near $\partial\Omega$. To this end, we consider

$$\left.\begin{aligned} w_\varepsilon^+ &= \frac{C_0}{d^\alpha} + \frac{(C_1+\varepsilon)}{d^{\alpha-1}} + C_\varepsilon, & w_\varepsilon^- &= \frac{C_0}{d^\alpha} + \frac{(C_1-\varepsilon)}{d^{\alpha-1}} - C_\varepsilon & \text{if } p<\frac{3}{2},\\ w_\varepsilon^- &= \frac{C_0}{d} - (C_1+\varepsilon)\operatorname{Log} d + C_\varepsilon, & w_\varepsilon^- &= \frac{C_0}{d} - (C_1-\varepsilon)\operatorname{Log} d - C_\varepsilon & \text{if } p=\frac{3}{2},\end{aligned}\right\} \tag{33}$$

where C_ε is a positive constant to be determined. Tedious computations show that, provided C_1 is given by (31) and C_ε is large enough, w_ε^+ (resp. w_ε^-) is a supersolution of (1) [resp. subsolution of (1)]. Therefore, $w_\varepsilon^- \leqq u \leqq w_\varepsilon^+$ in Ω for all $\varepsilon>0$ and (32) is proved.

Next, if $\frac{3}{2}<p\leqq 2$, we also want to build convenient sub and supersolutions. However, in this case, the choices are not straightforward as above. Indeed, recalling that $\alpha=\frac{2-p}{p-1}$ we choose

$$w_\varepsilon^+ = \frac{C_0}{d^\alpha} - (C_1+\varepsilon)d^{1-\alpha} + C_\varepsilon, \qquad w_\varepsilon^- = \frac{C_0}{d^\alpha} - (C_1-\varepsilon)d^{1-\alpha} - C_\varepsilon, \tag{34}$$

where

$$C_1 = -\frac{1}{2}\frac{\alpha}{1-\alpha} C_0 \Delta d \quad \text{if} \quad p<2, \qquad C_1 = -\frac{1}{2}\Delta d \quad \text{if} \quad p=2.$$

Again, one can check that w_ε^+, w_ε^- for conveniently large C_ε are sub and supersolutions of (1) and since they go to $+\infty$ at $\partial\Omega$ we deduce that $w_\varepsilon^- \leqq u \leqq w_\varepsilon^+$ in Ω and we conclude. □

Remark II.6. In the various bounds on the behaviour of explosive solutions near the boundary, it may seem strange that the leading terms are not continuous with respect to p (as p goes to 2 for example). Similarly, in (34) the term $d^{1-\alpha}$ vanishes and could seem to be irrelevant. However – and this fits well with the stochastic control interpretation – these questions disappear if we look for formal expansions of the gradient obtained by differentiating these expansions for the solution: indeed, in Theorem II.1, u behaves like

$$(p-1)^{\frac{p-2}{p-1}} \frac{1}{2-p} d(x)^{-\frac{2-p}{p-1}}$$

so $\nabla u(x)$ should behave like $-(p-1)^{-1/(p-1)}\nabla d(x) d^{-1/(p-1)}$ and when p goes to 2 this quantity goes to $-\nabla d(x) d^{-1}$ which is precisely the gradient of $-\log d$. A similar explanation holds for (34).

III. Infinite Boundary Conditions and Blowing up Data

In this section, we consider the case of data f blowing up at the boundary fast enough to force solutions of (1) bounded from below to blow up at the boundary. This also will yield some uniqueness results. The results of this section correspond to Theorem I.3.

III.1. Forced Infinite Boundary Conditions

Theorem III.1. *Assume that* $f \in L^\infty_{\text{loc}}(\Omega)$ *satisfies* (18). *Then, any solution* u *of* (1) $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ *which is bounded from below converges to* $+\infty$ *as* $d(x)$ *goes to* 0.

Remark III.1. The proof below may be adapted to treat the case of $f \in L^r_{\text{loc}}(\Omega)$ satisfying (18) with $r > N$.

Remark III.2. In general, there may exist solutions of (1) which are not bounded from below. For instance, take

$$f(x) = -\frac{\alpha C_0 \Delta d}{d^{\alpha+1}} + \frac{C_0 \alpha(\alpha+1)}{d^{\alpha+2}} |\nabla d|^2 + C_0^p \alpha^p |\nabla d|^p d^{-(\alpha+1)p} - \lambda C_0 d^{-\alpha}$$

with α, $C_0 > 0$, $u(x) = -\dfrac{C_0}{d^\alpha}$ is obviously a solution of (1) and f satisfies (18) with $\beta = \max((\alpha+1)p, \alpha+2)$. And it is easy to check that any $\beta > \max(p,q)$ can be reached with a convenient α [in fact even $\beta = \max(p,q)$] may be reached provided we replace $-C_0 d^{-\alpha}$ by $C_0 \log d$ for $\beta = p \geqq q$. It is also worth noticing that such solutions may exist for linear equations like

$$-\Delta u + u = f \quad \text{in} \quad \Omega$$

provided f behaves like $\dfrac{C_1}{d^\beta}$ with $\beta \geqq 2$ near the boundary.

Proof of Theorem III.1. Even if the arguments are very much similar, we will have to consider two different cases namely $\beta \geqq p > 2$ and $\beta \geqq q \geqq p$. In both cases, the strategy of proof consists in picking a point x_0 at a distance $2r$ of the boundary, working in the ball $B(x_0, r)$ rescaling the equation conveniently in order to deduce that $\liminf \{u(x) \mid d(x) \to 0_+\}$ is more than a fixed constant K_0 and then reiterating the argument to show that $\liminf \{u(x) \mid d(x) \to 0_+\} \geqq nK_0$ for all $n \geqq 1$.

Without loss of generality (add a large constant to u) we may assume that $u \geqq 0$ in Ω and that $f \geqq C_2 d^{-\beta}$ for some $C_2 > 0$, with $\beta = \max(p,q)$. Next, let $r > 0$ and let x_0 be any point in Ω such that $d(x_0) = 2r$. Clearly, we have

$$-\Delta u + |\nabla u|^p + \lambda u \geqq C_3 r^{-\beta} \quad \text{in} \quad B(x_0, r), \qquad u|_{\partial B(x_0,r)} \geqq 0, \tag{35}$$

where $C_3 = C_2 2^{-\beta}$. Using the existence results of Lions [16], we deduce that $u \geqq \tilde{u}_r(x - x_0)$ in $B(x_0, r)$ where $\tilde{u}_r \in C^2(\overline{B(0,r)})$ solves

$$-\Delta \tilde{u}_r + |\nabla \tilde{u}_r|^p + \lambda \tilde{u}_r = C_3 r^{-\beta} \quad \text{in} \quad B(0,r), \qquad \tilde{u}|_{\partial B(0,r)} = 0. \tag{36}$$

Next, in the case when $1 < p \leqq 2 \leqq q = \beta$, we introduce $u_r(x) = r^\alpha \tilde{u}_r(rx)$ for $x \in B(0,1)$ where $\alpha = (2-p)/(p-1)$ so that u_r solves

$$-\Delta u_r + |\nabla u_r|^p + \lambda r^2 u_r = C_3 \quad \text{in} \quad B(0,1), \qquad u_r|_{\partial B(0,1)} = 0. \tag{37}$$

And using the estimates of [21], one checks easily that u_r, as r goes to 0, converges uniformly to the solution u_0 of

$$-\Delta u_0 + |\nabla u_0|^p = C_3 \quad \text{in} \quad B(0,1), \qquad u_0|_{\partial B(0,1)} = 0. \tag{38}$$

Observing that $u_0>0$ in $B(0,1)$ (strong maximum principle) and so $u_0(0)>0$, we deduce easily that if $p<2<q=\beta$ then u blows up at $\partial\Omega$ and $\liminf\{u(x)d(x)^\alpha \mid d(x)\to 0_+\}>0$.

Now, if $p=2=q=\beta$, the above argument only shows

$$\liminf\{u(x)\mid d(x)\to 0_+\}\geqq K_0>0, \tag{39}$$

where $K_0=u_0(0)$.

In the other case i.e. $2<p=\beta$, we introduce $u_r(x)=\tilde{u}_r(rx)$ for $x\in B(0,1)$ so that $u_r\in C^2(\overline{B(0,1)})$ solves

$$-r^{p-2}\Delta u_r+|\nabla u_r|^p+\lambda r^p u_r=C_3 \quad \text{in} \quad B(0,1), \qquad u_r|_{\partial B(0,1)}=0.$$

And using the results of Lions [21], one sees that u_r converges uniformly to the unique viscosity solution u_0 in $C(\overline{B(0,1)})$ of

$$|\nabla u_0|^p=C_3 \quad \text{in} \quad B(0,1), \qquad u_0|_{\partial B(0,1)}=0$$

which is in fact explicitely given by

$$u_0(x)=C_3^{1/p}(1-|x|).$$

Therefore, in this case also, we prove that (39) holds with $K_0=C_3^{1/p}$.

In particular, for any $\varepsilon>0$, there exists $s_\varepsilon>0$ such that for $x\in\Omega$, $d(x)<s_\varepsilon$ then $u(x)\geqq K_0-\varepsilon$. Then, we go back to (35) replacing the boundary inequality by $u|_{\partial B(x_0,r)}\geqq K_0-\varepsilon$ if $r<s_\varepsilon/2$. And we go through the above proof to deduce finally

$$\liminf\{u(x)\mid d(x)\to 0_+\}\geqq K_0+K_0-\varepsilon=2K_0-\varepsilon$$

for all $\varepsilon>0$: indeed, the limit functions u_0' now satisfy the boundary conditions $u_0'=K_0-\varepsilon$ on $\partial B(0,1)$ i.e. $u_0'=u_0+K_0-\varepsilon$. Letting ε go to 0 and iterating the above argument, Theorem III.1 is proved. □

Remark III.3. Considering $w_{\varepsilon,\delta}(x)=-\varepsilon\log(d(x)+\delta)+\delta\log d(n)-C$, we see that $u\geqq w_{\varepsilon,\delta}$ near $d\Omega$ and this proves Theorem III.1 even if $\beta\geqq 2>p>1$.

III.2. Uniqueness Results

Theorem III.2. *Let $f\in L^\infty_{\text{loc}}(\Omega)$ satisfy* (18). *Then, there exists a maximum solution of* (1) *in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ which goes to $+\infty$ on $\partial\Omega$ and any $v\in L^1_{\text{loc}}(\Omega)$ satisfying* (21) *satisfies $v\leqq u$ a.e. in Ω. Among all solutions of* (1) *in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ which go to $+\infty$ on $\partial\Omega$, or equivalently that are bounded from below on Ω, there exists a minimum one which is the increasing limit of sequence of subsolutions of* (1) *(i.e. satisfying* (21)*) in $W^{2,r}(\Omega)$ $(\forall r<\infty)$.*

If we impose further restrictions on f, when we have the

Theorem III.3. *Let $f\in L^\infty_{\text{loc}}(\Omega)$ satisfy* (18′). *Then, there exists a unique solution of* (1) *in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ bounded from below. In addition, this solution satisfies* (19).

Proof of Theorem III.2. Let $C>0$ be a constant such that

$$f(x)\geqq Cd(x)^{-\beta}-C,$$

where $\beta=\max(p,q)$. Then, we set $\underline{w}_\delta=-M\,\mathrm{Log}(d+\delta)-K$ if $p\geqq 2\geqq q$, $\underline{w}=M(d+\delta)^{-\alpha}-K$ if $p<2<q$, where $\alpha=(q-p)/p$, M, K are positive constants

chosen in such a way that for δ small enough $\underline{w}_\delta$ is a subsolution of (1). In fact, we may find $R(\delta)\downarrow +\infty$ as $\delta\uparrow 0_+$ such that (with $a\wedge b=\inf(a,b)$):

$$+\Delta \underline{w}_\delta+|\nabla \underline{w}_\delta|^p+\lambda \underline{w}_\delta \leqq f\wedge R(\delta) \quad \text{in} \quad \Omega .$$

Then, using the existence results of Lions [16, 19] we deduce that there exists $\underline{u}_\delta \in W^{2,r}(\Omega)$ $(\forall r<\infty)$ solution of

$$-\Delta \underline{u}_\delta+|\nabla \underline{u}_\delta|^p+\lambda \underline{u}_\delta=f\wedge R(\delta) \quad \text{in} \quad \Omega, \qquad \underline{u}_\delta=\underline{w}_\delta \quad \text{on} \quad \partial\Omega;$$

and by the maximum principle $\underline{u}_\delta \geqq \underline{w}_\delta$ in Ω.

The remainder of the proof consists in passing to the limit as δ goes to 0 in order to build the minimum solution. To do so we need local upper bounds on $\underline{u}_\delta$: we will achieve this by building a supersolution. We first observe that it is possible to find $\Phi\in C^1(0,\infty)$ such that $\Phi(t)\to+\infty$ as $t\to 0_+$, $\Phi'(t)<0$ if $t>0$, $\Phi(t)>0$ if $t>0$ and

$$(\Phi^{-1/q})'\to 0 \quad \text{as} \quad t\to 0_+, \qquad f(x)\leqq \Phi(d(x)) \qquad \text{a.e. in } \Omega .$$

Now let $R=\sup_{\Omega} d$, $C_0=\sup_{[0,R]}(\Phi^{-1/q})'$. We denote by

$$\Psi_1=\mu\Phi^{1/p}, \qquad \Psi(t)=\int_t^R \Psi_1(s)ds,$$

where μ is a positive constant to be determined. We finally set

$$\bar{w}(x)=\Psi(d)+K$$

where K is a positive constant to be determined. We claim next that for large μ and K, $\bar{w}$ is a supersolution of (1) which of course blows up at $\partial\Omega$. Indeed, we find, denoting by $C=\|\Delta d\|_\infty$, that if $d(x)\leqq\delta_0$

$$\begin{aligned}
-\Delta\bar{w}+|\nabla\bar{w}|^p+\lambda\bar{w} &\geqq -\Psi''(d)-C|\Psi'(d)|+|\Psi'(d)|^p\\
&=\frac{\mu}{p}\,\Phi^{\frac{1}{p}-1}\Phi'-C\mu\Phi^{1/p}+\mu^p\Phi\\
&\geqq \mu^p\Phi-C\mu\Phi^{1/p}-\mu\frac{C_0}{p}\,\Phi^{\frac{1}{p}-1}\Phi^{\frac{1}{q}+1}\\
&=\left(\mu^p-\mu\frac{C_0}{p}\right)\Phi-C\mu\Phi^{1/p}\geqq f,
\end{aligned}$$

if μ is large enough, say $\mu\geqq\mu_0>0$. We then fix $\mu=\mu_0$ and we consider on the set $d(x)>\delta_0$

$$-\Delta\bar{w}+|\nabla\bar{w}|^p+\lambda\bar{w}\geqq -M+\lambda K$$

for some constant M, and choosing $K\geqq\frac{1}{\lambda}\left(M+\sup_{\Omega_{\delta_0}}|f|\right)$ we conclude.

In particular, we see that $\underline{u}_\delta\leqq\bar{w}$ and thus $\underline{u}_\delta$ is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$. Furthermore, by the bounds proved in the appendix, this implies that $\underline{u}_\delta$ is also bounded in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and thus in $W^{2,r}_{\mathrm{loc}}(\Omega)$ by elliptic regularity. And, letting δ go to 0, $\underline{u}_\delta$ increases to a solution of (1) $\underline{u}$ which is above $\underline{w}$. The fact that $\underline{u}$ is the minimum solution of (1) which goes to $+\infty$ on $\partial\Omega$ is an easy consequence of the fact that any such solution is above $\underline{u}_\delta$ by the maximum principle.

To prove the existence of a maximum solution of (1) going to $+\infty$ on $\partial\Omega$, we first observe that $\bar{w}_\delta = \Psi(d(x)-\delta)+K$ is also a supersolution of (1) with Ω replaced by Ω_δ. Therefore, by maximum principle, any solution of (1) is below $\bar{w}_\delta$ and, passing to the limit in δ, thus below $\bar{w}$.

To build the maximum solution, several arguments are possible. One way to do it consists in maximizing $u(x_0)$ for some fixed $x_0 \in \Omega$ among all solutions of (1) bounded from below on Ω (or equivalently going to $+\infty$ on $\partial\Omega$). Then, observe that if u_1, u_2 are two such solutions then there exists another one, say u_3, above u_1 and u_2: indeed $\max(u_1, u_2)$ is a subsolution of (1) and we may solve for

$$-\Delta u_3^\delta + |\nabla u_3^\delta|^p + \lambda u_3^\delta = f \quad \text{in} \quad \Omega_\delta, \qquad u_3^\delta = \max(u_1, u_2) \quad \text{on} \quad \partial\Omega_\delta$$

the existence follows from [19]. Then $u_3^\delta \leqq \bar{w}_\delta$ and thus is bounded in $W^{2,r}_{\mathrm{loc}}(\Omega)$ by arguments we already made several times. Using several times the maximum principle, we see that u_3^δ converges (and increases) to a solution u_3 of (1) which is above u_1 and u_2. This observation implies that there exists a maximizing sequence (u_n) of solutions of (1) which maximizes $u_n(x_0)$ and which is nondecreasing. Then, since $u_n \leqq \bar{w}$, u_n converges (use again the a priori estimates) to a solution $\bar{u}$ of (1) which is bounded from below on Ω and thus blows up at $\partial\Omega$. Furthermore, the above construction of u_3 shows that the fact that $\bar{u}$ maximizes $u(x_0)$ among all solutions implies in fact that $\bar{u}$ is the maximum solution of (1).

Proof of Theorem III.3. Using the results of Theorem III.2 and their proofs, it is now easy to mimick the proofs of Theorems II.1–II.2 in order to obtain the uniqueness. Indeed, if we use (18′), we may replace the functions $\bar{w}_\delta$, $\underline{w}_\delta$ built above by the ones given by (22) provided one takes the values for C_0, α which are given in Theorem I.3. Then, this implies that, by the same proof as above, the minimum solution $\underline{u}$ and the maximum solution $\bar{u}$ of (1) going to $+\infty$ on $\partial\Omega$ satisfy

$$(C_0-\varepsilon)d(x)^{-\alpha} - C_\varepsilon \leqq \underline{u}(x) \leqq \bar{u}(x) \leqq (C_0+\varepsilon)d(x)^{-\alpha} + C_\varepsilon \quad \text{in} \quad \Omega$$

and we may now conclude using the same proof as in Theorem I.1. □

We now conclude this section with an improved uniqueness result where however no precise behaviour of the solution is given.

Theorem III.4. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ satisfy*

$$C'd^{-\beta} - C' \leqq f \leqq Cd^{-\beta} + C \quad \text{for some} \quad C \geqq C' > 0, \qquad \beta \geqq \max(p, q). \tag{40}$$

Then, there exists a unique solution of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$) which is bounded from below. Denoting by u this solution, we have for some $M \geqq 1$

$$\frac{1}{M} d^{-\alpha} - M \leqq u \leqq Md^{-\alpha} + M \quad in \quad \Omega,$$

where $\alpha = \dfrac{\beta}{p} - 1$ if $\beta > p$, and $d^{-\alpha}$ is replaced by $|\mathrm{Log}\, d|$ if $\beta = p \geqq q$.

Proof. By similar arguments to the ones given above, the maximum solution $\bar{u}$ and the minimum solution $\underline{u}$ satisfy for some $M \geqq 1$

$$\frac{1}{M} d^{-\alpha} - M \leqq u \leqq Md^{-\alpha} + M \quad \text{in} \quad \Omega.$$

Without loss of generality (adding a large constant to $f, \underline{u}, \bar{u}$) we may assume that $\bar{u} \geqq \underline{u} \geqq 1$, $f \geqq 1$ a.e. in Ω. Therefore, there exists $\theta \in (0,1)$ small enough such that $\underline{u} \geqq \theta \bar{u}$ in Ω. Let then $\theta_0 = \sup\{\theta \in (0,1]/\underline{u} \geqq \theta \bar{u}$ in $\Omega\}$ – we follow a uniqueness argument which was introduced in a different context by Laetsch [14]. If $\theta_0 = 1$, we are done. We thus argue by contradiction and assume that $\theta_0 < 1$. Of course, we have $\underline{u} \geqq \theta_0 \bar{u}$ in Ω. We then consider $z = \varepsilon d^{-\alpha}$ and we observe that z satisfies

$$-\Delta z + |\nabla z|^p + \lambda z \leqq \varepsilon^p d^{-\beta} + C_\varepsilon d^{-\beta+1}$$

and this is less than f for ε small enough say $\varepsilon \leqq \varepsilon_0$. We choose $\varepsilon = \varepsilon_0$. In fact $z_\delta = \varepsilon(d+\delta)^{-\alpha}$ also satisfies

$$-\Delta z_\delta + |\nabla z_\delta|^p + \lambda z_\delta \leqq f \quad \text{in} \quad \Omega.$$

And we consider $w_{\gamma,\delta} = (\theta_0 - \gamma)\bar{u} + (1 - \theta_0 + \gamma) z_\delta$; $w_{\gamma,\delta}$ satisfies for $\gamma < \theta_0$

$$-\Delta w_{\gamma,\delta} + |\nabla w_{\gamma,\delta}|^p + \lambda w_{\gamma,\delta} \leqq (\theta_0 - \gamma) f + (1 - \theta_0 + \gamma) f \equiv f \quad \text{in} \quad \Omega$$

and since $\underline{u}$, $\bar{u}$ blow up near the boundary we have $w_{\gamma,\delta} \leqq \theta_0 \bar{u} \leqq \underline{u}$ near the boundary. Therefore, by the maximum principle, $w_{\gamma,\delta} \leqq \underline{u}$ in Ω. We now let γ go to 0_+ and then δ go to 0_+ to find

$$\theta_0 \bar{u} + (1 - \theta_0) z \leqq \underline{u} \quad \text{in} \quad \Omega$$

but we obviously have $z \geqq \nu \bar{u}$ for some $\nu > 0$. Hence,

$$(\theta_0 + (1 - \theta_0)\nu)\bar{u} \leqq \underline{u} \quad \text{in} \quad \Omega$$

and this contradicts the definition of θ_0. □

IV. Superquadratic Hamiltonians

IV.1. Interior Gradient Bounds and Maximum Solutions

We begin with a result which gives interior gradient bounds for solutions of (1): similar bounds were first derived in [16, 19] and the proofs are recalled in the appendix. We only remark here that a sharper form of these bounds may be obtained by a simple scaling argument.

Theorem IV.1. *Let $f \in L^\infty_{\text{loc}}(\Omega)$ be bounded from below on Ω and satisfy*

$$|f(x)| \leqq C_1 d(x)^{-\beta} \quad \textit{for some} \quad \beta \geqq 0, \qquad C_1 \geqq 0. \tag{41}$$

Let $u \in W^{2,r}_{\text{loc}}(\Omega)$ ($\forall r < \infty$) be a solution of (1) satisfying

$$\lambda u \geqq -C_2 \quad \textit{for some} \quad C_2 \geqq 0. \tag{42}$$

Then, we set $\gamma = \dfrac{1}{p-1}$ if $\beta \leqq q$, γ arbitrary in $\left(\dfrac{\beta}{p}, 1\right)$ if $\beta > q$ and $\gamma = \dfrac{\beta}{p}$ if $f \in W^{1,\infty}_{\text{loc}}(\Omega)$ and $|\nabla f(x)| d(x)^{-\beta-1} \in L^\infty(\Omega)$. With these notations and assumptions we have

$$|\nabla u(x)| \leqq C_3 d(x)^{-\gamma} \quad \textit{in} \quad \Omega, \quad \square \tag{43}$$

where C_3 only depends on C_1, C_2, γ, β and the diameter of Ω.