nonlinearities in the gradient and singular boundary conditions. The uniqueness question may be solved directly by p.d.e. methods or by checking that any solution is the value function. Finally, one builds an optimal control using, whenever it is possible, the solution of the HJB equation. This is why some sections below deal with purely p.d.e. questions while others are concerned with the stochastic interpretation. Another distinction is made below between what we call the model problem (1) and more general equations. This artificial distinction is made only to simplify the exposition. In fact, in all sections below, we adopt a layered presentation with gradual generalizations where we just explain the required modifications of proofs.

II. Subquadratic Hamiltonians

We will be dealing here with (1) in the case when $1<p\leqq 2$.

II.1. Bounded Data

We begin with the case of bounded data i.e. we assume that $f\in L^\infty(\Omega)$.

Theorem II.1. *There is a unique solution $u\in W^{2,r}(\Omega)$ $(\forall r<\infty)$ of* (1) *such that $u(x)\to+\infty$ as $d(x)\to 0_+$. In addition, if $C_0=(p-1)^{\frac{p-2}{p-1}}(2-p)^{-1}$ when $p<2$, $C_0=1$ when $p=2$, then* (15) *holds. Finally, let $v\in L^1_{\mathrm{loc}}(\Omega)$ satisfy*

$$-\Delta v+p|\xi|^{p-2}\xi\cdot\nabla v+\lambda v\leqq f+(p-1)|\xi|^p \quad \textit{in} \quad \mathscr{D}'(\Omega), \qquad \forall\xi\in\mathbb{R}^n \tag{21}$$

then $v\leqq u$ a.e. in Ω; in other words, u is the maximum L^1_{loc} subsolution. □

Corollary II.1. *Let $f_1, f_2\in L^\infty(\Omega)$ and let u_1, u_2 be the corresponding solutions of* (1) *which go to $+\infty$ on $\partial\Omega$. Then, we have*

$$\sup_\Omega (u_1-u_2)^+\leqq\frac{1}{\lambda}\sup_\Omega (f_1-f_2)^+ . \quad \square$$

Proof of Corollary II.1. $u_1-\frac{1}{\lambda}\sup_\Omega (f_1-f_2)^+$ is a subsolution of (1) with f replaced by f_2 so by Theorem II.1 $u_1\leqq u_2+\frac{1}{\lambda}\sup_\Omega (f_1-f_2)^+$. □

The proof of Theorem II.1 is unfortunately a bit longer and we split it into several parts. First (step 1), we compute the explosion rate of such a solution and this trivial computation leads to families of super and subsolutions. Next (step 2), we build a minimum and a maximum "explosive" solution which have the same leading behaviour near the boundary. Then (step 3), we prove the uniqueness and (15). Finally (step 4), we prove the "maximal subsolution" property.

Step 1. It is reasonable to try to obtain the leading term in an expansion of a solution of (1) blowing up at the boundary by the following ansatz near the boundary: $u(x)\simeq C_0 d(x)^{-\alpha}$. The most explosive term in $[-\Delta u+|\nabla u|^p+\lambda u-f]$ is then

$$-C_0\alpha(\alpha+1)d^{-\alpha-2}+C_0^p\alpha^p d^{-(\alpha+1)p},$$

where we used (twice) the fact that $|\nabla d|=1$ near the boundary (in fact, as it is well-known: $|\nabla d|=1$ at each differentiability point of d, and d is smooth near the boundary if Ω is smooth). This leads to the choices

$$\alpha=\frac{2-p}{p-1}, \quad C_0=\alpha^{-1}(\alpha+1)^{1/(p-1)} \quad \text{if} \quad p<2.$$

Of course, if $p=2$ one replaces $C_0 d^{-\alpha}$ by $-C_0 \operatorname{Log} d$ and one finds $C_0=1$.

In order to use in a meaningful way the above formal consideration, we build two families of "approximations of $C_0 d^{-\alpha}$", each of which is a two-parameter family, where we first denote by d any smooth function, say $C^2(\bar{\Omega})$, on $\bar{\Omega}$ equal to $\operatorname{dist}(x,\partial\Omega)$ near the boundary, say for $\operatorname{dist}(x,\partial\Omega)\leqq\delta_0$ with $\delta_0>0$. Then, we introduce for $\varepsilon,\delta\geqq 0$

$$\left.\begin{aligned}\bar{w}_{\varepsilon,\delta}&=(C_0+\varepsilon)(d-\delta)^{-\alpha}+C_\varepsilon\\ \underline{w}_{\varepsilon,\delta}&=(C_0-\varepsilon)(d+\delta)^{-\alpha}-C_\varepsilon\end{aligned}\right\} \tag{22}$$

for some large constant C_ε to be determined. Of course, if $p=2$ then $(d\pm\delta)^{-\alpha}$ is replaced by $-\operatorname{Log}(d\pm\delta)$. Notice also that if $\underline{w}_{\varepsilon,\delta}$ is defined and smooth on Ω, $\bar{w}_{\varepsilon,\delta}$ is only defined on $\Omega_\delta=\{x\in\Omega, \operatorname{dist}(x,\partial\Omega)>\delta\}$ at least for $\delta<\delta_0$ (δ_0 to be choosen small enough; $0\leqq\delta<\delta_0$ will always be assumed in this proof). In fact, it will be handy to consider d as a smooth function on $\mathbb{R}^N$, say $C^2(\mathbb{R}^N)$, such that: $d(x)=\operatorname{dist}(x,\partial\Omega)$ if $x\in\bar{\Omega}$, $\operatorname{dist}(x,\partial\Omega)\leqq\delta_0$; $d(x)\geqq\delta_0$ if $\operatorname{dist}(x,\partial\Omega)\geqq\delta_0$, $x\in\Omega$; $d(x)=-\operatorname{dist}(x,\partial\Omega)$ if $x\notin\Omega$ and $\operatorname{dist}(x,\partial\Omega)\leqq\delta_0$; $d(x)\leqq-\delta_0$ if $\operatorname{dist}(x,\partial\Omega)\geqq\delta_0$, $x\notin\bar{\Omega}$. Observe of course that $|\nabla d|=1$ in $\{\operatorname{dist}(x,\partial\Omega)\leqq\delta_0\}$ and that $d(x)=-\delta=\operatorname{dist}(x,\partial\Omega_\delta)$ if $\operatorname{dist}(x,\partial\Omega)\leqq\delta_0$ while $d(x)+\delta=\operatorname{dist}(x,\partial\Omega^\delta)$ if $\operatorname{dist}(x,\partial\Omega)\leqq\delta_0$, where

$$\Omega^\delta=\{x\in\mathbb{R}^N, \operatorname{dist}(x,\bar{\Omega})\leqq\delta\}=\{x\in\mathbb{R}^N/d(x)\geqq-\delta\}.$$

So that, we may consider $\underline{w}_{\varepsilon,\delta}$ to be defined on Ω^δ. (Notice that such a function d exists as soon as Ω is open bounded and has a C^2-regular boundary $\partial\Omega$.)

We conclude these preliminaries with the following computations

$$\begin{aligned}&-\Delta\bar{w}_{\varepsilon,\delta}+|\nabla\bar{w}_{\varepsilon,\delta}|^p+\lambda\bar{w}_{\varepsilon,\delta}-f\\ &\quad=-\alpha(\alpha+1)(C_0+\varepsilon)(d-\delta)^{-\alpha-2}|\nabla d|^2+\alpha(C_0+\varepsilon)(d-\delta)^{-\alpha-1}\Delta d\\ &\qquad+\alpha^p(C_0+\varepsilon)^p(d-\delta)^{-p(\alpha+1)}|\nabla d|^p+\lambda(C_0+\varepsilon)(d-\delta)^{-\alpha}+\lambda C_\varepsilon-f.\end{aligned}$$

Recalling that $\alpha+2=(\alpha+1)p$ and $\alpha^p C_0^p=\alpha(\alpha+1)C_0$, we deduce easily for $\varepsilon\leqq 1$, $\delta\leqq\delta_0$

$$-\Delta\bar{w}_{\varepsilon,\delta}+|\nabla\bar{w}_{\varepsilon,\delta}|^p+\lambda\bar{w}_{\varepsilon,\delta}-f\geqq\nu\varepsilon(d-\delta)^{-\alpha-2}+\lambda C-C(1+(d-\delta)^{-\alpha-1})$$

for some $\nu>0$, $C\geqq 0$. And we can choose C_ε large enough in order to find

$$-\Delta\bar{w}_{\varepsilon,\delta}+|\nabla\bar{w}_{\varepsilon,\delta}|^p+\lambda\bar{w}_{\varepsilon,\delta}\geqq f \quad \text{in} \quad \Omega_\delta. \tag{23}$$

Similarly, one shows that C_ε can be choosen large enough to have:

$$-\Delta\underline{w}_{\varepsilon,\delta}+|\nabla\underline{w}_{\varepsilon,\delta}|^p+\lambda\underline{w}_{\varepsilon,\delta}\leqq f \quad \text{in} \quad \Omega^\delta. \tag{24}$$

Step 2. Building a minimum "explosive" solution is easy in the subquadratic case. Indeed, one solves

$$-\Delta u_R+|\nabla u_R|^p+\lambda u_R=f \quad \text{in} \quad \Omega, \qquad u_R\in W^{2,r}(\Omega) \quad (\forall r<\infty) \tag{25}$$

with boundary conditions going to infinity (as $R\to\infty$) like for instance

$$u_R=R \quad \text{on} \quad \partial\Omega \tag{26}$$

or

$$u_R=\underline{w}_{\varepsilon,1/R} \quad \text{on} \quad \partial\Omega, \quad \text{for any fixed} \quad \varepsilon>0. \tag{27}$$

Since $p\leqq 2$, the existence follows from standard results on subquadratic quasilinear equations (see for example Amann and Crandall [1]). In view of the maximum principle (we have to use here the slightly more general form of maximum principle in Sobolev spaces – see for example Bony [5] and Lions [23]) we deduce in the case of (27) for example

$$\underline{w}_{\varepsilon,1/R}\leqq u_R\leqq u_{R'}\leqq \bar{w}_{\varepsilon'} \quad \text{if} \quad 0<R<R', \quad \forall\varepsilon'>0$$

and where $\bar{w}_\varepsilon=\bar{w}_{\varepsilon,0}$. The last inequality of this string comes from the maximum principle provided we observe that $u_{R'}<\bar{w}_\varepsilon$ near $\partial\Omega$ since $\bar{w}_\varepsilon$ blows up at the boundary.

Hence, u_R is bounded in L^∞_{loc}. This combined with (25) implies that u_R is bounded in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$: this can be deduced either from [1] using again the fact that we are dealing with a subquadratic Hamiltonian or by using the gradient estimates of the appendix (see Lions [16, 19]) which yield bounds in $W^{1,\infty}_{\text{loc}}(\Omega)$ and then using (25). Anyway, u_R converges (as $R\to\infty$) to a solution $\underline{u}$ of (1) in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ which also satisfies $\underline{w}_\varepsilon\leqq\underline{u}\leqq\bar{w}_{\varepsilon'}$ $\forall\varepsilon'>0$. Next, we claim that $\underline{u}\geqq\underline{w}_{\varepsilon'}$ for all $\varepsilon'>0$. Indeed for any $R'>0$, we can find R such that $\underline{w}_{\varepsilon',1/R'}\leqq\underline{w}_{\varepsilon',1/R}$ and letting R' go to $+\infty$, we conclude easily.

We now claim that $\underline{u}$ is the minimum "explosive" solution of (1). Indeed, let u be another solution of (1) in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ such that $u\to+\infty$ as $d(x)\to 0_+$ then by maximum principle $u\geqq u_R$ in Ω and thus passing to the limit we obtain $u\geqq\underline{u}$ in Ω.

To build a maximum explosive solution, we consider the preceding minimum explosive solution $\underline{u}_\delta$ in Ω_δ and we let δ go to 0. Recall that we have

$$(C_0-\varepsilon)(d+\delta)^{-\alpha}-C_\varepsilon\leqq\underline{u}_\delta\leqq\bar{w}_{\varepsilon,\delta}, \qquad \forall\varepsilon>0$$

and clearly enough $\underline{u}_\delta\geqq\underline{u}_{\delta'}$ if $0<\delta'<\delta$. Therefore, passing to the limit, exactly as above we find a solution $\bar{u}$ of (1) such that

$$\underline{w}_\varepsilon\leqq\bar{u}\leqq\bar{w}_\varepsilon \quad \text{in} \quad \Omega.$$

The fact that $\bar{u}$ is the maximum explosive solution is proved by using again the maximum principle to show (with the above notations)

$$u\leqq\underline{u}_\delta\to\bar{u} \quad \text{as} \quad \delta \text{ goes to } 0.$$

In conclusion, we found solutions $\underline{u}$, $\bar{u}\in W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ of (1) such that

$$\underline{w}_\varepsilon\leqq\underline{u}\leqq u\leqq\bar{u}\leqq\bar{w}_\varepsilon \quad \text{in} \quad \Omega, \quad \text{for all} \quad \varepsilon>0, \tag{28}$$

where u is any solution of (1) in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ such that $u\to\infty$ as $d\to 0_+$.

Step 3: Uniqueness. It is of course enough to show that $\underline{u} \equiv \bar{u}$ in Ω. We first observe that (28) implies that $\bar{u}(x)(\underline{u}(x))^{-1}$ converges to 1 as $d(x) \to 0_+$. Therefore, if we denote by $m = \inf_{\Omega} f(x)$, we deduce that for all $\theta \in (0,1)$

$$\underline{u}(x) > \theta \bar{u}(x) + (1-\theta) m/\lambda \quad \text{in a neighbourhood of} \quad \partial\Omega .$$

In addition, $w = \theta \bar{u} + (1-\theta) m/\lambda$ satisfies in Ω

$$-\Delta w + |\nabla w|^p + \lambda w \leqq \theta f + (1-\theta) m \leqq f .$$

Therefore, we deduce easily by the maximum principle

$$w \leqq \underline{u} \quad \text{in} \quad \Omega$$

and we conclude letting θ go to 1.

Step 4. We wish to prove that the unique explosive solution $\underline{u}$ of (1) that we built above is also the maximum L^1_{loc} subsolution. Let $v \in L^1_{\text{loc}}(\Omega)$ satisfy (21). In order to avoid some rather unpleasant technicalities, we begin with the case $f \in C(\bar{\Omega})$: in that case, we smooth v by convolution i.e. we consider $v_n = v * \varrho_n$ where $\varrho \in \mathscr{D}(\mathbb{R}^N)$, $0 \leqq \varrho$, $\int_{\mathbb{R}^N} \varrho dx = 1$, $\text{supp}(\varrho) \subset B_1$ and $\varrho_n = n^N \varrho(n \cdot)$. Then, if $\delta > \frac{1}{n}$, we find easily

$$-\Delta v_n + |\nabla v_n|^p + \lambda v_n \leqq f * \varrho_n \quad \text{in} \quad \Omega_\delta$$

and $f * \varrho_n \leqq f + \varepsilon_n$ where $\varepsilon_n \underset{n}{\to} 0$. Therefore, we deduce

$$\left(v_n - \frac{\varepsilon_n}{\lambda_n} \right) \leqq \underline{u}_\delta \quad \text{if} \quad \delta > \frac{1}{n}$$

and we conclude letting n go to $+\infty$ and then δ go to 0_+.

If $f \in L^\infty(\Omega)$, we obtain by the above proof that $v_n \leqq \underline{u}^n_\delta$ where $\underline{u}^n_\delta$ is the explosive solution in Ω_δ corresponding to $f * \varrho_n$ $\left(\text{still if } \delta > \frac{1}{n} \right)$. In addition, the proof made above also shows that $\underline{u}^n_\delta$ is bounded in $L^\infty_{\text{loc}}(\Omega_\delta)$ and thus in $W^{2,r}_{\text{loc}}(\Omega_\delta)$ $(\forall r < \infty)$ since $f * \varrho_n$ is bounded in $L^\infty(\Omega)$: in fact, one may even choose C_ε such that

$$(C_0 - \varepsilon)(d - \delta)^{-\alpha} - C_\varepsilon \leqq \underline{u}^n_\delta \leqq (C_0 + \varepsilon)(d + \delta)^{-\alpha} + C_\varepsilon$$

(with the usual modifications if $p = 2$). Then, we may pass to the limit as n goes to $+\infty$ and $\underline{u}^n_\delta$ (or subsequences) converges to a solution of (1) in Ω_δ thus below $\underline{u}_\delta$ (in fact it is $\underline{u}_\delta$ because the above inequality shows it blows up at $\partial\Omega_\delta$). Therefore, $v \leqq \underline{u}_\delta$ in Ω_δ and we conclude letting δ go to 0. □

Remark II.1. One may deduce from the above arguments the "continuity" of the explosive solution with respect to Ω, p or f (for the weak $L^\infty *$ topology).

Remark II.2. By a convenient (and technical) variation of the above method one can show that it is possible to replace $f \in L^\infty(\Omega)$ by $f \in L^p_{\text{loc}}(\Omega)$ $\left(p > \frac{N}{2} \right)$, f bounded from below and f bounded near $\partial\Omega$.

II.2. General Data

We now wish to allow some data f which may not be bounded near $\partial\Omega$.

Theorem II.2. *Let $f \in L^\infty_{loc}(\Omega)$, assume that f is bounded from below and that f satisfies* (14). *Then, Theorem II.1 still holds provided one replaces C_0 by the unique positive solution of the equation $\left(\frac{2-p}{p-1}\right)^p C_0^p - \frac{2-p}{(p-1)^2} C_0 - C_1 = 0$ if $p<2$, $C_0^2 - C_0 - C_1 = 0$ if $p=2$.*

Proof. We only present the main modifications in the preceding proof. With the above new value of C_0, one builds exactly as in the proof of Theorem II.1 a maximum explosive solution $\bar{u}$ of (1) such that

$$(C_0-\varepsilon)d^{-\alpha} - C_\varepsilon \leqq \bar{u} \leqq (C_0+\varepsilon)d^{-\alpha} + C_\varepsilon \quad \text{in} \quad \Omega, \quad \forall \varepsilon > 0. \tag{29}$$

The above equation for C_0 comes into the picture when making the formal computations of Step 1 and balancing the various leading terms in $d^{-\alpha-2} = d^{-q}$. The only modification in the proof of Theorem II.1 consists in proving that there exists a minimum explosive solution $\underline{u}$ which also satisfies (29). To this end, we observe that $\underline{w}_{\varepsilon,\delta}$ is a subsolution of (1) when Ω is replaced by Ω^δ and f is replaced by

$$f_\delta = \min(f, C_2 + C_3(d+\delta)^{-q}) \quad \text{in} \quad \Omega, \quad = C_2 + C_3(d+\delta)^{-q} \quad \text{in} \quad \Omega^\delta - \Omega,$$

where C_3, C_2 are positive constants such that $C_3 > C_1$, $C_2 + C_3 d^{-q} > f$ in Ω. Obviously, $f_\delta \in L^\infty(\Omega)$. Therefore, by Theorem II.1 and its proof, there exists a unique explosive solution u_δ of (1) with f replaced by f_δ, [obtained by an increasing limit of solutions of (1) with finite boundary values] and $u_\delta \geqq \underline{w}_{\varepsilon,\delta}$. Since $f \geqq f_\delta$, any explosive solution of (1) is above u [use the maximum principle with the approximating bounded solutions of (1)] and thus in particular $\bar{u} \geqq u_\delta$. From this, we deduce easily letting δ go to 0 the existence of a minimum explosive solution of (1) $\underline{u}$ satisfying (29).

Remark II.3. The analogues of Remarks II.1–II.2 still hold: notice only that the stability with respect to f holds with respect to the weak $*$ L^∞ topology provided the data f are uniformly bounded from below and, satisfy (14) with C_1 bounded and $f(x) \leqq Cd^{-q} + C$ for some $C \geqq 0$.

Remark II.4. The proof also shows that if $\bar{w}$ is a supersolution of (1) which blows up on $\partial\Omega$ i.e.

$$-\Delta\bar{w} + |\nabla\bar{w}|^p + \lambda\bar{w} \geqq f \quad \text{in} \quad \Omega, \quad \bar{w} \to 0 \quad \text{as} \quad d(x) \to 0_+$$

then $\bar{w} \geqq u$.

Remark II.5. If we allow f to go to $-\infty$ near $\partial\Omega$ (or some points of $\partial\Omega$) then the situation is a bit more complex. Let $f \in L^\infty_{loc}(\Omega)$, if we assume (14) with $C_1 = 0$ then the above result is no longer true. In that case, there still exists a maximum explosive solution which behaves as $C_0 d^{-\alpha}$ and is the unique solution going to $+\infty$ as $C_0 d^{-\alpha}$. However, in general, there may exist other solutions going to $+\infty$

less rapidly: indeed, consider

$$f(x)=\frac{\Delta d}{d}-\frac{|\nabla d|^2}{d^2}+\frac{|\nabla d|^p}{d^p}-\lambda\,\mathrm{Log}\,d\,.$$

If $1<p<2$, f behaves like $-\frac{1}{d^2}$ near $\partial\Omega$ and thus satisfies (14). And notice that $u(x)=-\mathrm{Log}\,d(x)$ is then a solution of (1) which goes to $+\infty$ as $d(x)$ goes to 0.

If we assume (14) and $C_1>0$, then f is bounded from below and Theorem II.2 applies. Now, if we assume (14) and $C_1<0$, then there are two positive solutions C_0 of the equation stated in Theorem II.2 say $0<C_0^-<C_0^+$ and $C_0^-\to 0$, $C_0^+\to C_0$ as $C_1\to 0_-$. Again, there exists a maximum explosive solution of (1) behaving near $\partial\Omega$ as $C_0^+d^{-\alpha}$ and it is the unique such solution. But there also exists in general another explosive solution of (1) behaving near $\partial\Omega$ as $C_0^-d^{-\alpha}$: for instance, consider $f=-\Delta w+|\nabla w|^p+\lambda w$ where $w=C_0^-d^{-\alpha}$. □

II.3. Asymptotic Expansions Near the Boundary

In this section, we want to precise a bit the behaviour near the boundary of solutions which blow up at the boundary. Even if we will not present a complete asymptotic expansion near the boundary (which should include $[q]-1$ singular terms plus a bounded term where $[q]$ denotes the integer part of q), the methods we use should give it and we leave the awful computations to a courageous reader. We will only prove the

Theorem II.3. *Let $f\in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below and assume that*

$$\lim\{f(x)d(x)^{q-1}/d(x)\to 0_+\}=0\,. \tag{30}$$

We denote by u the unique solution of (1) *in $W^{2,r}(\Omega)\,(\forall r<\infty)$ which goes to $+\infty$ on $\partial\Omega$. Then, if $p\in(\frac{3}{2},2]$ i.e. $q\in[2,3)$, $u-\frac{C_0}{d^\alpha}$ is bounded on Ω when $p<2$ while $u+\log d$ is bounded on Ω when $p=2$. Next, if $p\in(1,\frac{3}{2}]$ we set*

$$\begin{aligned} C_1(x)&=-\frac{1}{2}\frac{\alpha}{\alpha-1}\,C_0\Delta d(x) \quad &&\text{if}\quad p<\frac{3}{2},\\ C_1(x)&=-\frac{1}{2}\,C_0\Delta d(x) &&\text{if}\quad p=\frac{3}{2}, \end{aligned} \tag{31}$$

and we have

$$\left.\begin{aligned} &\left\{u-\frac{C_0}{d^\alpha}\right\}d^{\alpha-1}\to C_1 \quad &&\text{as}\quad d\to 0_+ \quad \text{if}\quad p<\frac{3}{2},\\ &\left\{u-\frac{C_0}{d}\right\}|\log d|^{-1}\to C_1 &&\text{as}\quad d\to 0_+ \quad \text{if}\quad p=\frac{3}{2}. \end{aligned}\right\} \tag{32}$$

Proof. We begin with the case $1<p\leqq\frac{3}{2}$. In view of the results of the previous sections, it is enough build appropriate sub and supersolutions which blow up

near $\partial\Omega$. To this end, we consider

$$\left.\begin{array}{lll} w_\varepsilon^+ = \dfrac{C_0}{d^\alpha} + \dfrac{(C_1+\varepsilon)}{d^{\alpha-1}} + C_\varepsilon, & w_\varepsilon^- = \dfrac{C_0}{d^\alpha} + \dfrac{(C_1-\varepsilon)}{d^{\alpha-1}} - C_\varepsilon & \text{if } p<\dfrac{3}{2}, \\ w_\varepsilon^- = \dfrac{C_0}{d} - (C_1+\varepsilon)\,\mathrm{Log}\, d + C_\varepsilon, & w_\varepsilon^- = \dfrac{C_0}{d} - (C_1-\varepsilon)\,\mathrm{Log}\, d - C_\varepsilon & \text{if } p=\dfrac{3}{2}, \end{array}\right\} \quad (33)$$

where C_ε is a positive constant to be determined. Tedious computations show that, provided C_1 is given by (31) and C_ε is large enough, w_ε^+ (resp. w_ε^-) is a supersolution of (1) [resp. subsolution of (1)]. Therefore, $w_\varepsilon^- \leqq u \leqq w_\varepsilon^+$ in Ω for all $\varepsilon>0$ and (32) is proved.

Next, if $\frac{3}{2}<p\leqq 2$, we also want to build convenient sub and supersolutions. However, in this case, the choices are not straightforward as above. Indeed, recalling that $\alpha=\dfrac{2-p}{p-1}$ we choose

$$w_\varepsilon^+ = \frac{C_0}{d^\alpha} - (C_1+\varepsilon)d^{1-\alpha} + C_\varepsilon, \quad w_\varepsilon^- = \frac{C_0}{d^\alpha} - (C_1-\varepsilon)d^{1-\alpha} - C_\varepsilon, \quad (34)$$

where

$$C_1 = -\frac{1}{2}\frac{\alpha}{1-\alpha}C_0\Delta d \quad \text{if} \quad p<2, \qquad C_1 = -\frac{1}{2}\Delta d \quad \text{if} \quad p=2.$$

Again, one can check that w_ε^+, w_ε^- for conveniently large C_ε are sub and supersolutions of (1) and since they go to $+\infty$ at $\partial\Omega$ we deduce that $w_\varepsilon^- \leqq u \leqq w_\varepsilon^+$ in Ω and we conclude. □

Remark II.6. In the various bounds on the behaviour of explosive solutions near the boundary, it may seem strange that the leading terms are not continuous with respect to p (as p goes to 2 for example). Similarly, in (34) the term $d^{1-\alpha}$ vanishes and could seem to be irrelevant. However – and this fits well with the stochastic control interpretation – these questions disappear if we look for formal expansions of the gradient obtained by differentiating these expansions for the solution: indeed, in Theorem II.1, u behaves like

$$(p-1)^{\frac{p-2}{p-1}}\frac{1}{2-p}\,d(x)^{-\frac{2-p}{p-1}}$$

so $\nabla u(x)$ should behave like $-(p-1)^{-1/(p-1)}\nabla d(x)d^{-1/(p-1)}$ and when p goes to 2 this quantity goes to $-\nabla d(x)d^{-1}$ which is precisely the gradient of $-\log d$. A similar explanation holds for (34).

III. Infinite Boundary Conditions and Blowing up Data

In this section, we consider the case of data f blowing up at the boundary fast enough to force solutions of (1) bounded from below to blow up at the boundary. This also will yield some uniqueness results. The results of this section correspond to Theorem I.3.