$ behave like $\frac{C_0}{d^\alpha}$, $(w - v) \to -\infty$ as $d \to 0_+$. Therefore, by the maximum principle,

$$\max_{\Omega - \Omega_\delta} (w - v) = \max_{\partial\Omega_\delta} (w - v) .$$

Hence, if we let θ go to 1, we find that

$$\sup_{\Omega - \Omega_\delta} (\tilde{v} - v) = \max_{\partial\Omega_\delta} (\tilde{v} - v) .$$

On the other hand, we also deduce from the maximum principle that

$$\max_{\bar{\Omega}_\delta} (\tilde{v} - v) = \max_{\partial\Omega_\delta} (\tilde{v} - v) .$$

Therefore, any maximum point $\bar{x}$ of $\tilde{v} - v$ on $\partial\Omega_\delta$ is in fact a global maximum point of $\tilde{v} - v$ on Ω. But, since $\tilde{v} - v = \psi$ satisfies the equation

$$-\Delta\psi + B \cdot \nabla\psi = 0 \quad \text{in} \quad \Omega$$

for some B $L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$, the strong maximum principle then yields

$$\tilde{v} - v \equiv (\tilde{v} - v)(\bar{x}) \quad \text{in} \quad \Omega$$

and we conclude. □

We would like to conclude this section with a few remarks on the case $p = 2$ which make a connection between our results and the interpretation of first eigenvalues in terms of optimal stochastic control that was considered by Holland [9, 10]. Indeed, if $p = 2$ and if v solves (79) with $v \to \infty$ as $d \to 0_+$ then we may perform the well known logarithmic transformation i.e. $v = -\mathrm{Log}\,\varphi$ and we find

$$-\Delta\varphi + f\varphi = u_0\varphi \quad \text{in} \quad \Omega, \qquad \varphi > 0 \quad \text{in} \quad \Omega, \qquad \varphi \to 0 \quad \text{as} \quad d \to 0_+ \tag{83}$$

or in other words u_0 is the minimum eigenvalue of the operator $(-\Delta + f)$ with Dirichlet boundary conditions. And the uniqueness of u_0 corresponds to the uniqueness of an eigenvalue with a positive eigenfunction, while the uniqueness of v up to an additive constant corresponds to the uniqueness of φ up to a multiplicative constant.

VI.2. Forced Infinite Boundary Conditions

We will be now concerned with the case when f grows so fast at the boundary that u_λ automatically has to blow up at $\partial\Omega$. To simplify the presentation, we will only consider the case when f satisfies (40) and therefore, by Theorem III.4, u_λ is the unique solution of (1) which is bounded from below.

Theorem VI.2. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ satisfy (40) and let $p>1$, we denote by u_λ the unique solution of (1) which is bounded from below. Then, ∇u_λ and λu_λ are bounded in $L^\infty_{\mathrm{loc}}(\Omega)$ and λu_λ, v_λ converge uniformly on compact subsets of Ω to $u_0 \in \mathbb{R}$, $v \in W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ such that $v(x_0)=0$, v satisfies (79) and*

$$M^{-1}d^{-\alpha}-M \leqq v \leqq Md^{-\alpha}+M \quad \text{in} \quad \Omega, \quad \textit{for some} \quad M \geqq 1,$$

$$\textit{where} \quad \alpha=\frac{\beta}{p}-1 \quad \textit{if} \quad \beta>p \tag{84}$$

and $d^{-\alpha}$ is replaced by $|\mathrm{Log}\, d|$ if $\beta=p\geqq q$. In addition, if f satisfies (18′), then v satisfies (19). Furthermore, if $(\tilde{u}_0, \tilde{v}) \in \mathbb{R} \times W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ satisfies (79) and $\tilde{v}$ is bounded from below, then $\tilde{u}_0=u_0$, $\tilde{v}=v+C$ for some $C \in \mathbb{R}$.

Remark VI.1. If we consider the special case $p=2$ and if we perform the same logarithmic transformation as in the preceding section $v=-\mathrm{Log}\,\varphi$, we see that we are dealing with bounded, positive solutions of (83) and that the very fact that f blows up fast enough at $\partial\Omega$ forces φ to vanish on the boundary. Again, the uniqueness part of the above result may be interpreted as a uniqueness for first eigenvalues and eigenfunctions of the operator $-\Delta+f$ where no boundary condition on φ is imposed except "φ is bounded".

Proof of Theorem VI.2. Most of the proof of Theorem VI.1 goes through in this case except for the uniqueness arguments which use the precise behaviours of v, $\tilde{v}$ near the boundary. Of course, if we assume (18′) then the proof of Theorem VI.1 applies with some rather easy adaptations. In the general case, however, we have to involve slightly more elaborate arguments to show the uniqueness part of the above result. We only prove as in Theorem VI.1 that v, $\tilde{v}$ both satisfy (84). Next, we prove that $u_0=\tilde{u}_0$. We see that the corresponding proof (Step 3) in the proof of Theorem VI.1 only uses the fact that $\theta\tilde{v} \leqq v+C_\theta$ for any two solutions (u_0, v), $(\tilde{u}_0, \tilde{v})$ and for all $\theta \in (0,1)$. But this can be deduced from Theorem III.4: indeed $w=\theta\tilde{v}$ is a subsolution of the equation

$$-\Delta w+|\nabla w|^p+w \leqq g=\theta f+\theta\tilde{v}-\tilde{u}_0 \quad \text{in} \quad \Omega.$$

But in view of (40) and (84) $\theta f+\theta\tilde{v}-\tilde{u}_0 \leqq f+C'_\theta \leqq f+v-u_0+C_\theta$ for some constants C'_θ, $C_\theta \geqq 0$. Therefore, by Theorem III.4, we deduce

$$w \leqq v+C_\theta \quad \text{in} \quad \Omega$$

and our claim is proved.

Finally, we have to show the uniqueness of v up to a constant. Then, we observe that the proof given in Step 4 of the proof of Theorem VI.1 still applies provided we take C_1 small enough, indeed the only difference comes into the verification that $w-v \to -\infty$ as $d \to 0_+$ where $w=\theta\tilde{v}+(1-\theta)\dfrac{C_1}{d^\alpha}$ with $0<\theta<1$. But the inequality we just proved shows that

$$w \leqq \frac{1+\theta}{2}\, v+C_\theta+(1-\theta)\frac{C_1}{d^\alpha}$$

and taking C_1 small enough so that $\frac{C_1}{d^\alpha} \leqq \frac{1}{4} v + C$ we find

$$w \leqq \left(\frac{1+\theta}{2} + \frac{1-\theta}{4}\right) v + C_\theta + C = \frac{3+\theta}{4} v + C_\theta + C$$

therefore $w - v \to -\infty$ as $d \to 0_+$ since $v \to +\infty$ as $d \to 0_+$.

Then, the proof of Theorem VI.1 applies and we may conclude the proof of Theorem VI.2. □

VI.3. Superquadratic Hamiltonians

We now conclude this section by examining the remaining case namely the case when $p>2$ and $f \in L^\infty_{\text{loc}}(\Omega)$, is bounded from below and satisfies (41) with $\beta < p$. Then, we know there exists a unique solution u_λ of (1) satisfying (17) or (62). As λ goes to 0_+, we obtain the

Theorem VI.3. *Let $p>2$, let $f \in L^\infty_{\text{loc}}(\Omega)$ be bounded from below and satisfy* (41) *for some $\beta < p$. We denote by u_λ the unique solution of* (1) *in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r < \infty)$ satisfying* (62). *Then, ∇u_λ is bounded in $L^\infty_{\text{loc}}(\Omega)$ and λu_λ is bounded in $L^\infty(\Omega)$. And λu_λ, v_λ converge uniformly on $\bar{\Omega}$ to $u_0 \in \mathbb{R}$, $v \in W^{2,r}_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$ $(\forall r < \infty)$ such that $v(x_0) = 0$, v satisfies* (17) *and* (79). *In addition, if $(\tilde{u}_0, \tilde{v}) \in \mathbb{R} \times W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r < \infty)$ satisfies* (79) *and if $\tilde{v}$ satisfies* (62), *then $\tilde{u}_0 = u_0$, $\tilde{v} = v + C$ for some $c \in \mathbb{R}$.*

Proof. Clearly, λu_λ is bounded from below. Then, we may apply the local gradient bound given in Theorem IV.1: hence, ∇u_λ is bounded in $L^\infty_{\text{loc}}(\Omega)$. But this bound (see Corollary IV.1) also implies that u_λ is bounded in $C^{0,\theta}(\bar{\Omega})$ for some $\theta \in (0,1)$ independent of λ. Therefore, up to subsequences, λu_λ and u_λ converge uniformly on $\bar{\Omega}$ to $u_0 \in \mathbb{R}$, $v \in W^{2,r}_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$ $(\forall r < \infty)$ such that $v(x_0) = 0$, v satisfies (79). Next, by Theorem V.2, v satisfies (62) and therefore, by Proposition V.1, v satisfies (17).

Notice also that v is the unique solution satisfying (62) of

$$-\Delta v - |\nabla v|^p + \lambda v = g_\lambda \quad \text{in} \quad \Omega,$$

where $g_\lambda = f - u_0 + \lambda v$.

Next, using Theorem V.1, the uniqueness of u_0 follows immediately as in Step 3 of the proof of Theorem VI.1 ($\theta = 1$ is enough in this case).

Finally, we want to prove the uniqueness of v_0 up to a constant. Again, the only fact we have to prove is the following

$$\sup_{\Omega - \Omega_\delta} (\tilde{v} - v) = \max_{\partial\Omega_\delta} (\tilde{v} - v).$$

To this end, we set $\alpha = \frac{p-2}{p-1}$ and we consider for $\theta \in (0,1)$, $w = \theta\tilde{v} - (1-\theta) C_1 d^\alpha$ then for $C_1 > 0$, δ small enough (independent of θ) we have

$$-\Delta w + |\nabla w|^p \leqq f - u_0 - 1 \quad \text{in} \quad \Omega - \Omega_\delta.$$

In particular, for λ small enough, we have

$$-\Delta w + |\nabla w|^p + \lambda w \leqq g_\lambda \quad \text{in} \quad \Omega - \Omega_\delta.$$

We may now adapt without any real modification the proof of Theorem V.1 to deduce

$$\sup_{\Omega-\Omega_\delta} (w-v) = \max_{\partial\Omega_\delta} (w-v)$$

and we conclude letting θ go to 1. □

VII. Optimal Stochastic Control with State Constraints

We now want to use the results obtained in the preceding sections in order to solve some optimal stochastic control problems with state constraints: a rather vague way to formulate our problem is to say that we want to "constrain a Brownian motion in a given domain Ω by controlling its drift". More precisely, we consider a system whose state is given by the solution of the following stochastic differential equation

$$dX_t = a_t dt + \sqrt{2}\, dB_t, \qquad X_0 = x \in \Omega, \tag{3}$$

where B_t is a Brownian motion on a standard probability space (Ω, F, F_t, P) and where a_t is the control process i.e. a progressively measurable stochastic process taking values in $\mathbb{R}^N$ for instance. In other words

$$X_t = x + \int_0^t a_s ds + \sqrt{2} B_t$$

and we assume (at least) that $\int_0^T |a_s| ds < \infty$ a.s. $(\forall r < \infty)$.

We will say that this control a is admissible if $X_t \in \Omega\ \forall t \geqq 0$ a.s. Even if we could work with general controls of the above form, we will restrict ourselves to feedbacks or Markovian controls which are defined as follows. Let $a \in C(\Omega; \mathbb{R}^N)$, we may solve the stochastic differential equation

$$dX_t = a(X_t) dt + \sqrt{2}\, dB_t \quad \text{for} \quad 0 \leqq t < \tau_x, \qquad X_0 = x \in \Omega, \tag{85}$$

where τ_x is the first exit time of X_t from Ω i.e., $\tau_x = \inf\{t \geqq 0, X_t \notin \Omega\}$ ($\tau = +\infty$ if $X_t \in \Omega$ for all $t \geqq 0$). Thus, $a(X_t)$ is really the control but we will ignore this minor point of terminology and we will say that $a(\cdot)$ is the control (or control policy). Next, we define an admissible control as a control $a(\cdot)$ such that

$$P(\tau_x < \infty) = 0 \quad \text{for all} \quad x \in \Omega. \tag{86}$$

And we will denote by $\mathscr{A}$ the class of all admissible controls.

For each a, we define a cost function

$$J_\lambda(x, a) = E \int_0^\infty \{f(X_t) + c|a(X_t)|^q\} e^{-\lambda t} dt, \tag{87}$$

where $c = c(p, q) = q^{-1} p^{-1/(p-1)}$, $\lambda > 0$ is the discount factor. Observe that the running cost $f(x) + c|a|^q$ contains two terms: one which measures the cost for the state to be at x, and the other measuring the cost for using the control a. All throughout this section $f \in L^\infty_{\mathrm{loc}}(\Omega)$ is bounded from below so that $J_\lambda(x, a)$ makes sense even if it may be infinite. This is of course a very special example but we will come back on part 2 on much more general problems for which similar results to those which follow still hold.

Finally, we want to minimize J_λ. We introduce the value function

$$u_\lambda(x) = \inf_{a \in \mathscr{A}} J_\lambda(x, a), \qquad \forall x \in \Omega. \tag{88}$$

The typical questions that one wants to solve in such problems is to determine u_λ and possibly an optimal control (here an optimal Markovian control or an optimal feedback), i.e. some a in $\mathscr{A}$ such that

$$u_\lambda(x) = J_\lambda(x, a), \qquad \forall x \in \Omega.$$

And this is precisely what we will achieve using the results of the preceding sections. Let us also observe that it is not completely obvious that $\mathscr{A} \neq \emptyset$, let alone that there exists $a \in \mathscr{A}$ such that $J(x, a) < \infty$ for $x \in \Omega$.

Finally, we will also consider the case of ergodic control which consists, roughly speaking, in taking $\lambda = 0$.

VII.1. Subquadratic Hamiltonians

Theorem VII.1. *Let $1 < p < 2$, let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below and satisfy* (78). *Then, the value function u_λ given by* (88) *is the unique solution of* (1) *in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$) such that $u_\lambda \to +\infty$ as $d \to 0_+$. Furthermore, $a_0(x) = p|\nabla u_\lambda|^{p-2} \nabla u_\lambda(x)$ is the unique optimal markovian control.*

Proof. We denote by $\tilde{u}_\lambda$ the unique solution of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$) such that $\tilde{u}_\lambda \to +\infty$ as $d \to 0_+$. We are first going to show that $\tilde{u}_\lambda \geqq u_\lambda$ and that $a_0 \in \mathscr{A}$. Indeed, for $\delta > 0$ let $\tau^\delta_x = \inf(t \geqq 0, X_t \notin \Omega_\delta)$, we apply Itô's formula on $[0, \tau_\delta]$ with the process X_t corresponding to the control $a_0(x) = -p|\nabla \tilde{u}_\lambda|^{p-2} \nabla u_\lambda$ and we find for all x Ω_δ

$$\tilde{u}_\lambda(x) = E \int_0^{\tau^\delta_x} \{-\Delta \tilde{u}_\lambda(X_t) + p|\nabla \tilde{u}_\lambda|^p(X_t) + \lambda \tilde{u}_\lambda(X_t)\} e^{-\lambda t} dt + E\tilde{u}_\lambda(X_{\tau^\delta_x}) e^{-\lambda \tau^\delta_x}$$

hence from the equation (1) this yields

$$\tilde{u}_\lambda(x) = E \int_0^{\tau^\delta_x} \{f(X_t) + c|a_0(X_t)|^q\} e^{-\lambda t} dt + E\tilde{u}_\lambda(X_{\tau^\delta_x}) e^{-\lambda \tau^\delta_x}, \qquad \forall x \in \Omega_\delta. \tag{89}$$

In particular, we may deduce from this quantity

$$\left(\inf_{\partial \Omega_\delta} \tilde{u}_\lambda\right) E[e^{-\lambda \tau^\delta_x}] \leqq C, \qquad \forall x \in \Omega_\delta$$

for some C independent of δ.

Now, since $\tilde{u}_\lambda \to +\infty$ as $d \to 0_+$, $\left(\inf_{\partial \Omega_\delta} \tilde{u}_\lambda\right) \to +\infty$ as $\delta \to 0_+$, therefore $E[e^{-\lambda \tau^\delta_x}] \to 0$ as $\delta \to 0_+$. Hence $E[e^{-\lambda \tau_x}] = 0$ for all $x \in \Omega$ and this precisely means that $a_0 \in \mathscr{A}$.

In addition, we may also deduce from (89) that for δ small enough and for $x \in \Omega_\delta$

$$\tilde{u}_\lambda(x) \geqq E \int_0^{\tau^\delta_x} \{f(X_t) + c|a_0(X_t)|^q\} e^{-\lambda t} dt$$

since $\tilde{u}_\lambda \geqq 0$ for $x \in \Omega - \Omega_\delta$. Now, if we let δ go to and if we use the fact that $a_0 \in \mathscr{A}$ and thus $\tau_x^\delta \to +\infty$ a.s. as $\delta \to 0_+$ for all $x \in \Omega$, this yields

$$\tilde{u}_\lambda(x) \geqq J(x, a_0), \quad \forall x \in \Omega. \tag{90}$$

We next show that $\tilde{u}_\lambda \equiv u_\lambda$. If this is the case, (90) then implies that a_0 is optimal. To show that $\tilde{u}_\lambda \equiv u_\lambda$, we first recall from section that there exist w_n subsolutions of (1) in $W^{2,r}(\Omega)$ ($\forall r < \infty$) such that $w_n \underset{n}{\to} \tilde{u}_\lambda$ uniformly on compact subsets of Ω. Therefore, if $a \in \mathscr{A}$, we find using again Itô's rule for all $x \in \Omega_\delta$

$$w_n(x) \leqq E \int_0^{\tau_x^\delta} \{f(X_t) + c|a(X_t)|^q\} e^{-\lambda t} dt + E[w_n(X_{\tau_x^\delta}) e^{-\lambda \tau_x^\delta}].$$

Now, if $J(x, a) = +\infty$, we obviously have $w_n(x) \leqq J(x, a)$, while if $J(x, a) < \infty$, we deduce from the above inequality letting δ go to 0_+

$$w_n \leqq E \int_0^\infty \{f(X_t) + c|a(X_t)|^q\} e^{-\lambda t} dt = J(x, a)$$

since $\tau_x^\delta \to +\infty$ a.s. as $\delta \to 0_+$ and thus

$$|E[w_n(X_{\tau_x^\delta}) e^{-\lambda \tau_x^\delta}]| \leqq \sup_{\bar{\Omega}} |w_n| E[e^{-\lambda \tau_x^\delta}] \to 0 \quad \text{as} \quad \delta \to 0_+.$$

Therefore, letting n go to $+\infty$, we finally deduce for all $x \in \Omega$, $a \in \mathscr{A}$

$$\tilde{u}_\lambda(x) \leqq J(x, a)$$

and our claim is proved. □

The uniqueness of the optimal control is a bit technical but very simple to understand so we just sketch the argument: assume that a is optimal then applying Itô's rule we find for all $\delta > 0$, $x \in \Omega_\delta$

$$\begin{aligned} u_\lambda(x) &= E \int_0^{\tau_x^\delta} \{f(X_t) - a(X_t) \cdot \nabla u_\lambda(X_t) - |\nabla u_\lambda(X_t)|^p\} e^{-\lambda t} dt + E u_\lambda(X_{\tau_x^\delta}) e^{-\lambda \tau_x^\delta} \\ &\leqq E \int_0^{\tau_x^\delta} \{f(X_t) + c|a(X_t)|^q\} e^{-\lambda t} + E u_\lambda(X_{\tau_x^\delta}) e^{-\lambda \tau_x^\delta}. \end{aligned}$$

But recalling that $u_\lambda(x) = J(x, a)$ for all $x \in \Omega$ and using the Markov property of X_t, we deduce that the above right-hand side is also equal to $u_\lambda(x)$. Therefore, the equality yields that for all $x \in \Omega_\delta$

$$a(X_t) = a_0(X_t) \quad \text{for all} \quad t \in (0, \tau_x^\delta) \quad \text{a.s.},$$

(where X_t is the solution corresponding to a) and letting δ go to 0_+ we finally find that for all $x \in \Omega$, $a(x) = a_0(x)$ (recall that a, a_0 are continuous on Ω). □

VII.2. Superquadratic Hamiltonians

Theorem VII.2. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below, satisfy* (41) *for some $\beta < p$ and let $p > 2$. Then, the value function u_λ given by* (88) *is the unique solution of* (1) *in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$) satisfying* (62).

Proof. We will approximate the Hamiltonian $|\xi|^p$ as follows: let $R \geqq 1$, consider some Hamiltonian H_R such that H_R is convex on $\mathbb{R}^N$, $H_R(\xi)=|\xi|^p$ if $|\xi| \leqq R$, $H_R(\xi)|\xi|^{-\alpha}$ is constant for $|\xi|$ large where $\frac{\alpha}{\alpha-1} > \beta$ and $1<\alpha<2$, H_R increases uniformly on compact subsets to $|\xi|^p$ as R goes to $+\infty$. And we denote by $L_R(\eta)$ the following convex function

$$-L_R(\eta)=\inf_{\xi \in \mathbb{R}^N} (\eta \cdot \xi + H_R(\xi))$$

so that $L_R(\eta) \geqq c|\eta|^q$ and L_R decreases uniformly on compact subsets to $c|\eta|^q$.

Then, because $H_R = C_R|\xi|^\alpha$ for $|\xi|$ large, it is not difficult to adapt the results and the proofs of Sect. II to show that there exists a unique solution u_R of

$$-\Delta u_R + H_R(\nabla u_R) + \lambda u_R = f(x) \quad \text{in} \quad \Omega, \qquad u_R \in W^{2,r}_{\mathrm{loc}}(\Omega) \quad (\forall r < \infty)$$

such that $u_R \to +\infty$ as $d \to 0_+$.

And exactly as in the preceding section, we can check that

$$u_R(x) = \inf_{a \in \mathscr{A}} E \int_0^\infty \{f(X_t) + L_R(a(X_t))\} e^{-\lambda t} dt, \qquad \forall x \in \Omega.$$

Of course, u_R decreases to the value function u_λ given by (88). On the other hand, we remark that we may choose $H_R \in C^2(\mathbb{R}^N)$ such that

$$|D^2 H_R(\xi)|\,|\xi|^2, \qquad |DH_R(\xi)|\,|\xi| \leqq C_0(H_R(\xi)+1), \qquad \forall \xi \in \mathbb{R}^N,$$

and

$$H_R(\xi) \geqq |\xi|^\alpha, \qquad \forall \xi \in \mathbb{R}^N$$

for some C_0 independent of R. And we may adapt the a priori estimates in the appendix (see also part 2) to deduce that u_R is bounded in $W^{1,\infty}_{\mathrm{loc}}$ and thus in $W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r < \infty)$. Hence, $u_\lambda \in W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r < \infty)$ and solves (1). But then by Corollary IV.1, u_λ extends continuously to $\bar{\Omega}$ and we may apply Theorem V.2 to deduce that u_λ is the unique solution of (1) satisfying (62). □

The question of the optimality of the control $a_0 = -p|\nabla u_\lambda|^{p-2}\nabla u_\lambda$ is much more delicate: in fact, if an optimal control exists, by a similar proof to the one made in the preceding section, it has to be a_0 and if we know that $a_0 \in \mathscr{A}$ then a_0 is the optimal Markovian control. Hence, the main problem is whether $a_0 \in \mathscr{A}$. We know how to prove that $a_0 \in \mathscr{A}$ only when (61) (or some easy variants) holds and we refer the reader to Sect. IV.3 where a few cases when (61) holds are given. Indeed, if (61) holds then we deal with a diffusion X_t satisfying

$$dX_t = \sqrt{2}\, dB_t + a_0(X_t)\, dt,$$

where a_0 satisfies

$$a_0(x) d(x) \to -\mu n \quad \text{as} \quad d(x) \to 0_+, \tag{91}$$

with $\mu = q > 1$. Then, we claim that for any diffusion process of the above form, if (91) holds and $\mu > 1$, then X_t never leaves Ω with probability 1, while if (91) holds and $\mu<1$, X_t hots $\partial\Omega$ in finite time with propability 1. Indeed, if $\mu>1$, we apply Itô's rule with $-\log d(x)$ and we find for all $T<\infty$

$$-\log d(x)=E\left[-\log d(X_{\tau\wedge T})+\int_0^{\tau\wedge T}\left\{\frac{\Delta d}{d}-\frac{|\nabla d|^2}{d^2}-\frac{a\cdot\Delta d}{d}\right\}(X_s)ds\right]$$

(in fact we should replace τ by τ_δ for $\delta>0\ldots$). And we observe that

$$\frac{\Delta d}{d}-\frac{|\nabla d|^2}{d^2}-\frac{a\cdot\nabla d}{d}$$

behaves like $(\mu-1)\frac{1}{d^2}$ near $\partial\Omega$ and so this quantity is bounded from below on Ω. Hence, we obtain

$$E[-\log d(X_{\tau\wedge T})]\leqq CT-\log d(x)$$

therefore for all $x\in\Omega$, $P(\tau<T)=0$ and we conclude since this holds for all $T<\infty$.

On the other hand if $\mu<1$ by a simple argument, showing that $E[\tau_x]\leqq C$ for all $x\in\Omega$ is easily done if we prove the existence of a supersolution of

$$-\Delta z-a\cdot\nabla z\geqq\varepsilon \quad \text{in} \quad \Omega, \qquad \text{for some} \quad \varepsilon>0, \quad z\in C(\bar\Omega), \quad z=0 \quad \text{on} \quad \partial\Omega.$$

But this is achieved by considering for $\mu_0\in(\mu,1)$ the function

$$z_1=d^{1-\mu_0}(1-\mu_0)^{-1}-d^2\{2(\mu_0+1)\}^{-1}$$

which satisfies in $\Omega-\Omega_\delta$ for some δ small enough

$$\begin{aligned}-\Delta z_1-a\cdot\nabla z_1&=\mu_0 d^{-\mu_0-1}-a\cdot\nabla d d^{-\mu_0}+\frac{1}{\mu_0+1}\{1+a\cdot\nabla d d\}\\&\quad+\left\{\frac{1}{\mu_0+1}-d^{-\mu_0}\right\}\Delta d\geqq K>0\end{aligned}$$

for some $K>0$. Then, we consider the solution z_2 of

$$-\Delta z_2-a\cdot\nabla z_2=1 \quad \text{in} \quad \Omega_\delta, \qquad z_2=0 \quad \text{on} \quad \partial\Omega_\delta.$$

Finally, we set $z=z_1$ in $\Omega-\Omega_\delta$, $=z_1|_{\partial\Omega_\delta}+\gamma z_2$ in $\bar\Omega_\delta$ where γ is small enough so that $\gamma\frac{\partial z_2}{\partial n}\geqq\frac{\partial z_1}{\partial n}$ on $\partial\Omega_\delta$. It is then easy to check that z satisfies the desired inequality with $\varepsilon=\min(K,\gamma)$.

VII.3. Forced Constraints

We first observe that by the results and methods of the preceding sections (and the interior estimates given in the Appendix), for any $p>1$ and for any $f\in L^\infty_{\mathrm{loc}}(\Omega)$ bounded from below there exists a solution $\underline{u}\in W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r<\infty$) of (1) such that for all $v\in W^{2,r}(\Omega)$ ($\forall r<\infty$) satisfying

$$-\Delta v+|\nabla v|^p+\lambda v\leqq f \quad \text{in} \quad \Omega \tag{92}$$

then $v \leqq \underline{u}$ in Ω. Of course, if $1<p\leqq 2$ and $f(x)\leqq Cd(x)^{-1}$ then (see Sect. II) $\underline{u}\to +\infty$ as $d\to 0_+$ and $\underline{u}$ is the minimum such solution, if $p>2$ and $f(x)\leqq Cd(x)^{-\beta}$ for some $\beta<p$ then $\underline{u}$ is the unique solution of (1) satisfying (62) (see Sects. IV and V), while if $f(x)\geqq cd(x)^{-\beta}-C$ for some $c>0$, $\beta\geqq \max(p,q)$ then $\underline{u}\to +\infty$ as $d\to 0_+$ and $\underline{u}$ is the minimum solution of (1) bounded from below [and we have uniqueness if f behaves like $C_1 d(x)^{-\beta}$]. In fact, if $1<p\leqq 2$, then $\underline{u}\to +\infty$ as $d\to 0_+$ and $\underline{u}$ is the minimum such solution.

We then have the following

Proposition VII.1. *Let $1<p\leqq 2$, or let $p>2$ and $f\geqq cd^{-\beta}-C$ for some $c>0$, $C\geqq 0$, $\beta\geqq p$. Then, the value function u_λ given by* (88) *is the above ("minimum explosive") solution $\underline{u}$. In addition, $a_0(x)=-p|\nabla u|^{p-2}\nabla u$ is the unique optimal Markovian control.*

In fact, since $\underline{u}\to\infty$ as $d\to 0_+$, the proof is exactly the same as the proof of Theorem VI.1: one shows that $\underline{u}\geqq u_\lambda$ and $a_0\in\mathscr{A}$, then $\underline{u}\leqq u_\lambda$ and a_0 is the unique optimal Markovian control.

Remark VII.1. These results show that for any f bounded from below the formula (88) yields a finite function (locally bounded) on Ω. This may be proved directly by a tedious probabilistic construction of a control $\bar{a}\in\mathscr{A}$ such that $J(x,\bar{a})<\infty$ for all $x\in\Omega$.

VII.4. Ergodic Control

We now want to explain in this section the control problems associated with the asymptotic problems solved in Sect. VI. We begin with the cases when solutions go to $+\infty$ as $d(x)$ goes to 0_+.

Theorem VII.3. *Let $f\in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below and satisfy* (78), *let $1<p\leqq 2$. We denote by (v,u_0) the solutions given by Theorem VI.1. Then, we have the following equalities: for any $a\in\mathscr{A}$, let θ_a be a stopping time bounded by some arbitrary $T\geqq 0$ (independent of a), then*

$$v(x)=\inf_{a\in\mathscr{A}} E\int_0^{\theta_a}\{f(X_t)+c|a(X_t)|^q\}dt+v(X_{\theta_a})-\theta_a u_0\,,\qquad \forall x\in\Omega\,, \tag{93}$$

$$u_0=\lim_{T\to\infty}\inf_{a\in\mathscr{A}} E\frac{1}{T}\int_0^T\{f(X_t)+c|a(X_t)|^q\}dt\,,\qquad \forall x\in\Omega \tag{94}$$

and the control $a_0=-p|\nabla v|^{p-2}\nabla v$ belongs to $\mathscr{A}$ and is the unique optimal Markovian control where optimal means that (93)–(94) *are equalities when we choose $a=a_0$.*

Theorem VII.4. *Let $f\in L^\infty_{\mathrm{loc}}(\Omega)$ satisfy* (40) *and let $p>1$. Denoting by (v,u_0) the solutions given by Theorem VI.2, Theorem VII.3 still holds.*

Since the proof of Theorem VII.4 is very similar to the one of Theorem VII.3 we will only prove the latter.

We first deduce from Itô's formula that if X_t^0 denotes the process corresponding to the choice a_0 then for all $\delta>0$, $x\in\Omega$

$$v(x)=E\int_0^{\theta_0\wedge\tau_x^\delta}\{f(X_t^0)+c|a_0(X_t^0)|^q\}dt+v(X^0_{\theta_0\wedge\tau_x^\delta})-\theta_0\wedge\tau_x^\delta u_0\,, \tag{95}$$

where θ_0 stands for θ_{a_0} and τ_x^δ is the first exit time from Ω_δ. In particular for $\theta_0 = T$, we deduce

$$E(v(X^0_{T\wedge\tau_x^\delta})) \leqq v(x) + CT, \quad \text{for some} \quad C \geqq 0.$$

Therefore, recalling that v is bounded from below, we obtain

$$\left(\inf_{\partial\Omega_\delta} v\right) P[\tau_x^\delta \leqq T] \leqq v(x) + C(1+T)$$

and since $v \to +\infty$ as $d \to 0_+$, we deduce that $a_0 \in \mathscr{A}$.

In addition, if we pass to the limit in (95) as δ goes to 0_+, we find for all $x \in \Omega$

$$v(x) = E\int_0^{\theta_0} \{f(X_t^0) + c|a_0(X_t^0)|^q\}dt - \theta_0 u_0 + \lim_{\delta\to 0_+} E[v(X^0_{\theta_0\wedge\tau_x^\delta})]$$

and

$$\lim_{\delta\to 0_+} E[v(X^0_{\theta_0\wedge\tau_x^\delta})] \geqq \lim_{\delta\to 0_+} \{E[(v+C)(X^0_{\theta_0})\mathbb{1}_{\theta_0 \leqq \tau_x^\delta}] - C\},$$

where $C \leqq \inf_\Omega v$, and this last expectation increases to $E[v(X^0_{\theta_0})]$. Hence, we finally obtain for all $x \in \Omega$

$$v(x) \geqq E\int_0^{\theta_0} \{f(X_t^0) + c|a_0(X_t^0)|^q\}dt + v(X^0_{\theta_0}) - \theta_0 u_0. \tag{96}$$

And taking $\theta_0 = T$, we also deduce for all $x \in \Omega$

$$u_0 \geqq \varliminf_{T\to\infty} \frac{1}{T} E\int_0^T \{f(X_T^0) + c|a_0(X_t^0)|^q\}dt \tag{97}$$

since $E\frac{1}{T}v(X_T^0) \geqq -\frac{C}{T} \to 0$ as $T \to \infty$.

To complete the proof of Theorem VII.3, we basically need to prove the complementary inequalities in (93)–(94). This will be achieved by first introducing some approximated problem: let (v^δ, u_0^δ) be the solution in $W^{2,r}_{\text{loc}}(\Omega^\delta) \times \mathbb{R}$ $(\forall r < \infty)$ of (79) with Ω replaced by Ω^δ such that $v^\delta(x_0) = 0$, $v^\delta \to +\infty$ as $d(x) \to 0_+$. With the techniques of Sect. VI one can show that $v^\delta \uparrow v$ as δ goes to 0_+ and converges uniformly on compact subsets of Ω, while $u_0^\delta \downarrow u_0$ as $\delta \downarrow 0_+$.

Using Itô's formula, we immediately obtain for all $x \in \Omega$

$$v^\delta(x) \leqq \inf_{a\in\mathscr{A}} E\int_0^{\theta_a} \{f(X_t) + c|a(X_t)|^q\}dt + v^\delta(X_{\theta_a}) - \theta_a u_0^\delta$$

and letting δ go to 0_+ we deduce since $v^\delta \uparrow v$ as $\delta \downarrow 0_+$

$$v(x) \leqq \inf_{a\in\mathscr{A}} E\int_0^{\theta_a} \{f(X_t) + c|a(X_t)|^q\}dt + v(X_{\theta_a}) - \theta_a u_0$$

and this combined with (96) yields (93). In addition taking $\theta_a = T$, we also deduce

$$u_0^\delta \leqq \inf_{a\in\mathscr{A}} E\frac{1}{T}\int_0^T \{f(X_t) + c|a(X_t)|^q\}dt + \frac{2}{T}\sup_{\bar\Omega} |v^\delta|$$

hence

$$u_0^\delta \leq \varliminf_{T\to\infty} \inf_{a\in\mathscr{A}} E\frac{1}{T}\int_0^T \{f(X_t)+c|a(X_t)|^q\}dt$$

and letting δ go to 0, the resulting inequality combined with (97) yields (94). This also shows the optimality of a_0 and the uniqueness is easy to prove as in the preceding sections.

By the same truncation argument as in the proof of Theorem VII.2, one deduces the

Theorem VII.5. *Let $p>2$, let $f\in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below and satisfy* (40). *We denote by (v, u_0) the solutions given by Theorem VI.2. Then, the identities* (93)–(94) *still hold.*

Appendix: On Some Local Gradient Bounds

We want to show here some local gradient bounds for solutions of

$$-\varepsilon\Delta u+|\nabla u|^p+\lambda u=f \quad \text{in} \quad \Omega, \qquad u\in W^{2,r}_{\mathrm{loc}}(\Omega) \quad (\forall r<\infty), \tag{A.1}$$

where $0<\varepsilon\leq 1$, $0\leq\lambda\leq 1$, $1<p<\infty$, and $f\in L^\infty(\Omega)$ or even $f\in W^{1,\infty}(\Omega)$ and Ω is a bounded open set in $\mathbb{R}^n$. These bounds are obtained by the method introduced in [16, 19]. For related local bounds concerning different equations, we refer to Bombieri et al. [4], Ladyzhenskaya and Ural'tseva [13], Simon [26–28]. Our main result is the

Theorem A.1. *For any $\delta>0$, we set $\Omega_\delta=\{x\in\Omega/\mathrm{dist}(x,\partial\Omega)>\delta\}$.*

1) *Let $f\in W^{1,\infty}(\Omega)$, then we have for all $\delta>0$*

$$|\nabla u(x)|\leq C_\delta \quad \text{if} \quad x\in\Omega_\delta, \tag{A.2}$$

where C_δ depends only on bounds on $|\nabla f|$, lower bounds on $\lambda u-f$, δ, and p.

2) *Let $f\in L^\infty(\Omega)$, then we have for all $r<\infty$, $\delta>0$*

$$\|\nabla u\|_{L^r(\Omega_\delta)}\leq C_\delta, \tag{A.3}$$

where C_δ depends only on bounds on f, lower bounds on $f-\lambda u$, δ, p, and r.

Proof. We begin with case 1) i.e. when $f\in W^{1,\infty}(\Omega)$. In both cases, we will ignore the fact that u is not assumed to be smooth and we will thus skip the tedious approximation argument required to make the proof below complete. Then, let $\theta\in(0,1)$ to be determined later on and let $\varphi\in\mathscr{D}(\Omega)$, $0\leq\varphi\leq 1$ in Ω, $\varphi\equiv 1$ on Ω_δ, be such that

$$|\Delta\varphi|\leq C\varphi^\theta, \qquad |\nabla\varphi|^2\leq C\varphi^{1+\theta} \quad \text{in} \quad \Omega$$

for some C (depending only on δ, θ).

We next consider $w=|\nabla u|^2$ and we compute easily on $\mathrm{Supp}\,\varphi$

$$\left.\begin{aligned} &-\varepsilon\Delta(\varphi w)+p|\nabla u|^{p-2}\nabla u\cdot\nabla(\varphi w)+2\lambda\varphi w+2\varepsilon\varphi|D^2u|^2+2\varepsilon\frac{\nabla\varphi}{\varphi}\cdot\nabla(\varphi w)\\ &=2\varphi\nabla f\cdot\nabla u+p|\nabla u|^{p-2}(\nabla u\cdot\nabla\varphi)w-\varepsilon(\nabla\varphi)w+2\varepsilon\frac{|\nabla\varphi|^2}{\varphi}w.\end{aligned}\right\} \tag{A.4}$$

Then, let $x_0 \in \Omega$ be a maximum point of φw: we may assume that $x_0 \in \operatorname{Supp} \varphi$ and by the classical maximum principle we deduce from (A.4) the following inequality where all functions are taken at x_0

$$2\varepsilon\varphi |D^2 u|^2 \leqq C\varphi w^{1/2} + C\varphi^\theta w^{\frac{p+1}{2}} + C\varepsilon\varphi^\theta w. \tag{A.5}$$

Now, from Cauchy-Schwarz inequality and (A.1)

$$|D^2 u|^2 \geqq \frac{1}{N}(\Delta u)^2 \geqq \frac{1}{N\varepsilon^2}(|\nabla u|^p + \lambda u - f)^2 \geqq \frac{1}{N\varepsilon^2}(|\nabla u|^p - C)^{+2}$$

and this combined with (A.5) yields

$$\varphi w^p \leqq C + C\varepsilon\varphi w^{1/2} + C\varepsilon\varphi^\theta w^{\frac{p+1}{2}} + C\varepsilon^2\varphi^\theta w. \tag{A.6}$$

Now, choosing $\theta \geqq \frac{3-p}{2}$, we deduce easily

$$\max_{\Omega} \varphi w = \varphi w(x_0) \leqq C.$$

In case 2), i.e. when $f \in L^\infty(\Omega)$ we use integral estimates as follows: let $m \geqq 1$, we multiply (A.4) by $(\varphi w)^m$ and we find

$$\begin{aligned}
&\varepsilon m \int |\nabla(\varphi w)|^2 (\varphi w)^{m-1} dx + p \int |\nabla u|^{p-2} \nabla u \cdot \nabla(\varphi w)(\varphi w)^m dx \\
&\quad + \varepsilon \int \varphi |D^2 u|^2 (\varphi w)^m dx + \frac{1}{N\varepsilon} \int \varphi(|\nabla u|^p - C)^{+2} \varphi^m w^m dx \\
&\quad + 2\varepsilon \int \varphi^{m-1} \nabla\varphi \cdot \nabla(\varphi^{m+1} w^{m+1})(m+1)^{-1} dx \\
&\leqq 2 \int \varphi^{m+1} w^m \nabla f \cdot \nabla u dx + C_p \int w^{\frac{p+1}{2}} \varphi^{\theta m} w^m dx + C\varepsilon \int \varphi^{m+\theta} w^m dx.
\end{aligned}$$

We now want to bound the following terms

$$\begin{aligned}
&2 \int \varphi^{m+1} w^m \nabla f \cdot \nabla u dx \leqq 2C \int \varphi^{m+1} w^m |D^2 u| dx + m \int \varphi |\nabla(\varphi w)| (\varphi w)^{m+1} \\
&\quad \times |\nabla u| dx + C \int \varphi^{m+\theta} w^{m+1/2} dx \\
&\leqq \varepsilon \int \varphi^{m+1} w^m |D^2 u|^2 dx + \frac{C}{\varepsilon} \int \varphi^{m+1} w^m dx + \varepsilon \frac{m}{2} \int |\nabla(\varphi w)|^2 (\varphi w)^{m-1} dx \\
&\quad + C\frac{m}{\varepsilon} \int \varphi^{m+1} w^m dx + C \int \varphi^{m+\theta} w^{m+1/2} dx;
\end{aligned}$$

$$\begin{aligned}
&p \int |\nabla u|^{p-2} \nabla u \cdot \nabla(\varphi w)(\varphi w)^m dx \leqq \frac{\varepsilon m}{2} \int |\nabla(\varphi w)|^2 (\varphi w)^{m-1} dx \\
&\quad + \frac{p^2}{\varepsilon m} \int \varphi^{m+1} w^{m+p} dx;
\end{aligned}$$

$$2\varepsilon(m+1)^{-1} \int \varphi^{-1} \nabla\varphi \cdot \nabla(\varphi^{m+1} w^{m+1}) dx \leqq \frac{C\varepsilon}{m+1} \int \varphi^{m+\theta} w^{m+1} dx.$$

And collecting all these bounds, we finally deduce

$$\frac{1}{N\varepsilon}\int \varphi(|\nabla u|^p - C)^{+2}\varphi^m w^m dx \leqq C_p \int w^{m+\frac{p+1}{2}}\varphi^{m+\theta}dx + C\varepsilon\int \varphi^{m+\theta}w^m dx$$

$$+C\int \varphi^{m+\theta}w^{m+1/2}dx + \frac{C}{\varepsilon}\int \varphi^{m+1}w^m dx + C\frac{m}{\varepsilon}\int \varphi^{m+1}w^m dx$$

$$+\frac{C\varepsilon}{m+1}\int \varphi^{m+\theta}w^{m+1}dx + \frac{p^2}{\varepsilon m}\int \varphi^{m+1}w^{m+p}dx.$$

To get rid of the last term, we choose m in $\left]\frac{p^2}{N}, \infty\right[$ and we find

$$\int \varphi^{m+1}w^{m+p}dx \leqq C + C\int w^{m+\frac{p+1}{2}}\varphi^{m+\theta}dx.$$

And we conclude choosing $\theta \geqq (m+p)^{-1}\{(p+1)/2 + m(3-p)/2\}$.

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