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Nonlinear Elliptic Equations with Singular Boundary Conditions and Stochastic Control with State Constraints

1. The Model Problem

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I. Introduction

I.1. General Introduction

One of the primary goals of this paper is to study various models of stochastic control problems involving constraints on the state of the system (state constraints). And by following the dynamic programming approach this is equivalent to study some nonlinear second-order elliptic equations. Then, the state constraints lead to highly singular boundary conditions. A typical example would be: let Ω be a bounded, smooth domain in $\mathbb{R}^N$, we look for a solution $u\in C^2(\Omega)$ of

$$-\Delta u+|\nabla u|^p+\lambda u=f \quad \text{in} \quad \Omega \tag{1}$$

where $p>1$, $\lambda>0$, f is a given smooth function in Ω, and the boundary condition is given by

$$u(x)\to+\infty \quad \text{as} \quad \operatorname{dist}(x,\partial\Omega)\to 0\,, \tag{2}$$

(in fact, this boundary condition will correspond to the case $1<p\leqq 2$).

We will show in this paper how such nonlinear, boundary value problems (1)–(2) can be solved and we will thus obtain existence, uniqueness and comparison results leading also to a complete solution of the stochastic control problem we are considering. It turns out that many cases have to be investigated and the results differ somewhat from one case to the other: typical behaviours involve the cases $1<p\leqq 2$, $p>2$, f blowing up near $\partial\Omega$, λ going to 0. Let us also mention that the methods introduced below allow us to treat more general nonlinear second-order elliptic equations, like more general quasilinear elliptic equations, Hamilton-Jacobi-Bellman equations, semilinear elliptic equations, first-order Hamilton-Jacobi equations, Monge-Ampère equations: in all those examples, singular boundary conditions may be encountered (and are even of a fundamental use) and we refer to Brézis [6], Crandall and Lions [7], Urbas [30], Simon [26–28], where such boundary conditions appear. And, our methods enable us to treat these equations with boundary conditions like (2).

I.2. Description of the Stochastic Control Problems

The basic model problem we are considering is a stochastic control problem where the *state of the controlled system is a diffusion process* and a typical example is the solution of the following *stochastic differential equation*

$$dX_t = a_t dt + dB_t\,, \qquad X_0 = x \in \mathbb{R}^N\,, \tag{3}$$

where B_t is a standard Brownian motion [in some probability space (Ω, F, F_t, P)...] and where a_t is the control process i.e. a progressively measurable stochastic process that we may choose as we wish (taking possibly into account restrictions on the controls such that, for instance, a_t takes its values into a given set A...). A very important, particular class of controls is given by the so-called feedback controls i.e., given a function $a(\cdot)$, one looks for a solution of

$$dX_t = a(X_t)dt + dB_t\,, \qquad X_0 = x\,. \tag{4}$$

This general class of problems occur in many contexts: however, depending on the particular examples of applications, it is possible to classify those problems in the following categories. For some problems, the state process X_t may take its value in $\mathbb{R}^N$ without any restriction while in other problems the state X_t should remain in some given region $\bar{\Omega}$. In the latter case, the model is to be complemented with boundary prescriptions in case the process reaches or crosses the boundary $\partial\Omega$. Let us immediately mention that if Ω is bounded, and a_t or $a(\cdot)$ are bounded then *for all* $t>0$ $P(X_t \in \partial\Omega)>0$. The most usual models in stochastic control theory yield the following boundary prescriptions: in the case of the so-called *exit problems*, one considers the first exit time of X_t from $\bar{\Omega}$ (or the first hitting time of X_t on $\partial\Omega$) and the process is stopped at this time. The other standard model consists in a boundary mechanism which prevents the state process from escaping from $\bar{\Omega}$; the simplest of which is the *reflecting boundary condition.*

Now, at least for deterministic problems, it is well-known that another way to enforce state constraints (i.e. $X_t \in \bar{\Omega}$) is simply to restrict our attention to controls such that X_t remains in $\bar{\Omega}$ or in Ω. In the case of (nondegenerate) stochastic models like (3) or (4), this possibility does not seem to have been explored. And this is precisely the type of problems we have in mind. In view of a remark made above, it is clear enough that in order to constrain a Brownian motion in a bounded domain Ω we need to use unbounded drifts a_t or $a(\cdot)$: in other words, we will have to choose feedbacks controls which, roughly speaking, push back the state process inside Ω when it gets near $\partial\Omega$ and with an intensity which blows up at the boundary. To be more specific, we will consider the class $\mathscr{A}$ of feedback controls i.e. of, say, continuous functions on Ω, $a(\cdot)$ such that the solution X_t of (4) stays in Ω with probability 1 for all $t \geqq 0$, (and for all initial points $x \in \Omega$).

Once, admissible control processes and thus state processes have been specified we may now describe a typical example of the optimal control problems we want to study. For each $a \in \mathscr{A}$, we will consider for example the *following cost function*

$$J(x,a) = E \int_0^\infty \left\{ f(X_t) + \frac{1}{q} |a_t|^q \right\} e^{-\lambda t} dt, \qquad \forall x \in \Omega, \tag{5}$$

where $q > 1$, f is a given function on Ω say bounded from below and $\lambda > 0$ is a given parameter – the so-called *discount factor*, and where we denote by $a_t = a(X_t)$. Let us emphasize that this particular form of the *running cost* $g(x,a) = f(x) + \frac{1}{q} |a|^q$ is by no means essential for the analysis which follows: it just provides a simple but general enough model problem. Let us also mention that this choice of cost functions corresponds to the so-called *infinite horizon* problems and that other cases are considered in this paper.

Finally, we wish to minimize J i.e. we want to determine the *value function* (or Bellman function):

$$u(x) = \inf_{a \in \mathscr{A}} J(x,a), \qquad \forall x \in \Omega \tag{6}$$

and optimal (feedback) controls $\underset{\sim}{a}$ such that $u(x) = J(x, \underset{\sim}{a})$.

I.3. Description of the Associated Boundary Value Problem

We want now, in this section, to follow the heuristic dynamic programming approach to such optimal stochastic control problems: the dynamic programming argument (which can be viewed as a modern, extended version of Hamilton-Jacobi-Carathéodory theories for problems in the calculus of variations), leads to a nonlinear partial differential equation. More precisely, the dynamic programming principle, due to R. Bellman, indicates that the value function u given by (6) should satisfy the following second-order, quasilinear, elliptic equation

$$-\frac{1}{2} \Delta u + \frac{1}{p} |\nabla u|^p + \lambda u = f \quad \text{in} \quad \Omega, \tag{7}$$

where p is the conjugate exponent of q i.e. $p = \frac{q}{q-1}$. In fact, such a claim, even if we forget the heuristic aspect of Bellman's derivation of (7) is by no means obvious here, in view of the restriction to feedback controls and of the state constraints. But nevertheless (7) is to be expected for the value function u. This equation is a very particular case of the so-called Hamilton-Jacobi-Bellman equations. And at least for problems like exit problems or the ones corresponding to reflecting boundary conditions (as described in the preceding section), a rigorous derivation of the

Hamilton-Jacobi-Bellman equation and the analysis of such nonlinear p.d.e. are now available: see Fleming and Rishel [8]; Bensoussan and Lions [2, 3]; Krylov [11, 12]; Lions [16–18]; Lions and Trudinger [24, 25] and the bibliography therein.

Let us add to this general description that the exit problems lead to Dirichlet type boundary conditions like

$$u=\varphi \quad \text{on} \quad \Omega, \tag{8}$$

where φ is the exit cost i.e. the price to be paid for hitting the boundary at a point x of $\partial\Omega$. On the other hand, reflecting type boundary conditions lead to Neumann (or oblique derivative) type boundary conditions like for instance

$$\frac{\partial u}{\partial n}=\psi \quad \text{on} \quad \partial\Omega, \tag{9}$$

where n is the unit outward normal to $\partial\Omega$ and ψ is the reflection cost i.e. the price to be paid for reflecting on the boundary $\partial\Omega$ at the point x.

Finally, let us mention that another aspect of Bellman's dynamic programming argument is a rule for finding an optimal feedback control which in the case of (7) reduces to the choice

$$a(x)=-|\nabla u|^{p-2}\nabla u(x) \quad \text{for} \quad x\in\Omega. \tag{10}$$

Now, we go back to the state-constraints problem described in the preceding section and we ask ourselves the following question: what is the boundary condition (or any other characterization at $\partial\Omega$) we may expect for the value function u given by (6)? From the above considerations it is tempting to say that to discourage hitting the boundary we should impose an infinite exit cost or reflection cost i.e.

$$u(x)\to+\infty \quad \text{as} \quad \operatorname{dist}(x,\partial\Omega\to 0 \tag{11}$$

or

$$\frac{\partial u}{\partial n}(x)\to+\infty \quad \text{as} \quad \operatorname{dist}(x,\partial\Omega)\to 0 \tag{12}$$

[where $n(x)$ is defined near $\partial\Omega$ by $-\nabla(\operatorname{dist}(x,\partial\Omega))$]. More sophisticated formulations, which are also very natural from the control viewpoint, are: u is the maximum solution (or even subsolution) of (7); or: u is the upperenvelope of bounded solutions of (7).... Finally, for readers experienced with viscosity solutions, a possible form of the boundary condition could be

$$u-\varphi \quad \text{achieves its minimum over} \quad \Omega \tag{13}$$

for all $\varphi\in C^2(\bar\Omega)$ [or $C^{1,1}(\bar\Omega)$, or $C^1(\bar\Omega)$, or even $C^{0,1}(\bar\Omega)$]: this "viscosity formulation" will be explained below in Sect. IVV, see also Lions [17, 29] for the deterministic case.

It turns out, and the precise results are given in the next section, that if the latter formulations are always true, the choice between the boundary conditions (11) or (12) requires some careful analysis and will in fact depend on the behaviour of f

near $\partial\Omega$ and on q. This can easily be "justified" by a vague economical argument: if f blows up fast enough near $\partial\Omega$ of if q is large (remember that a has to blow up near $\partial\Omega$) then the cost functions will blow up at $\partial\Omega$ and so will u. Then, we should expect (11). On the other hand, if f, say, is bounded and if q is near 1 then it does not cost much to drive the state off $\partial\Omega$ and we may expect now u to be bounded on Ω. On the other hand, recalling Bellman's rule (10) for the optimal control and the fact that a cannot remain bounded if we want X_t to stay in Ω, we should expect that some condition like (12) holds. Of course, the reason for which we insist on conditions like (11) or (12) compared to a "maximum solution" characterization is because of the specific information contained in those formulations (which could turn out to be crucial for numerical purposes). Finally, note that $p=1$ is excluded in the p.d.e. results (see next Sect. I.4). This corresponds to the fact that it is impossible to force state-constraints with bounded controls. All these heuristic considerations will find their mathematical counter parts in the results presented in the next section.

I.4. Short Review of the Results

In this section, we present some of the results obtained in this paper on the simple example of the model equation (1) [equivalent to (7) after an obvious scaling]. In doing so, we follow the order of the sections below. To simplify the presentation we will always assume at least that $f\in C^1(\Omega)$, is bounded *from below*. We will denote by $d(x)=\operatorname{dist}(x,\partial\Omega)$ for all $x\in\bar{\Omega}$.

We begin with the case when the running cost f is not too large, while the other term in the cost function is quite large since we will assume $1<p\leqq 2$ i.e. $q\geqq 2$.

Theorem I.1. *Assume that* $1<p\leqq 2$ *and that* f *satisfies*

$$\lim\{f(x)d(x)^q/d(x)\to 0_+\}=C_1\geqq 0. \tag{14}$$

Then, there is a unique solution $u\in C^2(\Omega)$ *of* (1) *such that* $u(x)\to+\infty$ *as* $d(x)\to 0_+$. *In addition, any solution* $v\in C^2(\Omega)$ *of* (1) *satisfies:* $u\geqq v$ *on* Ω. *Finally, if* C_0 *is the unique positive root of* $\left(\frac{2-p}{p-1}\right)^p C_0^p-\frac{2-p}{(p-1)^2}C_0-C_1=0$ *if* $p<2$, $C_0^2-C_0-C_1=0$ *if* $p=2$, *then* u *satisfies*

$$\left.\begin{aligned}&\lim\left\{u(x)d(x)^{\frac{2-p}{p-1}}/d(x)\to 0_+\right\}=C_0 \quad \text{if} \quad p<2\\ &\lim\{u(x)\,|\mathrm{Log}\,d(x)|^{-1}/d(x)\to O_+\}=C_0 \quad \text{if} \quad p=2.\end{aligned}\right\} \tag{15}$$

□

We now turn to the case when both terms in the running cost are not too large: in particular we assume that $p>2$ i.e. $1<q\leqq 2$.

Theorem I.2. *Assume that* $p>2$ *and that* f *satisfies*

$$\lim\{f(x)d(x)/d(x)\to 0_+\}=0, \quad \textit{for some} \quad \beta\in(0,p). \tag{16}$$

Then, all solutions $v\in C^2(\Omega)$ *of* (1) *bounded from below are bounded and may be extended continuously to* $\bar{\Omega}$. *And there exists a maximum solution* $u\in C^2(\Omega)$ *of* (1). *In* *addition, u satisfies*

$$\liminf_{y\in\Omega,\, y\to x} \{u(y)-u(x)\}\,|y-x|^{-\alpha}<0\,, \quad \textit{for all} \quad x\in\partial\Omega \tag{17}$$

where $\alpha=(p-2)/(p-1)$.

Furthermore, if $\liminf\{f(x)d(x)^{\gamma}/d(x)+0_+\}>0$ *for some* $\gamma\in(q,\beta)$, *then* (17) *holds with* $\alpha=1-\gamma/p$. □

Also, if additional assumptions on Ω or f are made, we are able to sharpen (17) or prove (12) [or even sharper estimates then (17) and (12)...].

The next case concerns the situation when the running cost f is blowing up near the boundary very fast. We have the

Theorem I.3. *Assume that* f *satisfies*

$$\liminf\{f(x)d(x)^{\beta}/d(x)\to 0_+\}>0\,, \quad \textit{for some} \quad \beta\geqq\max(p,q)\,. \tag{18}$$

Then, any solution $v\in C^2(\Omega)$ *of* (1) *bounded from below converges to* $+\infty$ *as* $d(x)$ *goes to* 0. *In addition, such a solution is unique if* (18) *is replaced by*

$$\lim\{f(x)d(x)^{\beta}/d(x)\to 0_+\}=C_1>0\,, \quad \textit{for some} \quad \beta\geqq\max(p,q) \tag{18'}$$

and this solution, denoted by u, satisfies

$$\lim\{u(x)d(x)^{\alpha}/d(x)\to 0_+\}=C_0\,, \tag{19}$$

where $d(x)^{\alpha}$ *is replaced by* $|\mathrm{Log}\,d(x)|^{-1}$ *if* $\beta=p\geqq q$; $\alpha=\dfrac{\beta}{p}-1$ *and* $C_0=\left(\dfrac{C_1}{\alpha}\right)^{1/p}$ *if* $\beta>\max(p,q)$; $C_0=C_1^{1/p}$ *if* $\beta=p>2$; $C_0=(1+C_1)^{1/2}$ *if* $\beta=p=2$. □

Roughly speaking, the combination of Theorems I.1–I.3 cover all possible situations. One way of unifying the above results is by the use of the viscosity formulation of the various boundary conditions encountered above namely

$$u-\varphi \quad \text{achieves its minimum over } \Omega\,, \quad \text{for all} \quad \varphi\in C^2(\bar{\Omega})\,. \tag{20}$$

Theorem I.4. *Assume that* $p>1$ *and* $\beta>0$ *and that either* f *is bounded or* $f(x)d(x)^{\beta}$ *converges to a positive constant as* $d(x)$ *converges to* 0_+. *Then, there is a unique* $u\in C^2(\Omega)$ *solution of* (1) *satisfying* (20). □

This is a nonexhaustive list of results since we will consider below many related questions like the stochastic interpretation of the above solutions, the existence of optimal controls, the ergodic problem, i.e. $\lambda\to 0_+$, the approximation of such solutions, extensions to more general data f or Hamiltonians. Finally, we will also briefly explain how the techniques we introduce allow us to treat similar boundary conditions for other types of nonlinear equations.

I.5. Organization of the Paper

As usual in stochastic control problems, various strategies are possible. One can use p.d.e. methods to derive the existence of a smooth solution of the associated HJB equation – here a second order quasilinear elliptic equation with strong nonlinearities in the gradient and singular boundary conditions. The uniqueness question may be solved directly by p.d.e. methods or by checking that any solution is the value function. Finally, one builds an optimal control using, whenever it is possible, the solution of the HJB equation. This is why some sections below deal with purely p.d.e. questions while others are concerned with the stochastic interpretation. Another distinction is made below between what we call the model problem (1) and more general equations. This artificial distinction is made only to simplify the exposition. In fact, in all sections below, we adopt a layered presentation with gradual generalizations where we just explain the required modifications of proofs.

II. Subquadratic Hamiltonians

We will be dealing here with (1) in the case when $1<p\leqq 2$.

II.1. Bounded Data

We begin with the case of bounded data i.e. we assume that $f\in L^\infty(\Omega)$.

Theorem II.1. *There is a unique solution $u\in W^{2,r}(\Omega)$ $(\forall r<\infty)$ of* (1) *such that $u(x)\to+\infty$ as $d(x)\to 0_+$. In addition, if $C_0=(p-1)^{\frac{p-2}{p-1}}(2-p)^{-1}$ when $p<2$, $C_0=1$ when $p=2$, then* (15) *holds. Finally, let $v\in L^1_{\mathrm{loc}}(\Omega)$ satisfy*

$$-\Delta v+p|\xi|^{p-2}\xi\cdot\nabla v+\lambda v\leqq f+(p-1)|\xi|^p \quad in \quad \mathscr{D}'(\Omega), \qquad \forall\xi\in\mathbb{R}^n \tag{21}$$

then $v\leqq u$ a.e. in Ω; in other words, u is the maximum L^1_{loc} subsolution. □

Corollary II.1. *Let $f_1, f_2\in L^\infty(\Omega)$ and let u_1, u_2 be the corresponding solutions of* (1) *which go to $+\infty$ on $\partial\Omega$. Then, we have*

$$\sup_\Omega\,(u_1-u_2)^+\leqq\frac{1}{\lambda}\sup_\Omega\,(f_1-f_2)^+ . \quad \square$$

Proof of Corollary II.1. $u_1-\frac{1}{\lambda}\sup_\Omega\,(f_1-f_2)^+$ is a subsolution of (1) with f replaced by f_2 so by Theorem II.1 $u_1\leqq u_2+\frac{1}{\lambda}\sup_\Omega\,(f_1-f_2)^+$. □

The proof of Theorem II.1 is unfortunately a bit longer and we split it into several parts. First (step 1), we compute the explosion rate of such a solution and this trivial computation leads to families of super and subsolutions. Next (step 2), we build a minimum and a maximum "explosive" solution which have the same leading behaviour near the boundary. Then (step 3), we prove the uniqueness and (15). Finally (step 4), we prove the "maximal subsolution" property.

Step 1. It is reasonable to try to obtain the leading term in an expansion of a solution of (1) blowing up at the boundary by the following ansatz near the boundary: $u(x)\simeq C_0 d(x)^{-\alpha}$. The most explosive term in $[-\Delta u+|\nabla u|^p+\lambda u-f]$ is then

$$-C_0\alpha(\alpha+1)d^{-\alpha-2}+C_0^p\alpha^p d^{-(\alpha+1)p},$$